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In the context of abstract interpretation based static analysis, we cope with the problem of correctness and optimality for logic program analysis. We propose a new framework equipped with a denotational, goal-dependent semantics which refines many goal-driven frameworks appeared in the literature. The key point is the introduction of two specialized concrete operators for forward and backward unification. We prove that our goal-dependent semantics is correct w.r.t. computed answers and we provide the best correct approximations of all the operators involved in the semantics for set-sharing analysis. We show that the precision of the overall analysis is strictly improved and that, in some cases, we gain precision w.r.t. more complex domains involving linearity and freeness information.

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1. INTRODUCTION

The theory of abstract interpretation [Cousot and Cousot 1977; 1979] provides a general framework for modeling and developing static analysis of programs. The basic idea of abstract interpretation is to use the formal semantics of languages to analyze and verify program properties. Given any (concrete) semantics over some (concrete) domain of objects expressing program denotation, an abstract interpretation is given by an abstract semantics over a domain of abstract objects, able to mimic the concrete semantics and to compute (in an approximated way) some program property. Thus, when designing an abstract interpretation, the concrete semantics and its concrete domain play a fundamental role. Once fixed a concrete semantics, the next step is to find a suitable abstraction (a function mapping concrete objects to abstract ones) and operators on the abstract domain able to compute a sound approximation of the concrete program semantics. Finding (good) abstraction mappings and (optimal) abstract operators are the two major difficulties in any abstract interpretation project.

In the context of declarative languages, abstract interpretation has been widely used to design static analysis for logic programs, by exploiting the declarative model underlying most of the logic program semantics in order to project sound and useful abstractions. The most interesting (and studied) properties for logic programs are arguably groundness and sharing. A term is said to be ground if it contains no variables. Groundness analysis aims at discovering ground variables in the answer substitutions, that is to say, variables that will be bind, in all the answer substitutions, to ground terms only. On the contrary, the goal of (set) sharing analysis is to detect sets of variables which share a common variable. For instance, in the answer substitution $\{x/f(z,a), y/g(z)\}$ for the goal $p(x,y)$, the variables x and y share the common variable z. Typical applications of sharing analysis are in the fields of optimization of unification [Søndergaard 1986] and parallelization of logic programs [Hermenegildo and Rossi 1995].

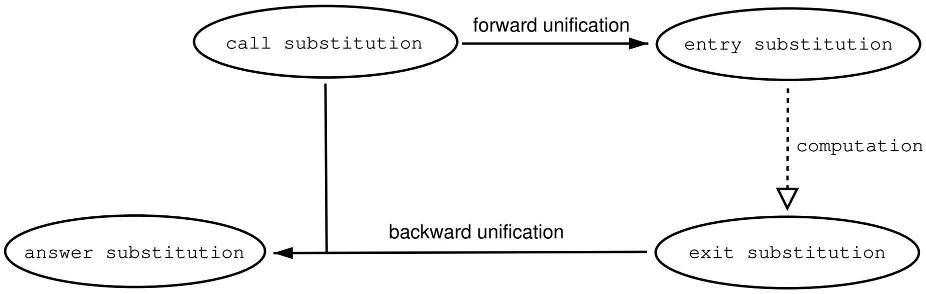
The property of sharing has been the object of many works, focusing on both the theoretical and practical aspects of the analysis. While the results on groundness analysis seem to converge toward the domain `Pos` [Armstrong et al. 1998; Cortesi et al. 1996] which is commonly recognized as the optimal domain for detecting the groundness property, the same does not happen for the sharing property. The problem of finding a "good" domain for sharing analysis is still open, as proved by the amount of work recently appeared for this specific topic. The common starting point of most of these proposals is the domain `Sharing` [Langen 1990; Jacobs and Langen 1992] (slightly modified in [Cortesi and Filè 1999]). It is widely recognized that the original domain `Sharing` is not very precise, so that it is often combined with other domains for treating freeness, linearity, groundness or structural information (see Bagnara et al. [2000] for a comparative evaluation). Among the various proposals, we find new domains for more efficient and/or more precise sharing analyses [Bagnara et al. 2002; Howe and King 2001], combinations of domains (e.g., including freeness and/or linearity information [Hans and Winkler 1992; Muthukumar and Hermenegildo 1992]), and many techniques for improving the abstract operators used in the analysis [King and Longley 1995; Hans and Winkler 1992]. The above solutions suggest to migrate toward more complex domains (and analysis) in order to gain more precision. On the contrary, we believe that some of the weak points of these abstract interpretations may be found already in the concrete domains and semantics used as starting point for the abstraction. Therefore, we plan to devise a new framework for the analysis of sharing properties, by using the classical `Sharing` domain, but being careful in the choice of the right concrete domain and operators, in order to gain the more precision is possible.

Typically, concrete semantics for abstract interpretation based analysis of logic programs are defined over a concrete domain of substitutions. But substitutions are often too informative for describing the semantics of logic programs. For example, consider the one-clause program `p(x,x)` and the goal $p(x,y)$. All of $\{x/y\}$, $\{y/x\}$, $\{x/u, y/v\}$, $\{x/v, y/v\}$ are computed answers, corresponding to different choices of most general unifiers and renamed clauses. But we are not interested in making any distinction among them. Thus, it would be more natural to adopt a domain of (equivalence) classes of substitutions. Many framework for abstract interpretation of logic programs (see [Jacobs and Langen 1992; Marriott et al. 1994; Levi and Spoto 2003]) have adopted similar solutions for avoiding redundancy and causality when choosing computed answers. Nevertheless, the standard semantics of logic programs, namely SLD resolution, computes substitutions. Thus, any framework for logic programming should relate, in some way, to computed answer substitutions, in order to prove that the semantics reflects the underlying operational behavior. However, all the above proposals depart from the standard notion of substitution, loosing the relationship between the semantics result and the standard set of computed answers given by SLD resolution. We would like to reconcile these approaches with the standard concept of substitution: in particular, we show that these domains are quotient sets of substitutions, w.r.t. suitable equivalence relations. Therefore, we propose a domain of classes of substitution, equipped with a set of functions for performing the basic operations of projection, renaming and unification, and show the relationship between our domain, the standard set of substitutions and one of the above domains.

Once fixed the concrete domain, we need to provide a concrete semantics. We are interested in goal-driven analysis of logic programs, therefore we need a goal-dependent semantics which is well suited for static analysis, i.e., a collecting semantics over computer answers. Unfortunately, using a collecting goal-dependent semantics may lead to a loss of precision already at the concrete level, as shown by Marriott et al. [1994]. Actually, all the semantics proposed in the above frameworks introduce a loss of precision in computing answer substitutions. Basically, in any goal-dependent semantics, the unification operator is used twice:

—For performing parameter passing by unifying the given goal and the call substitution with the head of the chosen clause. The result is a new goal and an entry substitution. This is called *forward unification*.

—For propagating back to the initial goal the exit substitution (that is, the result of the sub-computation), so obtaining the answer substitution (or success substitution) for the initial goal. This is called *backward unification*.



For instance, given the initial goal $p(x)$ and the call substitution $\{x/f(y)\}$, we unify with the head of the clause `p(z)` $\leftarrow$ `q(z)` by computing the most general unifier $\{x/f(y), z/f(y)\}$, which, projected on the variables of the clause, is simply $\{z/f(y)\}$. Projection is needed in order to avoid an unbounded growing of the set of variables in the entry substitution, which is acceptable at the concrete level, but not at the abstract level, where it may lead to non-terminating analysis. The new goal and entry substitution become $q(z)$ and $\{z/f(y)\}$. Once obtained an exit substitution for the goal $q(z)$, for instance $\{z/f(a)\}$, we have to relate this result to the original goal $p(x)$. Thus we need a so-called backward unification, which allows us to conclude that $\{x/f(a)\}$ is an answer for $p(x)$ with call substitution $\{x/f(y)\}$.

Using a backward unification operator in a collecting semantics introduce a loss of precision, due to the fact that we deal with a set of call substitutions, from which we possibly obtain a set of exit substitutions. Now, when we go backward to obtain the answer substitutions, we may unify a call substitution with an exit substitution which does not pertain to the same computational path (see Section 4.3 or [Marriott et al. 1994]). It is possible to reduce this problem by using two different operators for forward and backward unification. In this way, backward unification can be realized using the operation of matching between substitutions, as already suggested in [Hans and Winkler 1992; King and Longley 1995]. In turn, using two different operators allows to improve also the forward unification, which can be specialized in order to exploit the peculiarity of this process: the variables which occur in the clause head are always renamed apart w.r.t. the goal and the calling substitutions, hence they are free and independent from the rest of the variables. This is the first time this idea is pursued to improve the precision of an abstract analysis. Thanks to these choices, we improve the precision of the collecting semantics already at the concrete level.

On the abstract side, we provide the forward and backward unification operators for set sharing analysis. We show that they are both correct and optimal w.r.t. the concrete operators, that is, they are the best correct approximations of the concrete operators. Optimality is one of the main points in the design of any abstract interpretation, guaranteeing that it is not possible to improve the precision of the analysis for a given abstract domain. Note that the abstract forward unification takes advantage of the fact that some variables are known to be free and independent from all the others. This is the first time that an abstract unification operator, exploiting freeness or linearity, is proved to be optimal. Note, however, that freeness and linearity informations are not encoded in the abstract domain, but are just used in the internal steps of the abstract unification algorithm. This means that the algorithm cannot be immediately extended to work with more complex domains, such as SFL [King and Longley 1995], retaining optimality. Nonetheless, the abstract unification is able to exploit freeness and linearity better than other algorithms: that could be used to improve the unification operation in more complex domains. This is also the first time that a backward abstract unification operator is proved to be correct and optimal w.r.t. a corresponding concrete operator. Moreover, it appears to be more precise than all the other backward unifications in literature.

These new operators allow us to obtain a strictly more precise sharing analysis than all the others which use the same domain `Sharing`. However, nothing of the ideas contained in this paper is tied to the abstract domain in use. The framework we propose may be coupled with better abstract domains to improve even more the result of the abstract analysis.

2. NOTATIONS

Given a set A, let $\wp(A)$ be the powerset of A and $\wp_f(A)$ be the set of finite subsets of A. Given two posets $(A, \leq_A)$ and $(B, \leq_B)$, we denote by $A \rightarrow B$ ($A \rightarrow^c B$) the space of monotonic (continuous) functions from A to B ordered pointwise. We use $\leq_B$ to denote the order relation over $A \rightarrow B$. When an order for A or B is not specified, we assume the least informative order ($x \leq y \iff x = y$). We also use $A \uplus B$ to denote disjoint union and $|A|$ for the cardinality of the set A.

Given A, C complete lattices, a *Galois Insertion* [Cousot and Cousot 1979] $\langle \alpha, \gamma \rangle$:

$C \leftrightharpoons A$ is given by a pair of maps $\alpha : C \to A$, $\gamma : A \to C$ such that $\alpha(c) \leq_A a \iff c \leq_C \gamma(a)$ and α is onto (or equivalently, γ is injective). We say that an abstract operator $f^\alpha : A \to A$ is *correct* w.r.t. a concrete operator $f : C \to C$ when $\alpha \circ f \leq_A f^\alpha \circ \alpha$, which is equivalent to $f \circ \gamma \leq_C \gamma \circ f^\alpha$ and to $f^\alpha \geq_A \alpha \circ f \circ \gamma$. The abstract operator is *optimal* when $f^\alpha = \alpha \circ f \circ \gamma$. In this case f^α is called the *best correct approximation* of f. When $\alpha \circ f = f^\alpha \circ \alpha$ then f^α is said to be *complete*, while if $f \circ \gamma = \gamma \circ f^\alpha$ then f^α is γ-*complete* (also called γ-optimal in [Cortesi et al. 1996]).

In the following, we fix a first order signature (Σ, Π) and an infinite set of variables $\mathcal{V}$. We use Terms and Atoms to denote the set of terms and atomic formulas (atoms) respectively. Moreover, we call *body* or *goal* a finite sequence of atomic formulas, *clause* an object $H \leftarrow B$ where H is an atom and B is a body, *program* a set of clauses. We use $\Box$ for the empty body and we write H as a short form for $H \leftarrow \Box$. We denote with Bodies, $\mathsf{Clauses}$ and Progs the set of bodies, clauses and programs respectively. Given a term t, we denote by $\mathrm{vars}(t)$ the set of variables occurring in t and by $\mathrm{uvars}(t)$ the subset of $\mathrm{vars}(t)$ whose elements only appear once in t (e.g., $\mathrm{uvars}(t(x, y) = t(y, z)) = \{x, z\}$). We apply vars and uvars to any syntactic object, with the obvious meaning. We abuse the notation and write a syntactic object o instead of the set of variables $\mathrm{vars}(o)$, when it is clear from the context (e.g., if t is a term and $x \in \mathcal{V}$, then $x \in t$ should be read as $x \in \mathrm{vars}(t)$).

We denote with ϵ the empty substitution and by $\{x_1/t_1, \ldots, x_n/t_n\}$ a substitution θ with $\theta(x_i) = t_i \neq x_i$. Let $\mathrm{vars}(\theta)$ be the set $\mathrm{dom}(\theta) \cup \mathrm{rng}(\theta)$ and, given $U \in \wp_f(\mathcal{V})$, let $\theta_{|U}$ be the projection of θ over U, i.e., the only substitution such that $\theta_{|U}(x) = \theta(x)$ if $x \in U$ and $\theta_{|U}(x) = x$ otherwise. We also write $\theta_{|-U}$ to denote the restriction of θ over all variables but those in U, i.e., $\theta_{|-U} = \theta_{|\mathrm{dom}(\theta) \setminus U}$. Given θ_1 and θ_2 two substitutions with disjoint domains, we denote by $\theta_1 \uplus \theta_2$ the substitution θ such that $\mathrm{dom}(\theta) = \mathrm{dom}(\theta_1) \cup \mathrm{dom}(\theta_2)$ and $\theta(x) = \theta_i(x)$ if $x \in \mathrm{dom}(\theta_i)$, for each $i \in \{1, 2\}$. The application of a substitution θ to a term t is written as $t\theta$ or $\theta(t)$. Given two substitutions θ and δ, their composition, denoted by $\theta \circ \delta$, is given by $(\theta \circ \delta)(x) = \theta(\delta(x))$. Instantiation induces a preorder on substitutions: θ is more general than δ, denoted by $\delta \leq \theta$, if there exists σ such that $\sigma \circ \theta = \delta$. If $\approx$ is the equivalence relation induced by $\leq$, we say that σ and θ are equal up to renaming when $\sigma \approx \theta$. The set of substitutions, idempotent substitutions and renamings are denoted by *Subst*, *ISubst* and *Ren* respectively.

Given a set of equations E, we write $\sigma = \mathrm{mgu}(E)$ to denote that σ is a most general unifier of E. Since σ is only defined up to renamings, we only use this notation in a context where it is not important the actual unifier which is chosen. Any idempotent substitution σ is a most general unifier of the corresponding set of equations $\mathrm{Eq}(\sigma) = \{x = \sigma(x) \mid x \in \mathrm{dom}(\sigma)\}$. In the following, we will abuse the notation and denote by $\mathrm{mgu}(\sigma_1, \ldots, \sigma_n)$, when it exists, the substitution $\mathrm{mgu}(\mathrm{Eq}(\sigma_1) \cup \ldots \cup \mathrm{Eq}(\sigma_n))$.

In the rest of the paper, we use: U, V, W to denote finite sets of variables; h, k, u, v, w, x, y, z for variables; c, s, t for term symbols or terms; a, b for constants; cl for clauses; $\eta, \theta, \sigma, \delta$ for substitutions; ρ for renamings. The same holds for derivatives of these symbols, obtained by adding subscripts, superscripts or both.

3. DOMAINS OF EXISTENTIALLY QUANTIFIED SUBSTITUTIONS

The first question we need to ask ourselves when analyzing the behavior of logic programs is what kind of observable we are interested in. Undoubtedly, computed answers have played a prominent role, since they are the result of the process of SLD-resolution. Moreover, they have several nice properties: and-compositionality, condensing and a bottom-up T_P-like characterization [van Emden and Kowalski 1976; Bossi et al. 1994]. However, we are really interested in computed answers only up to renaming. For example, consider the one-clause program $\mathtt{p(x,x)}$. and the goal $p(x,y)$. All of $p(x,x)$, $p(y,y)$, $p(u,u)$ and $p(v,v)$ are computed answers, corresponding to different choices of most general unifiers and renamed clauses, but we are not interested in making any distinction among them. For instance, the s-semantics [Bossi et al. 1994] is defined over equivalence classes of atoms modulo renaming. Moreover, when we consider a denotational semantics suited for program analysis, computed answer *substitutions* are much more useful than computed answers, since most of the domains are expressed as abstractions of sets of substitutions. Again, we are not really interested in the substitutions, but in their quotient-set w.r.t. an appropriate equivalence relation. However, this time we cannot take renaming as the relevant equivalence relation. Let us consider the answer substitutions corresponding to the computed answers in the previous example: we obtain $\theta_1 = \{y/x\}$, $\theta_2 = \{x/y\}$, $\theta_3 = \{x/u, y/u\}$ and $\theta_4 = \{x/v, y/v\}$. Although θ_1 and θ_2 are equal up to renaming, the same does not hold for θ_3 and θ_4. Nonetheless, they essentially represent the same answer, since u and v are just two different variables we chose when renaming apart the clause $\mathtt{p(x,y)}$ from the goal $p(x,y)$, and therefore are not relevant to the user. On the other side, if θ_3 and θ_4 are answer substitutions for the goal $p(x,y,u)$, then they correspond to computed answers $p(u,u,u)$ and $p(v,v,u)$ and therefore are fundamentally different. As a consequence, the equivalence relation we need to consider must be coarser then renaming, and must take into account the set of variables of interest, i.e., the set of variables which appear in the goal.

A semantics for computed answer (substitutions) may follow one of three possible directions:

(1) it may compute only part of the answer substitutions, provided that at least one substitution in each equivalence class is given as a result (e.g., [Cortesi et al. 1994]);

(2) it may compute all the computed answer substitutions;

(3) it may directly work with a (suitable) quotient domain of substitutions (e.g., [Marriott et al. 1994]).

The problem with the first two solutions is that they work by directly manipulating substitutions. Everyone knows that this is quite tedious and error prone. This happens because substitutions are too much related to syntax, so that the intuition of what should happen is often betrayed by the reality, when we need to handle problems such as variable clashes and renamings. Actually, at least one framework of the first kind, namely the widely used one in [Cortesi and Filè 1999], has a small flaw due to an unsound treatment of variable clashes (this will be discussed in details in Section 8.2).

Moreover, the first approach is generally pursued by choosing a particular most general unifier and a fixed way of renaming apart terms and substitutions. The semantics is then parametric with respect to these choices. As stated by Jacobs and Langen [1992], this makes difficult to compare different semantics, since each of them may use different conventions for mgu and renaming. We would like to add that this also makes difficult to state properties of a given semantics (such as compositionality properties), since they only hold up to suitable equivalence relations.

For these reasons, we think that the best solution is to move towards a domain of equivalence classes of substitutions. This does not mean we can avoid to work with substitutions altogether, but all the difficulties which arise, such as renaming apart and variables clashes, may be dealt once and for all at the domain level, reducing the opportunities for subtle mistakes to appear.

3.1 Yet another Domain of Existentially Quantified Substitutions

There are several semantics for logic programming in literature, working on different domains of equivalence classes of substitutions: *ESubst* [Jacobs and Langen 1992], ex-equations [Marriott et al. 1994] and existential Herbrand constraints [Levi and Spoto 2003]. For all of them, the basic idea is that some variables, in a substitution or equation, are existentially quantified, so that their names become irrelevant. However, all these proposals depart from the standard notion of substitution. As a result, the relationship between what they compute and the standard set of computed answers for a goal has never been proved. We would like to reconcile these approaches with the standard concept of substitution: in particular, we want to prove that these domains are quotient sets of substitutions, w.r.t. suitable equivalence relations.

We begin, in this section, by introducing a new equivalence relation $\sim$ over substitutions, which capture the extended notion of renaming which is needed to work with computed answers. Moreover, we introduce a new domain $\mathit{Subst}_\sim$ of classes of substitutions modulo $\sim$, which will be used in the rest of the paper. We are inspired, in this, by the seminal paper of Palamidessi [1990]. Later, in Section 8.1, we will prove that $\mathit{Subst}_\sim$ and the domain *ESubst* [Jacobs and Langen 1992] are isomorphic. Similar proofs may be developed for ex-equations and Herbrand constraints.

Given $\theta_1, \theta_2 \in \mathit{Subst}$ and $U \in \wp_f(\mathcal{V})$, we define the preorder:

$$\theta_1 \preceq_U \theta_2 \iff \exists \delta \in \mathit{Subst}. \forall v \in U.\ \theta_1(v) = \delta(\theta_2(v)) \ . \tag{1}$$

Intuitively, if $\theta_1 \preceq_U \theta_2$, then θ_1 is an instance of θ_2, provided we are only interested in the variables in U.

Example 3.1. It is easy to check that $\{x/a, y/u\} \preceq_{\{x,y\}} \{y/v\}$, since we may choose $\delta = \{x/a, v/u\}$ in (1). Note that the same does not happen if we consider the standard ordering on substitutions, i.e., $\{x/a, y/u\} \not\leq \{y/v\}$. Moreover, if we enlarge the set U of variables of interest, we obtain $\{x/a, y/u\} \not\preceq_{\{x,y,v\}} \{y/v\}$.

Note that, in Equation (1), it is important that δ is a generic substitution. If we restrict δ to be idempotent, some equivalences do not hold anymore. For example, $\{x/t(u), y/t(v)\} \preceq_{\{x,y\}} \{x/v, y/u\}$ and this is what we intuitively want, since the names of the variables u and v are not relevant. However, to prove this relation, we choose $\delta = \{u/t(v), v/t(u)\}$ in (1), and it is not an idempotent substitution.

PROPOSITION 3.2. *$\preceq_U$ is a preorder for any $U \in \wp_f(\mathcal{V})$.*

PROOF. Let $U \in \wp_f(\mathcal{V})$. By definition, $\theta \preceq_U \theta \iff \exists \delta \in \mathit{Subst}.\forall v \in U.\ \theta(v) = \delta(\theta(v))$, which is a tautology by choosing as δ the empty substitution. Now assume $\theta_1 \preceq_U \theta_2$ and $\theta_2 \preceq_U \theta_3$. Therefore, there exist δ_1 and δ_2 such that, $\forall v \in U$, $\theta_1(v) = \delta_1(\theta_2(v))$ and $\theta_2(v) = \delta_2(\theta_3(v))$. Therefore, $\forall v \in U$, it holds $\theta_1(v) = \delta_1(\theta_2(v)) = \delta_1(\delta_2(\theta_3(v)))$. Therefore, by choosing as δ the composition $\delta_1 \circ \delta_2$ we have that $\theta_1 \preceq_U \theta_3$. $\square$

The next step is to define the relation:

$$\theta_1 \sim_U \theta_2 \iff \exists \rho \in \mathit{Ren}.\forall v \in U.\ \theta_1(v) = \rho(\theta_2(v)) \ , \tag{2}$$

which may be proved to be the equivalence relation induced by the preorder $\preceq_U$.

Example 3.3. It is easy to check that $\{x/v, y/u\} \sim_{\{x,y\}} \epsilon$ by choosing $\rho = \{x/v, v/x, y/u, u/y\}$. Note that $\sim_U$ is coarser than the standard equivalence relation $\approx$: there is no renaming ρ such that $\epsilon = \rho \circ \{x/v, y/u\}$. As it happens for $\preceq$, if we enlarge the set of variables of interest, not all equivalences between substitutions are preserved: for instance, $\{x/v, y/u\} \not\sim_{\{x,y,v\}} \epsilon$.

LEMMA 3.4. *Let $\theta : V \to \mathcal{V}$ an injective map of variables. Then there exists $\rho \in \mathit{Ren}$ such that $\rho(x) = \theta(x)$ for each $x \in V$ and* $\mathrm{vars}(\rho) = V \cup \theta(V)$.

PROOF. Since θ is injective, $|V| = |\theta(V)|$, from which it follows that $|V \setminus \theta(V)| = |\theta(V) \setminus V|$. Let f be any bijective map from $|\theta(V) \setminus V|$ to $|V \setminus \theta(V)|$, and let us define a substitution ρ as follows:

$$\rho(v) = \begin{cases} \theta(v) & \text{if } v \in V \\ f(v) & \text{if } v \in \theta(V) \setminus V \\ v & \text{otherwise.} \end{cases}$$

Note that, if $x \in V$, $\rho(x) = \theta(x)$ by definition. Moreover, it is easy to check that ρ is bijective, therefore, it is a renaming. Finally, $\mathrm{vars}(\rho) = \mathrm{dom}(\rho) = V \cup (\theta(V) \setminus V) = V \cup \theta(V)$. $\square$

PROPOSITION 3.5. *$\sim_U$ is the equivalence relation induced by $\preceq_U$.*

PROOF. If $\theta_1 \sim_U \theta_2$ there exists $\rho \in \mathit{Ren}$ such that $\forall v \in U.\ \theta_1(v) = \rho(\theta_2(v))$. By definition of $\preceq_U$, we have that $\theta_1 \preceq_U \theta_2$ by choosing as δ in (1) the renaming ρ. Symmetrically, by choosing as δ the renaming ρ^{-1} (the inverse of ρ), it follows that $\theta_2 \preceq_U \theta_1$.

Now assume that $\theta_1 \preceq_U \theta_2$ and $\theta_2 \preceq_U \theta_1$. Therefore there exist $\delta, \delta' \in \mathit{Subst}$ such that $\theta_2(x) = \delta'(\theta_1(x))$ and $\theta_1(x) = \delta(\theta_2(x))$, thus $\theta_2(x) = \delta'(\delta(\theta_2(x)))$ for each $x \in U$. In general, δ and δ' might not be renamings. Our goal is to build a renaming ρ, obtained by modifying δ, such that $\theta_1(x) = \rho(\theta_2(x))$. Let $V = \mathrm{vars}(\theta_2(U))$. Since each $v \in V$ belongs to $\mathrm{vars}(\theta_2(x))$ for some $x \in U$, it follows that $(\delta' \circ \delta)(v) = v$ for all $v \in V$. Therefore, $\delta_{|V}$ may be viewed as an injective map from V to $\mathcal{V}$. By Lemma 3.4, there exists $\rho \in \mathit{Ren}$ such that $\rho_{|V} = \delta_{|V}$. Therefore, for each $x \in U$, $\rho(\theta_2(x)) = \delta(\theta_2(x)) = \theta_1(x)$. $\square$

It is worth noting that $\preceq_U$ is coarser than $\leq$ and that $\sim_U$ is coarser than renaming. It is also easy to check that, given $\rho \in \mathit{Ren}$ and $\delta \in \mathit{Subst}$, $\rho \circ \theta \sim_U \theta$ and $\delta \circ \theta \preceq_U \theta$.

Now, let $\mathit{ISubst}_{\sim_U}$ be the quotient set of ISubst w.r.t. $\sim_U$. We define a new domain $\mathit{ISubst}_{\sim}$ of *existential substitutions* as the disjoint union of all the $\mathit{ISubst}_{\sim_U}$ for $U \in \wp_f(\mathcal{V})$, in formulas:

$$\mathit{ISubst}_{\sim} = \biguplus_{U \in \wp_f(\mathcal{V})} \mathit{ISubst}_{\sim_U} \ . \tag{3}$$

In the following we write $[\theta]_U$ for the equivalence class of θ w.r.t. $\sim_U$. We call *canonical representatives* of the equivalence class $[\theta]_U \in \mathit{ISubst}_{\sim}$ the substitutions $\theta' \in \mathit{ISubst}$ such that $\theta' \sim_U \theta$ and $\mathrm{dom}(\theta') = U$. It is immediate to see that every existential substitution has a canonical representative, although it is not unique. For example, two canonical representatives of $[\{y/f(x)\}]_{x,y,z}$ are $\{y/f(h), x/h, z/k\}$ and $\{y/f(u), x/u, z/v\}$. Working with canonical representatives is of great help, especially in the proofs, since we are sure they have no variables of interest in the range.

By definition of $\preceq_U$, when $\theta \preceq_U \theta'$ then, for all $W \subseteq U$, it holds that $\theta \preceq_W \theta'$. This allows us to define a partial order $\preceq$ over $\mathit{ISubst}_{\sim}$ given by:

$$[\theta]_U \preceq [\theta']_V \iff U \supseteq V \wedge \theta \preceq_V \theta' \ . \tag{4}$$

Intuitively, $[\theta]_U \preceq [\theta']_V$ means that θ is an instance of θ' w.r.t. the variables in V, provided that they are all variables of interest of θ. It is easy to show that $\preceq$ is well-defined in $\mathit{ISubst}_{\sim}$, that is it does not depend on the choice of the representatives.

Note that, although we use equivalence classes of idempotent substitutions, we could build an isomorphic domain by working with equivalence classes of the set of all the substitution. In other words, if we define $\mathit{Subst}_{\sim} = \biguplus_{U \in \wp_f(\mathcal{V})} \mathit{Subst}_{\sim_U}$, we obtain the following:

PROPOSITION 3.6. *The posets* $(\mathit{Subst}_{\sim}, \preceq)$ *and* $(\mathit{ISubst}_{\sim}, \preceq)$ *are isomorphic.*

PROOF. It is enough to prove that, for each $U \in \wp_f(\mathcal{V})$ and $\theta \in \mathit{Subst}$, there exists $\theta' \in \mathit{ISubst}$ such that $\theta \sim_U \theta'$. Let $V = \mathrm{rng}(\theta) \cap \mathrm{dom}(\theta)$ and $W \subseteq \mathcal{V}$ such that $W \cap (U \cup \mathrm{vars}(\theta)) = \emptyset$ and $|V| = |W|$. Moreover, we take a renaming ρ such that $\mathrm{vars}(\rho) = V \cup W$ and $\rho(V) = W$. Then, we may define a substitution θ' such that

$$\theta' = (\rho \circ \theta)_{|U} \ .$$

Note that $\mathrm{dom}(\theta') = (\mathrm{dom}(\theta) \cup W) \cap U \subseteq \mathrm{dom}(\theta)$ and $\mathrm{rng}(\theta') \subseteq \mathrm{rng}(\theta) \setminus V \cup W$. Therefore, $\mathrm{dom}(\theta') \cap \mathrm{rng}(\theta') = \emptyset$, i.e., $\theta' \in \mathit{ISubst}$. Moreover, by definition, $\theta' \sim_U \theta$. $\square$

The isomorphism between $\mathit{Subst}_{\sim}$ and $\mathit{ISubst}_{\sim}$ holds since a variable in $\mathrm{rng}(\theta)$ is considered not of interest if it also occurs in $\mathrm{dom}(\theta)$. Therefore $\{x/y, y/x\} \sim_{\{x,y\}} \{x/u, x/v\}$, since y and x in the range of $\{x/y, y/x\}$ are just names for existential quantified variables. Obviously $\{x/y\} \not\sim_{\{x,y\}} \{x/u\}$ since here y only appears in the range and is therefore considered as a variable of interest.

3.2 Operations on the new Domain

It is now time to define some useful operations over $ISubst_{\sim}$. These will be used as building blocks for the semantics to be defined further away in the paper. They will also give some more insights over the structure of $ISubst_{\sim}$. To ease notation, we often omit braces from the sets of variables of interest when they are given extensionally. So we write $[\theta]_{x,y}$ instead of $[\theta]_{\{x,y\}}$ and $\sim_{x,y,z}$ instead of $\sim_{\{x,y,z\}}$. When the set of variables of interest is clear from the context or it is not relevant, it will be omitted. Finally, we omit the braces which enclose the bindings of a substitution when it occurs inside an equivalence class, i.e., we write $[x/y]_U$ instead of $[\{x/y\}]_U$.

3.2.1 *Projection.* We define an operator which projects an element of $ISubst_{\sim}$ over a given set of variables V, given by

$$\pi_V([\sigma]_U) = [\sigma]_{U\cap V} \ , \tag{5}$$

which can be easily proved to be well-defined. Moreover, the following properties hold:

(1) $\pi_U \circ \pi_V = \pi_{U\cap V}$;
(2) $\pi_U([\sigma]_U) = [\sigma]_U$;
(3) π_V is monotonic w.r.t. $\preceq$.

3.2.2 *Renaming.* Another useful operation on classes of substitutions is renaming. We first define the application of a renaming $\rho \in Ren$ to a substitution $\theta \in Subst$ as

$$\rho(\theta) = \{\rho(x)/\rho(\theta(x)) \mid x \in \mathrm{dom}(\theta)\} \ . \tag{6}$$

Intuitively, we treat θ as a syntactic object and apply the renaming to both left and right hand sides. Note that $\rho(\theta)$ can be equivalently defined as $\rho \circ \theta \circ \rho^{-1}$.

Proposition 3.7. *Given $\rho \in Ren$ and $\theta \in Subst$ it holds that $\rho(\theta) = \rho \circ \theta \circ \rho^{-1}$.*

Proof. Let $\theta' = \rho(\theta)$. Since $x \neq \theta(x)$ for all $x \in \mathrm{dom}(\theta)$, then $\rho(x) \neq \rho(\theta(x))$ by injectivity of ρ. It follows that $\mathrm{dom}(\theta') = \rho(\mathrm{dom}(\theta))$. Given $x \notin \mathrm{dom}(\theta')$, it follows that $x \notin \rho(\mathrm{dom}(\theta))$ and thus $\rho^{-1}(x) \notin \mathrm{dom}(\theta)$. As a consequence, $\rho(\theta(\rho^{-1}(x))) = \rho(\rho^{-1}(x)) = x = \theta'(x)$. Now assume that $x \in \mathrm{dom}(\theta')$. Therefore there exists $y \in \mathrm{dom}(\theta)$ such that $\rho(y) = x$ and $\theta'(x) = \rho(\theta(y))$. Then $\rho(\theta(\rho^{-1}(x))) = \rho(\theta(y)) = \theta'(x)$. □

We may lift this definition to classes of substitutions in the standard way as follows:

$$\rho([\sigma]_U) = [\rho(\sigma)]_{\rho(U)} \ . \tag{7}$$

For example, let $\sigma = \{x/k, y/t(z,k)\}$, $U = \{x, y, z\}$ and consider the renaming:

$$\rho = \{x/u, u/x, y/z, z/y, k/h, h/k\} \ .$$

If we apply ρ to $[\sigma]_U$ we obtain $\rho([\sigma]_U) = [\{u/h, z/t(y,h)\}]_{u,y,z}$. Note that we do not need to worry about variable clashes.

Theorem 3.8. *The renaming operation is well defined.*

PROOF. It is enough to prove monotonicity w.r.t. the preorder $\preceq_U$. Given $\theta_1, \theta_2 \in$ *Subst* such that $\theta_1 \preceq_U \theta_2$, we prove that $\rho(\theta_1) \preceq_{\rho(U)} \rho(\theta_2)$. By Proposition 3.7, we need to show that $\rho \circ \theta_1 \circ \rho^{-1} \preceq_{\rho(U)} \rho \circ \theta_2 \circ \rho^{-1}$. which is equivalent to $\theta_1 \circ \rho^{-1} \preceq_{\rho(U)} \theta_2 \circ \rho^{-1}$. By hypothesis, there exists a substitution $\delta \in$ *Subst* such that $\theta_1(x) = \delta(\theta_2(x))$ for all $x \in U$. Therefore, for all $v \in \rho(U)$, it holds $\theta_1(\rho^{-1}(v)) = \delta(\theta_2(\rho^{-1}(v)))$, which is the thesis. □

Several properties hold for the renaming operation:

(1) $(\rho_1 \circ \rho_2)([\theta]_V) = \rho_1(\rho_2([\theta]_V))$;
(2) ρ is monotonic w.r.t. $\preceq$;
(3) $\rho(\pi_V([\theta]_U)) = \pi_{\rho(V)}(\rho([\theta]_U))$;
(4) $\rho([\theta]_U) = [\theta]_U$ if $\rho_{|U} = id$.

We just prove the last two, since the first is trivial and the second one immediately follows from the proof of Theorem 3.8. Note that the first point implies that $\rho : ISubst_\sim \to ISubst_\sim$ is invertible.

PROPOSITION 3.9. *Renaming is a congruence w.r.t.* π, *i.e.*,

$$\rho(\pi_V([\theta]_U)) = \pi_{\rho(V)}(\rho([\theta]_U)) \ .$$

for $[\theta]_U \in ISubst_\sim$ *and* $\rho \in Ren$. *Moreover, if* $\rho_{|U} = id$ *then* $\rho([\theta]_U) = [\theta]_U$.

PROOF. By definition $\rho(\pi_V([\theta]_U)) = \rho([\theta]_{U\cap V}) = [\rho(\theta)]_{\rho(U\cap V)}$. Since ρ is bijective, $\rho(U \cap V) = \rho(U) \cap \rho(V)$ and therefore $\rho(\pi_V([\theta]_U)) = \pi_{\rho(V)}([\rho(\theta)]_{\rho(U)}) = \pi_{\rho(V)}(\rho([\theta]_U))$. Now assume that $\rho_{|U} = id$. We know that $\rho([\theta]_U) = [\rho \circ \theta \circ \rho^{-1}]_{\rho(U)} = [\theta \circ \rho^{-1}]_U$ since $\rho(U) = U$. Note that, for each $x \in U$, $\theta(\rho^{-1}(x)) = \theta(x)$ since $\rho(x) = x$. This gives the thesis. □

3.2.3 *Unification.* Given $U, V \in \wp_f(\mathcal{V})$, $[\theta_1]_U, [\theta_2]_V \in ISubst_\sim$, we define the most general unifier between these two classes as the mgu of suitably chosen representatives, where variables not of interest are renamed apart. In formulas:

$$\mathrm{mgu}([\theta_1]_U, [\theta_2]_V) = [\mathrm{mgu}(\theta'_1, \theta'_2)]_{U\cup V}$$

where $\theta_1 \sim_U \theta'_1 \in ISubst$, $\theta_2 \sim_V \theta'_2 \in ISubst$ and $(U \cup \mathrm{vars}(\theta'_1)) \cap (V \cup \mathrm{vars}(\theta'_2)) \subseteq U \cap V$. The last condition is needed to avoid variables clashes between the chosen representatives θ'_1 and θ'_2.

Example 3.10. Let $\theta_1 = \{x/a, y/t(v_1, v_1, v_2)\}$ and $\theta_2 = \{y/t(a, v_2, v_1), z/b\}$. Then

$$\mathrm{mgu}([\theta_1]_{x,y}, [\theta_2]_{y,z}) = [\{x/a, y/t(a, a, v), z/b\}]_{x,y,z}$$

by choosing $\theta'_1 = \theta_1$ and $\theta'_2 = \{y/t(a, w, v), z/b\}$. In this case we have

$$\{x/a, y/t(a, a, v), z/b\} \sim_{x,y,z} \mathrm{mgu}(\theta'_1, \theta'_2) = \{x/a, y/t(a, a, v), z/b, v_1/a, w/a, v_2/v\} \ .$$

We may prove that mgu over $ISubst_\sim$ is well defined and that $\mathrm{mgu}([\theta_1]_U, [\theta_2]_V)$ is the greatest lower bound of $[\theta_1]_U$ and $[\theta_2]_V$ w.r.t. $\preceq$.

THEOREM 3.11. mgu *is well-defined.*

PROOF. We begin by proving that, given $\sigma, \sigma'\delta \in \mathit{ISubst}$, if $\sigma \sim_U \sigma'$ with $(U \cup \text{vars}(\sigma)) \cap (V \cup \text{vars}(\delta)) \subseteq U \cap V$ and $(U \cup \text{vars}(\sigma')) \cap (V \cup \text{vars}(\delta)) \subseteq U \cap V$, then $\text{mgu}(\sigma, \delta) \sim_{U \cup V} \text{mgu}(\sigma', \delta)$. We have the following equalities:

$$\begin{aligned}
& \text{mgu}(\sigma, \delta) \\
\sim_{U\cup V}\ & \text{mgu}(\sigma, \delta)_{|U\cup V} \\
=\ & \text{mgu}(\sigma_{|U}, \delta, \sigma_{|-U})_{|U\cup V} \\
=\ & (\text{mgu}(\sigma_{|U}, \delta) \circ \sigma_{|-U})_{|U\cup V} \\
& \quad [\text{since dom}(\sigma_{|-U}) \text{ is disjoint from vars}(\sigma_{|U}) \text{ and vars}(\delta)] \\
=\ & \text{mgu}(\sigma_{|U}, \delta)_{|U\cup V} \enspace .
\end{aligned}$$

By hypothesis there is $\rho \in \mathit{Ren}$ such that $(\rho \circ \sigma')_{|U} = \sigma_{|U}$. We may choose a ρ such that $\text{vars}(\rho) \subseteq \sigma_{|U} \cup \sigma'_{|U}$ by using Lemma 3.4. Then $\text{vars}(\rho) \cap V \subseteq U$. We have:

$$\begin{aligned}
& \text{mgu}(\sigma_{|U}, \delta)_{|U\cup V} \\
=\ & \text{mgu}((\rho\circ\sigma')_{|U}, \delta)_{|U\cup V} \\
=\ & (\text{mgu}((\rho\circ\sigma')_{|U}, \delta) \circ \theta')_{|U\cup V} \qquad [\text{for each } \theta' \text{ s.t. dom}(\theta') \cap (U \cup V) = \emptyset] \\
=\ & \text{mgu}((\rho\circ\sigma')_{|U}, \delta, (\rho\circ\sigma')_{|-U})_{|U\cup V} \qquad [\text{by choosing } \theta' = (\rho\circ\sigma')_{|-U}] \\
=\ & \text{mgu}(\rho\circ\sigma', \delta)_{|U\cup V} \\
\sim_{U\cup V}\ & \text{mgu}(\sigma', \delta)_{|U\cup V} \quad [\text{mgu is a congruence w.r.t. renaming [Palamidessi 1990]}] \\
\sim_{U\cup V}\ & \text{mgu}(\sigma', \delta) \enspace .
\end{aligned}$$

which proves the required property. Now, to prove the general theorem, assume there are $\sigma \sim_U \sigma'$, $\delta \sim_V \delta'$ with $(U \cup \text{vars}(\sigma)) \cap (V \cup \text{vars}(\delta)) \subseteq U \cap V$ and $(U \cup \text{vars}(\sigma')) \cap (V \cup \text{vars}(\delta')) \subseteq U \cap V$. Then, consider a new substitution $\sigma'' \sim_U \sigma'$ such that $(U \cup \text{vars}(\sigma'')) \cap (V \cup \text{vars}(\delta)) \subseteq U \cap V$, $(U \cup \text{vars}(\sigma'')) \cap (V \cup \text{vars}(\delta')) \subseteq U \cap V$ and we repeatedly apply the previous property, obtaining

$$\text{mgu}(\sigma, \delta) \sim_{U\cup V} \text{mgu}(\sigma'', \delta) \sim_{U\cup V} \text{mgu}(\sigma'', \delta') \sim_{U\cup V} \text{mgu}(\sigma', \delta') \enspace . \quad \square$$

Note that, in the proof, the condition $(U \cup \text{vars}(\theta'_1)) \cap (V \cup \text{vars}(\theta'_2)) \subseteq U \cap V$ implies that $\text{vars}(\theta'_1) \cap V \subseteq U \cap V$ and $\text{vars}(\theta'_2) \cap U \subseteq U \cap V$. If we relax the condition to $\text{vars}(\theta'_1) \cap \text{vars}(\theta'_2) \subseteq U \cap V$ then this property no longer holds and mgu ceases to be well defined. This is actually the origin of the flaw in [Cortesi and Filè 1999] which we will examine in Section 8.2.

Example 3.12. Consider $\theta_1 = \{x/a\}$ and $\theta_2 = \{u/b\}$. Assume we have a relaxed definition of mgu as stated above. Then, to compute $\text{mgu}([\theta_1]_x, [\theta_2]_{u,v})$ we may choose $\theta'_1 = \theta_1$ and $\theta'_2 = \theta_2$ to obtain $\{x/a, u/b\}$. But with the relaxed condition we might also choose $\theta'_1 = \{x/a, v/a\}$ and $\theta_2 = \theta'_2$ since it is true that $\text{vars}(\theta'_1) \cap \text{vars}(\theta'_2) = \emptyset$. However $\text{mgu}(\theta'_1, \theta'_2) = \{x/a, v/a, u/b\} \not\sim_{x,u,v} \{x/a, u/b\}$.

THEOREM 3.13. mgu *is the greatest lower bound of* $(\mathit{ISubst}_\sim, \preceq)$.

PROOF. If $[\theta]_{U\cup V} = \text{mgu}([\theta_1]_U, [\theta_2]_V)$, we may assume, without loss of generality, that $\theta = \text{mgu}(\theta_1, \theta_2)$ and θ_1, θ_2 are canonical representatives. It immediately follows that $\theta \leq \theta_1$ and therefore $\theta \preceq_U \theta_1$. In the same way, $\theta \preceq_V \theta_2$.

Now, assume $[\eta]_{U \cup V} \preceq [\theta_1]_U$ and $[\eta]_{U \cup V} \preceq [\theta_2]_V$. We want to prove that $[\eta]_{U \cup V} \preceq [\theta]_{U \cup V}$. By definition of $\preceq$, there is a σ_1 such that $\eta(x) = \sigma_1(\theta_1(x))$ for each $x \in U$. We may choose σ_1 such that $\mathrm{dom}(\sigma_1) \subseteq \mathrm{rng}(\theta_1)$. In the same way, there is σ_2 such that $\mathrm{dom}(\sigma_2) \subseteq \mathrm{rng}(\theta_2(x))$ and $\eta(x) = \sigma_2(\theta_2)$ for each $x \in V$. We may define a new substitution σ such that

$$\sigma(x) = \begin{cases} \sigma_1(\theta_1(x)) & \text{if } x \in U \cup \mathrm{dom}(\sigma_1), \\ \sigma_2(\theta_2(x)) & \text{if } x \in V \cup \mathrm{dom}(\sigma_2), \\ x & \text{otherwise.} \end{cases}$$

Note that this definition is correct, since the first two cases may occur simultaneously only if $x \in U \cap V$, which implies $\sigma_1(\theta_1(x)) = \sigma_2(\theta_2(x)) = \eta(x)$. It is easy to check that $\eta \sim_{U \cup V} \sigma$ and $\sigma = \sigma \circ \theta_1 = \sigma \circ \theta_2$. Therefore

$$\eta \sim_{U \cap V} \sigma \leq \mathrm{mgu}(\theta_1, \theta_2) = \theta \ ,$$

i.e., $\eta \preceq_{U \cup V} \theta$, which proves the thesis. □

We now give some properties which relate the mgu with the other operations on *ISubst*$_\sim$, namely renaming and projection.

Proposition 3.14. *ρ is a congruence w.r.t. unification. In formulas, if E is a set of equations and $[\theta_1]_{U_1}, [\theta_2]_{U_2} \in$ ISubst$_\sim$ then it holds that:*

—$\mathrm{mgu}(\rho(E)) = \rho(\mathrm{mgu}(E))$
—$\rho(\mathrm{mgu}([\theta_1]_{U_1}, [\theta_2]_{U_2})) = \mathrm{mgu}(\rho([\theta_1]_{U_1}), \rho([\theta_2]_{U_2}))$.

Proof. The first property is trivial since the unification algorithm does not depend from the actual name of variables. Therefore, to prove the second property, we only need to check that $\mathrm{mgu}([\theta_1]_{U_1}, [\theta_2]_{U_2}) = [\mathrm{mgu}(\theta'_1, \theta'_2)]_{U_1 \cup U_2}$ implies $\mathrm{mgu}(\rho([\theta_1]_{U_1}), \rho([\theta_2]_{U_2})) = [\mathrm{mgu}(\rho(\theta'_1), \rho(\theta'_2))]_{\rho(U_1) \cup \rho(U_2)}$. First of all, since $\theta'_1 \sim_{U_1} \theta_1$, then $\rho(\theta'_1) \sim_{\rho(U_1)} \rho(\theta_1)$, by Theorem 3.8. The same holds for θ_2. Then, we prove that $(\rho(U_1) \cup \mathrm{vars}(\rho(\theta'_1))) \cap (\rho(U_2) \cup \mathrm{vars}(\rho(\theta'_2))) \subseteq \rho(U_1) \cap \rho(U_2)$. It is obvious that $\rho(\mathrm{vars}(\theta)) = \mathrm{vars}(\rho(\theta))$. Therefore, since ρ is bijective,

$$\begin{aligned}&(\rho(U_1) \cup \mathrm{vars}(\rho(\theta'_1))) \cap (\rho(U_2) \cup \mathrm{vars}(\rho(\theta'_2))) \\ &\quad = \rho((U_1 \cup \mathrm{vars}(\theta'_1)) \cap (U_2 \cup \mathrm{vars}(\theta'_2))) \subseteq \rho(U_1 \cap U_2) = \rho(U_1) \cap \rho(U_2) \ . \ \square \end{aligned} \tag{8}$$

Proposition 3.15. *Given a set of variables V and $[\theta_1]_{U_1}, [\theta_2]_{U_2} \in$ ISubst, we have that*

$$\pi_V(\mathrm{mgu}(\pi_V([\theta_1]_{U_1}), [\theta_2]_{U_2})) = \mathrm{mgu}(\pi_V([\theta_1]_{U_1}), \pi_V([\theta_2]_{U_2})) \ . \tag{9}$$

Proof. First observe that $\pi_V(\mathrm{mgu}(\pi_V([\theta_1]_{U_1}), [\theta_2]_{U_2}))) = [\theta]_{V \cap ((V \cap U_1) \cup U_2)} = [\theta]_{V \cap (U_1 \cup U_2)}$ where $\theta \in \mathrm{mgu}(\theta'_1, \theta'_2)$, θ'_1 and θ'_2 are canonical representatives of $[\theta_1]_{V \cap U_1}$ and $[\theta_2]_{U_2}$ and $\mathrm{vars}(\theta'_1) \cap \mathrm{vars}(\theta'_2) \subseteq V \cap U_1 \cap U_2$. Note that $\theta'_2 \sim_{U_2} \theta_2$ and therefore $\theta'_2 \sim_{V \cap U_2} \theta_2$. Moreover $(\mathrm{vars}(\theta'_1) \cup (V \cap U_1)) \cap (\mathrm{vars}(\theta'_2) \cup (V \cap U_2)) \subseteq V \cap U_1 \cap U_2$, and therefore θ'_1 and θ'_2 are valid representatives to compute $\mathrm{mgu}(\pi_V([\theta_1]_{U_1}), \pi_V([\theta_2]_{U_2}))$ according to (3.2.3). Therefore $[\theta]_{V \cap (U_1 \cup U_2)} = \mathrm{mgu}(\pi_V([\theta_1]_{U_1}), \pi_V([\theta_2]_{U_2}))$ and this proves the thesis. □

Thanks to all the above properties, the domain *ISubst*$_\sim$, excepting for the fact that it is not a boolean algebra, has the same structure of a (locally finite) cylindric algebra [Henkin et al. 1971]. In particular, $[\epsilon]_\emptyset$ is the unit element, the diagonal elements are given by the substitutions $[x/y]_{\{x,y\}}$ and cylindrification may be defined as $c_x([\theta]_V) = \pi_{V\setminus\{x\}}([\theta]_V)$.

It would be possible, like in [Palamidessi 1990], to define a "least common anti-instance" operator which corresponds to the least upper bound in $\mathit{ISubst}_\sim$. However, since it has no point for the semantic framework we are going to define, we omit the details.

4. CONCRETE SEMANTICS

Since we are interested in goal-dependent analysis of logic programs, we need a goal-dependent semantics which is well suited for static analysis, i.e., a collecting semantics over computer answers. Unfortunately, using a collecting goal-dependent semantics may lead to a loss of precision already at the concrete level, as shown by Marriott et al. [1994]. It is possible to reduce this problem by using two different operators for forward and backward unification. In particular, it turns out that backward unification may be realized using the operation of matching between substitutions, as already suggested in [Hans and Winkler 1992; King and Longley 1995]. Our semantics is inspired by similar ones such as those in [Marriott et al. 1994; Jacobs and Langen 1992] and especially in [Cortesi et al. 1994], but it is adapted according to what stated above.

4.1 Concrete Domain

We start to define the concrete domain for the semantics. A concrete object is essentially a set of existential substitutions with a fixed set of variables of interest. In formulas:

$$\mathtt{Psub} = \{[\Sigma, U] \mid \Sigma \subseteq \mathit{ISubst}_{\sim_U}, U \in \wp_f(\mathcal{V})\} \cup \{\bot_{\mathrm{Ps}}, \top_{\mathrm{Ps}}\}$$

where $\top_{\mathrm{Ps}}$ and $\bot_{\mathrm{Ps}}$ are the top and bottom elements respectively and

$$[\Sigma_1, U_1] \leq_{\mathrm{Ps}} [\Sigma_2, U_2] \iff U_1 = U_2 \text{ and } \Sigma_1 \subseteq \Sigma_2 \ .$$

The notation we adopt may appear clumsy, since the set of variables of interest U in $[\Sigma, U]$ may be derived from Σ. However, when we move to the abstract domain, we need to explicitly keep track of this set U. By using $[\Sigma, U]$ in $\mathtt{Psub}$, we want to keep a consistent notation for both concrete and abstract domains.

It turns out that $(\mathtt{Psub}, \leq_{\mathrm{Ps}})$ is a complete lattice, and we denote by $\sqcup_{\mathrm{Ps}}$ its least upper bound, which is given by

$$\begin{aligned}
\top_{\mathrm{Ps}} \sqcup_{\mathrm{Ps}} x = x \sqcup_{\mathrm{Ps}} \top_{\mathrm{Ps}} &= \top_{\mathrm{Ps}} \\
\bot_{\mathrm{Ps}} \sqcup_{\mathrm{Ps}} x = x \sqcup_{\mathrm{Ps}} \bot_{\mathrm{Ps}} &= x \\
[\Sigma_1, U_1] \sqcup_{\mathrm{Ps}} [\Sigma_2, U_2] &= \begin{cases} [\Sigma_1 \cup \Sigma_2, U_1] & \text{if } U_1 = U_2, \\ \top_{\mathrm{Ps}} & \text{otherwise.} \end{cases}
\end{aligned} \tag{10}$$

We now define the main operations over $\mathtt{Psub}$, that is: projection over a set of variables, unification of an object with a single substitution and the operation for matching two objects of $\mathtt{Psub}$. All the operations are strict: when one of the argument is $\bot_{\mathrm{Ps}}$ the result is $\bot_{\mathrm{Ps}}$. If no argument is $\bot_{\mathrm{Ps}}$ and at least one of the argument is $\top_{\mathrm{Ps}}$ the result is $\top_{\mathrm{Ps}}$. Therefore, in the following, we will omit the cases for the objects $\bot_{\mathrm{Ps}}$ and $\top_{\mathrm{Ps}}$.

Given $[\Sigma, U] \in \mathtt{Psub}$ and $V \subseteq \mathcal{V}$, we define the projection of $[\Sigma, U]$ on the set of variables V as

$$\pi_{\mathrm{Ps}}([\Sigma, U], V) = [\{\pi_V(\sigma) \mid \sigma \in \Sigma\}, U \cap V] \ . \tag{11}$$

The concrete unification $\mathsf{unif}_{\mathrm{Ps}} : \mathtt{Psub} \times \mathit{ISubst} \to \mathtt{Psub}$ is given by:

$$\mathsf{unif}_{\mathrm{Ps}}([\Sigma, U], \delta) = [\{\mathrm{mgu}([\sigma]_U, [\delta]_V) \mid [\sigma]_U \in \Sigma, V = \mathrm{vars}(\delta)\}, U \cup V]. \tag{12}$$

The operations π_{Ps} and $\mathsf{unif}_{\mathrm{Ps}}$ are just the pointwise extensions of π_V and mgu. Note that, in $\mathsf{unif}_{\mathrm{Ps}}$, the argument δ may have variables which do not appear in U. This is not always the case in literature. For example, in [Cortesi and Filè 1999; Bagnara et al. 2000] we find a variant of $\mathsf{unif}_{\mathrm{Ps}}$ which only consider the case when $\mathrm{vars}(\delta) \subseteq U$. When this does not happen, the same effect is obtained by first enlarging the set of variables of interest U, and then applying unification. Although nothing changes at the concrete level, this gives a loss of precision when when we move to the abstract side, since the composition of two optimal abstract operators is generally less precise then the optimal abstract counterpart of the whole $\mathsf{unif}_{\mathrm{Ps}}$ (see Section 6).

Finally, we define the matching operation. The idea is to design an operator which performs unification between two substitutions $[\theta_1]_{U_1}$ and $[\theta_2]_{U_2}$ only if the process of unification does not instantiate the first substitution. In other words, we require that if we compute $\mathrm{mgu}([\theta_1]_{U_1}, [\theta_2]_{U_2})$ and we only observe variables in U_1, that is $\pi_{U_1}(\mathrm{mgu}([\theta_1]_{U_1}, [\theta_2]_{U_2}))$, then we obtain exactly $[\theta_1]_{U_1}$. The next proposition shows this is equivalent to require that $\theta_1 \preceq_{U_1 \cap U_2} \theta_2$.

Proposition 4.1. *Given two substitutions $[\theta_1]_{U_1}$ and $[\theta_2]_{U_2}$, then $\theta_1 \preceq_{U_1 \cap U_2} \theta_2$ iff $[\theta_1]_{U_1} = \pi_{U_1}(\mathrm{mgu}([\theta_1]_{U_1}, [\theta_2]_{U_2}))$.*

Proof. By Theorem 3.13, mgu is the glb of $(\mathit{ISubst}_\sim, \preceq)$ and π is monotonic, from which it follows that the condition $\pi_{U_1}(\mathrm{mgu}([\theta_1]_{U_1}, [\theta_2]_{U_2})) = [\theta_1]_{U_1}$ is equivalent to $[\theta_1]_{U_1} \preceq \pi_{U_1}(\mathrm{mgu}([\theta_1]_{U_1}, [\theta_2]_{U_2}))$. Moreover, it holds that $\theta_1 \preceq_{U_1 \cap U_2} \theta_2$ iff $\pi_{U_1}([\theta_1]_{U_1}) \preceq \pi_{U_1}([\theta_2]_{U_2})$. By Prop. 3.15 we have that $\pi_{U_1}(\mathrm{mgu}([\theta_1]_{U_1}, [\theta_2]_{U_2})) = \mathrm{mgu}(\pi_{U_1}([\theta_1]_{U_1}), \pi_{U_1}([\theta_2]_{U_2}))$. Therefore, the thesis follows by properties of greatest lower bounds. □

We can now define the matching operator $\mathsf{match}_{\mathrm{Ps}} : \mathtt{Psub} \times \mathtt{Psub} \to \mathtt{Psub}$ as follows:

$$\begin{aligned}\mathsf{match}_{\mathrm{Ps}}([\Sigma_1, U_1], [\Sigma_2, U_2]) = [\{&\mathrm{mgu}([\theta_1]_{U_1}, [\theta_2]_{U_2}) \mid \\ &\theta_1 \preceq_{U_1 \cap U_2} \theta_2, [\theta_1]_{U_1} \in \Sigma_1, [\theta_2]_{U_2} \in \Sigma_2\}, U_1 \cup U_2] \ .\end{aligned} \tag{13}$$

The above operator allows us to unify all the pairs of substitutions $[\theta_1]_{U_1} \in \Sigma_1$ and $[\theta_2]_{U_2} \in \Sigma_2$, under the condition that the common variables in U_1 and U_2 may not be further instantiated w.r.t. their value in θ_1.

Example 4.2. Let $\Sigma_1 = \{[x/y]_{x,y}\}$ and $\Sigma_2 = \{[u/x]_{u,x}, [x/t(u)]_{u,x}\}$. Then

$$\mathsf{match}_{\mathrm{Ps}}([\Sigma_1, \{x, y\}], [\Sigma_2, \{u, x\}]) = [\{[x/y, x/u]_{x,y,u}\}, \{x, y, u\}] \ .$$

Note that $[y/t(u), x/t(u)]_{u,x,y}$, obtained by unifying $[x/y]_{x,y}$ with $[x/t(u)]_{u,x}$, is not in the result of matching. This is because $[x/t(u)]_{u,x}$ is strictly more instantiated then $[x/y]_{x,y}$ w.r.t. the variable x and therefore $\{x/y\} \not\preceq_x \{x/t(u)\}$.

PROPOSITION 4.3. *The operations* π_{Ps}, $\mathsf{unif}_{\mathrm{Ps}}$ *and* $\mathsf{match}_{\mathrm{Ps}}$ *are continuous over* $\mathtt{Psub}$.

PROOF. Trivial from their definitions. If we do not consider the element $\top_{\mathrm{Ps}}$, they are actually additives. □

4.2 Semantics

Using the operators defined so far, we introduce a denotational semantics for logic programs. It computes, for a given goal G, the set of computed answers for a program w.r.t. G modulo the equivalence relation $\sim_{\mathrm{vars}(G)}$. It is a goal-dependent *collecting semantics* [Cousot and Cousot 1994], in that it works by computing the set of possibly entry ed exit substitutions at each point in the program.

We call *denotation* an element in the set of continuous maps:

$$\mathcal{D}en = \mathsf{Atoms} \to \mathtt{Psub} \xrightarrow{c} \mathtt{Psub} \ . \tag{14}$$

We have the following semantic functions:

$$\begin{aligned}
\mathcal{P} &: \mathsf{Progs} \to \mathcal{D}en \\
\mathcal{C} &: \mathsf{Clauses} \to \mathcal{D}en \xrightarrow{c} \mathcal{D}en \\
\mathcal{B} &: \mathsf{Bodies} \to \mathcal{D}en \xrightarrow{c} \mathtt{Psub} \xrightarrow{c} \mathtt{Psub} \ .
\end{aligned}$$

The corresponding definitions[1] are:

$$\begin{aligned}
&\mathcal{P}[\![P]\!] = \mathit{lfp}\ \lambda d. \left(\bigsqcup_{cl \in P} \mathcal{C}[\![cl]\!] d \right) \\
&\mathcal{C}[\![H \leftarrow B]\!] dAx = \mathbf{U}^b_{\mathrm{Ps}}(\mathcal{B}[\![B]\!] d(\mathbf{U}^f_{\mathrm{Ps}}(x, A, H)), x, H, A) \\
&\mathcal{B}[\![\Box]\!] dx = x \\
&\mathcal{B}[\![A, B]\!] dx = \mathcal{B}[\![B]\!] d(dAx)
\end{aligned}$$

given by means of the following operators:

$$\begin{aligned}
\mathbf{U}^f_{\mathrm{Ps}} &: \mathtt{Psub} \times \mathsf{Atoms} \times \mathsf{Atoms} \to \mathtt{Psub} \ , \\
\mathbf{U}^b_{\mathrm{Ps}} &: \mathtt{Psub} \times \mathtt{Psub} \times \mathsf{Atoms} \times \mathsf{Atoms} \to \mathtt{Psub} \ .
\end{aligned}$$

$\mathbf{U}^f_{\mathrm{Ps}}$ and $\mathbf{U}^b_{\mathrm{Ps}}$ are respectively the *forward* and *backward* unification [Muthukumar and Hermenegildo 1992]. They are used according to the following pattern:

—the *forward unification*, in order to compute the collecting *entry substitution* $\mathbf{U}^f_{\mathrm{Ps}}(x, A, H)$ from the collecting *call substitution* x;

—the *backward unification*, in order to compute the collecting *answer substitution* $\mathbf{U}^b_{\mathrm{Ps}}(\mathcal{B}[\![B]\!] d(\mathbf{U}^f_{\mathrm{Ps}}(x, A, H)), x, H, A)$ starting from the collecting *exit substitution* $d(\mathbf{U}^f_{\mathrm{Ps}}(x, A, H))$.

The formal definitions of $\mathbf{U}^f_{\mathrm{Ps}}$ and $\mathbf{U}^b_{\mathrm{Ps}}$ are the following:

$$\mathbf{U}^f_{\mathrm{Ps}}([\Sigma, U], A_1, A_2) = \pi_{\mathrm{Ps}}(\mathsf{unif}_{\mathrm{Ps}}(\rho([\Sigma, U]), \mathrm{mgu}(\rho(A_1) = A_2)), \mathrm{vars}(A_2)) \ , \tag{15}$$

[1] Here we use the lambda notation, writing $\mathit{lfp}\ \lambda x.E(x)$ to denote the least fixed point of the function f given by $f(x) = E(x)$.

where ρ is a renaming such that $\rho(U \cup \text{vars}(A_1)) \cap \text{vars}(A_2) = \emptyset$ and $\rho([\Sigma, U]) = [\{\rho([\sigma]_U) \mid [\sigma]_U \in \Sigma\}, \rho(U)]$ is the obvious lifting of renamings from $\mathit{ISubst}_\sim$ to Psub.

$$\mathbf{U}^b_{\text{Ps}}([\Sigma_1, U_1], [\Sigma_2, U_2], A_1, A_2) = \\ \pi_{\text{Ps}}(\mathsf{match}_{\text{Ps}}(\rho([\Sigma_1, U_1]), \mathsf{unif}_{\text{Ps}}([\Sigma_2, U_2], \text{mgu}(\rho(A_1) = A_2))), U_2 \cup \text{vars}(A_2)) \tag{16}$$

where ρ is a renaming such that $\rho(U_1 \cup \text{vars}(A_1)) \cap (U_2 \cup \text{vars}(A_2)) = \emptyset$. If $\rho(A_1)$ and A_2 do not unify, the results for both the operations is assumed to be $\bot_{\text{Ps}}$.

Theorem 4.4. *$\mathbf{U}^f_{\text{Ps}}$ and $\mathbf{U}^b_{\text{Ps}}$ are well defined, in that they are independent from the choice of ρ. Moreover, they are continuous.*

Proof. Continuity is trivial from their definition, therefore we only need to prove the independence from the choice of the renaming ρ. We only consider the case when none of the argument is $\bot_{\text{Ps}}$ or $\top_{\text{Ps}}$, since otherwise the result is always $\bot_{\text{Ps}}$ or $\top_{\text{Ps}}$. Moreover, note that, given atoms A_1 and A_2, if ρ_1 and ρ_2 are renamings such that $\rho_i(\text{vars}(A_1)) \cap \text{vars}(A_2) = \emptyset$ for $i \in \{1, 2\}$, then $\rho_1(A_1)$ and A_2 unify iff $\rho_2(A_1)$ and A_2 unify. Therefore, we can restrict ourselves to the case where the two atoms given as arguments, appropriately renamed, do unify. Otherwise, the result is always $\bot_{\text{Ps}}$.

Observe that, by Prop. 3.14, given $\rho \in \mathit{Ren}$, $[\theta_1]_{U_1}, [\theta_2]_{U_2} \in \mathit{ISubst}$, we have that $\rho(\text{mgu}([\theta_1]_{U_1}, [\theta_2]_{U_2})) = \text{mgu}(\rho([\theta_1]_{U_1}), \rho([\theta_2]_{U_2})$. By definition of $\mathsf{unif}_{\text{Ps}}$, it follows that $\rho(\mathsf{unif}_{\text{Ps}}([\Sigma, U], \delta)) = \mathsf{unif}_{\text{Ps}}(\rho([\Sigma, U]), \rho(\delta))$, since $\text{vars}(\rho(\delta)) = \rho(\text{vars}(\delta))$.

Let ρ_1, ρ_2 be renamings. We first show that

$$\pi_{\text{Ps}}(\mathsf{unif}_{\text{Ps}}(\rho_1([\Sigma, U]), \text{mgu}(\rho_1(A_1) = A_2), \text{vars}(A_2))) = \\ \pi_{\text{Ps}}(\mathsf{unif}_{\text{Ps}}(\rho_2([\Sigma, U]), \text{mgu}(\rho_2(A_1) = A_2), \text{vars}(A_2)))$$

provided that $(\rho_i(U) \cup \text{vars}(A_1)) \cap \text{vars}(A_2) = \emptyset$, for $i \in \{1, 2\}$. Let $W = \text{vars}(\rho_1(U) \cup \rho_1(A_1))$ and $\delta = (\rho_2 \circ \rho_1^{-1})_{|W}$. Then δ may be viewed as an injective map from V to $\mathcal{V}$, since it is the composition of injective functions. By Lemma 3.4 there exists a renaming ρ such that $\rho_{|W} = \delta$ and $\text{vars}(\rho) = \text{vars}(\delta) \subseteq W \cup \text{rng}(\delta) \subseteq W \cup \rho_2(U) \cup \rho_2(A_1)$. Observe that $\text{vars}(\rho) \cap \text{vars}(A_2) = \emptyset$ since, by hypothesis, for each $i \in \{1, 2\}$ it is the case that $\rho_i(U \cup \text{vars}(A_1)) \cap \text{vars}(A_2) = \emptyset$. Thus the following equivalences hold:

$$\begin{aligned}
& \pi_{\text{Ps}}(\mathsf{unif}_{\text{Ps}}(\rho_1([\Sigma, U]), \text{mgu}(\rho_1(A_1) = A_2), \text{vars}(A_2))) \\
= \; & \rho(\pi_{\text{Ps}}(\mathsf{unif}_{\text{Ps}}(\rho_1([\Sigma, U]), \text{mgu}(\rho_1(A_1) = A_2), \text{vars}(A_2)))) \\
& \quad [\text{since } \rho_{|\text{vars}(A_2)} = \mathit{id} \text{ and by Prop. 3.9}] \\
= \; & \pi_{\text{Ps}}(\rho(\mathsf{unif}_{\text{Ps}}(\rho_1([\Sigma, U]), \text{mgu}(\rho_1(A_1) = A_2), \text{vars}(A_2)))) \\
& \quad [\text{by Prop. 3.9}] \\
= \; & \pi_{\text{Ps}}(\mathsf{unif}_{\text{Ps}}(\rho(\rho_1([\Sigma, U])), \rho(\text{mgu}(\rho_1(A_1) = A_2), \text{vars}(A_2)))) \\
= \; & \pi_{\text{Ps}}(\mathsf{unif}_{\text{Ps}}(\rho_2([\Sigma, U]), \text{mgu}(\rho(\rho_1(A_1)) = \rho(A_2)), \text{vars}(A_2))) \\
& \quad [\text{by Prop. 3.14}] \\
= \; & \pi_{\text{Ps}}(\mathsf{unif}_{\text{Ps}}(\rho_2([\Sigma, U]), \text{mgu}(\rho_2(A_1)) = A_2), \text{vars}(A_2)) \; .
\end{aligned}$$

We now show that $\mathbf{U}^b_{\mathrm{Ps}}$ is independent from the choice of the renaming. First of all, note that by Proposition 3.14 and Theorem 3.8 the following follows:

$$\rho(\mathsf{match}_{\mathrm{Ps}}([\Sigma_1, U_1], [\Sigma_2, U_2])) = \mathsf{match}_{\mathrm{Ps}}(\rho([\Sigma_1, U_1]), \rho([\Sigma_2, U_2])) \ .$$

Assume given $\rho_1, \rho_2 \in \mathit{Ren}$ such that $\rho_i((U_1 \cup \mathrm{vars}(A_1)) \cap (U_2 \cup \mathrm{vars}(A_2))) = \emptyset$, for $i \in \{1, 2\}$. Let $W = \mathrm{vars}(\rho_1(U_1) \cup \rho_1(A_1))$ and $\delta = (\rho_2 \circ \rho_1^{-1})_{|W}$. As shown above, there exists $\rho \in \mathit{Ren}$ such that $\rho_{|W} = \delta$ and $\mathrm{vars}(\rho) = \mathrm{vars}(\delta) \subseteq W \cup \rho_2(U_1) \cup \rho_2(A_1)$. Observe that $\delta_{|U_2 \cup \mathrm{vars}(A_2)} = \mathit{id}$. Thus the following equivalences hold:

$$\begin{aligned}
&\pi_{\mathrm{Ps}}(\mathsf{match}_{\mathrm{Ps}}(\rho_1([\Sigma_1, U_1]), \mathsf{unif}_{\mathrm{Ps}}([\Sigma_2, U_2], \mathrm{mgu}(\rho_1(A_1) = A_2))), U_2 \cup \mathrm{vars}(A_2))\\
= \ &\rho(\pi_{\mathrm{Ps}}(\mathsf{match}_{\mathrm{Ps}}(\rho_1([\Sigma_1, U_1]), \mathsf{unif}_{\mathrm{Ps}}([\Sigma_2, U_2], \mathrm{mgu}(\rho_1(A_1) = A_2))), \ldots))\\
= \ &\pi_{\mathrm{Ps}}(\mathsf{match}_{\mathrm{Ps}}(\rho(\rho_1([\Sigma_1, U_1])), \mathsf{unif}_{\mathrm{Ps}}(\rho([\Sigma_2, U_2]), \mathrm{mgu}(\rho(\rho_1(A_1) = \rho(A_2))))), \ldots)\\
= \ &\pi_{\mathrm{Ps}}(\mathsf{match}_{\mathrm{Ps}}(\rho_2([\Sigma_1, U_1]), \mathsf{unif}_{\mathrm{Ps}}([\Sigma_2, U_2], \mathrm{mgu}(\rho_2(A_1) = A_2))), \ldots) \ .
\end{aligned}$$

This concludes the proof of the theorem. □

THEOREM 4.5. *All the semantic functions are well defined and continuous.*

PROOF. The proof is trivial since the semantic functions are obtained by composition, application, projection and tupling of continuous functions. Therefore, they are continuous and compute continuous denotations. Moreover, they do not depend on the choice of ρ in $\mathbf{U}^f_{\mathrm{Ps}}$ and $\mathbf{U}^b_{\mathrm{Ps}}$, as proved in Theorem 4.4. □

Note that several frameworks have been developed for logic programs, and not all of them use the same operators for forward and backward unification. We will discuss the benefits of our choices later, when we introduce the abstract operators, since the relative merits of the different proposal mainly arise when speaking about abstractions.

4.3 Correctness and Completeness

The semantics we have defined in this section is significant only up to the point that, studying its properties, it is possible to derive some conclusions about the properties of the real operational behavior of logic programs. We said before that we consider as the relevant operational observable of our analysis the set of classes of computed answers for a goal. Therefore, the best we can expect from our collecting semantics is that it enables us to recover the set of computed answer for each goal. Our first theorem is a partial positive answer to this question.

THEOREM 4.6. (SEMANTIC CORRECTNESS) *Given a program P and an goal G, if θ is a computed answer for the goal G, then* $\mathcal{B}[\![G]\!](\mathcal{P}[\![P]\!])G[\{\epsilon\}, \mathrm{vars}(G)] \sqsupseteq [\{[\phi]\}, \mathrm{vars}(G)]$.

PROOF. The proof, quite long and tedious, may be found in the appendix. □

Therefore, we know that all the computed answers may be obtained by our semantics. However, the opposite is not true: the semantics given in this paper, although more precise than a semantics which only uses unification, is not complete w.r.t. computed answers. Actually, Marriott et al. [1994, Section 5.5] give an example where a collecting goal-dependent semantics computes a substitution which is not a computed answer. When matching is used to compute the backward unification, as it is the case in our framework, that example does not work anymore (see Example 7.3).

However, also with the use of matching, the collecting semantics computes substitutions which are not computed answers. Consider the program P given by the following clauses:

```
p(x,y) :- q(x).
q(x).
```

We want to compute $\mathcal{P}[\![P]\!]p(x,y)[\Phi,\{x,y\}]$ where $\Phi = \{[x/y],[x/a]\}$. It is easy to check that

$$\mathcal{P}[\![P]\!]q(x)[\Theta,\{x\}] = [\Theta,\{x\}]$$

for each $[\Theta,\{x\}] \in$ `Psub`. Therefore, this implies that

$$\mathcal{P}[\![P]\!]p(x,y)[\Phi,\{x,y\}] = [\{[x/y],[x/a],[x/a,y/a]\},\{x,y\}] \ .$$

The substitution $[x/a,y/a]$ arises from calling $q(x)$ with the substitution $[x/a]$ and matching the result with $[x/y]$, which is not forbidden by matching. However, there is no substitution in the class of $[\{x/a,y/a\}]_{x,y}$ which is a computed answer for the goal G in the program P with entry substitution in Θ.

While this loss of precisions is not relevant for downward-closed abstract domains, where goal-dependent collecting semantics is more precise than goal-independent ones, this is not the case for upward-closed abstract domain, where goal-independent semantics are more precise. For instance, de La Banda et al. [1998] show several semantics which combine a goal-dependent and a goal-independent computation to improve precision over all the conditions.

5. ABSTRACT SEMANTICS

Several abstract domains have been used for analyses of sharing and aliasing. We use the domain `Sharing` [Jacobs and Langen 1992; Cortesi and Filè 1999] which computes set-sharing information.

5.1 Abstract Domain and Operations

$$\texttt{Sharing} = \{[A,U] \mid A \subseteq \wp(U), (A \neq \emptyset \Rightarrow \emptyset \in A), U \in \wp_f(\mathcal{V})\} \cup \{\top_{Sh}, \bot_{Sh}\} \ .$$

Intuitively, an abstract object $[A,U]$ describes the relations between the variables in U: if $S \in A$, the variables in S are allowed to share a common variable. For instance, $[\{\{x,y\},\{z\},\emptyset\},\{x,y,z\}]$ represents the (equivalence classes of) substitutions where x and y may possibly share, while z is independent from both x and y: $\{x/y\}$ and ϵ are two of such substitutions while $\{x/z\}$ is not.

The domain is ordered like `Psub`, with $\top_{Sh}$ and $\bot_{Sh}$ as the greatest and least element respectively, and $[A_1,U_1] \sqsubseteq_{Sh} [A_2,U_2]$ iff $A_1 = A_2$ and $U_1 \subseteq U_2$. The lowest upper bound satisfies the following property:

$$[A_1,U_1] \sqcup_{Sh} [A_2,U_2] = \begin{cases} [A_1 \cup A_2, U_1] & \text{if } U_1 = U_2, \\ \top_{Sh} & \text{otherwise.} \end{cases} \tag{17}$$

To design the abstraction from `Psub` to `Sharing`, we first define a map $\alpha_{\mathrm{Sh}} : \mathit{ISubst}_{\sim} \rightarrow \texttt{Sharing}$ as

$$\alpha_{\mathrm{Sh}}([\sigma]_V) = [\{\mathit{occ}(\sigma,y) \cap V \mid y \in \mathcal{V}\}, V] \ . \tag{18}$$

where $occ(\sigma, y) = \{z \in \mathcal{V} \mid y \in \mathrm{vars}(\sigma(z))\}$ is the set of variables z such that y occurs in $\sigma(z)$. For instance, $occ(\{x/t(y,z), x'/z, y'/z'\}, z) = \{x, x'\}$. We call *sharing group* an element of $\wp_f(\mathcal{V})$.

We say that x is *independent* from y in $[\sigma]_V$ when, given $\alpha_{\mathrm{Sh}}([\sigma]_V) = [S, U]$, there is no $X \in S$ such that $\{x, y\} \subseteq X$. Given $U \in \wp(\mathcal{V})$, we say that x is independent from U in $[\sigma]_V$ when it is independent from y for each $y \in U$ different from x. Finally, x is independent in $[\sigma]_V$ if it is independent from V in $[\sigma]_V$.

Proposition 5.1. *The map $\alpha_{\mathrm{Sh}} : \mathit{ISubst}_\sim \to \mathtt{Sharing}$ is well defined, i.e., it does not depend on the choice of representatives.*

Proof. If $\sigma \sim_V \sigma'$ and $X \in occ(\sigma, y) \cap V$, let $\rho \in \mathit{Ren}$ such that $\sigma'(x) = \rho(\sigma(x))$ for each $x \in V$. Then

$$\begin{aligned} occ(\sigma', \rho(y)) \cap V &= \{z \in V \mid \rho(y) \in \mathrm{vars}(\sigma'(z))\} \\ &= \{z \in V \mid y \in \rho^{-1}(\mathrm{vars}(\rho(\sigma(z))))\} \\ &= \{z \in V \mid y \in \mathrm{vars}(\sigma(z))\} \\ &= occ(\sigma, y) \cap V \ . \end{aligned}$$

Therefore $X \in occ(\sigma', \rho(y)) \cap V$, which proves the thesis. □

The abstraction function just defined may be lifted pointwise to $\alpha_{\mathrm{Sh}} : \mathtt{Psub} \to \mathtt{Sharing}$ as follows:

$$\begin{aligned} &\alpha_{\mathrm{Sh}}(\bot_{\mathrm{Rs}}) = \bot_{Sh} \qquad\qquad \alpha_{\mathrm{Sh}}(\top_{\mathrm{Rs}}) = \top_{Sh} \\ &\alpha_{\mathrm{Sh}}([\Sigma, U]) = \bigsqcup_{[\sigma]_U \in \Sigma} \alpha_{\mathrm{Sh}}([\sigma]_U) \end{aligned} \tag{19}$$

To ease the notation, often we will write a sharing group as the sequence of its elements in any order (e.g., $\mathtt{xyz}$ represents $\{x, y, z\}$) and we omit the empty set when clear from the context. For example:

$$\begin{aligned} \alpha_{\mathrm{Sh}}([\{[\epsilon]\}, \{x,y,z\}]) &= [\{\mathtt{x}, \mathtt{y}, \mathtt{z}\}, \{x,y,z\}] \\ \alpha_{\mathrm{Sh}}([\{[x/y, z/a]\}, \{x,y,z\}]) &= [\{\mathtt{xy}\}, \{x,y,z\}] \\ \alpha_{\mathrm{Sh}}([\{[\epsilon], [x/y, z/a]\}, \{x,y,z\}]) &= [\{\mathtt{xy}, \mathtt{x}, \mathtt{y}, \mathtt{z}\}, \{x,y,z\}] \ . \end{aligned}$$

Since α_{Sh} is additive, there is an induced concretization function γ_{Sh}, the right adjoint of α_{Sh}, which maps each abstract object to the set of substitutions it represents:

$$\gamma_{\mathrm{Sh}}([S, U]) = [\{[\theta]_U \mid \alpha_{\mathrm{Sh}}([\theta]_U) \sqsubseteq_{Sh} [S, U]\}, U] \ . \tag{20}$$

Note that each abstract object represents the *possible* relations between variables: given an object $[A, U]$, a substitution where all the variables in U are ground is in $\gamma_{\mathrm{Sh}}([A, U])$ independently from A.

Proposition 5.2. *$\langle \alpha_{\mathrm{Sh}}, \gamma_{\mathrm{Sh}} \rangle : \mathtt{Psub} \leftrightharpoons \mathtt{Sharing}$ defines a Galois connection. Moreover, if the term signature has at least a constant symbol, we have a Galois insertion.*

Proof. That $\langle \alpha_{\mathrm{Sh}}, \gamma_{\mathrm{Sh}} \rangle$ is a Galois connection immediately follows from the fact they are an adjoint pair. Now, we want to prove that, in presence of a constant symbol a, α_{Sh} is onto. Given $[S,V] \in \mathtt{Sharing}$ and $X \in S$, consider the substitution θ_X defined as

$$\theta_X(x) = \begin{cases} w & \text{if } x \in X \\ a & \text{if } x \in V \setminus X \\ v & \text{otherwise.} \end{cases}$$

It is easy to check that $\alpha_{\mathrm{Sh}}([\theta_X]_V) = [\{X\}, S]$ and therefore $\alpha_{\mathrm{Sh}}([\{[\theta_X]_V \mid X \in S\}, V]) = [S,V]$. Moreover, we have $\alpha_{\mathrm{Sh}}(\bot_{\mathrm{Ps}}) = \bot_{\mathit{Sh}}$ and $\alpha_{\mathrm{Sh}}(\top_{\mathrm{Ps}}) = \top_{\mathit{Sh}}$. $\square$

It is worth noting that, in general, when the signature does not have constant symbols, $\langle \alpha_{\mathrm{Sh}}, \gamma_{\mathrm{Sh}} \rangle$ is not a Galois insertion. In fact, it happens that

$$\gamma_{\mathrm{Sh}}([\{\mathtt{x}\}, \{x,y\}]) = [\emptyset, \{x,y\}] = \gamma_{\mathrm{Sh}}([\{\mathtt{y}\}, \{x,y\}]) \ , \tag{21}$$

since $[\{\mathtt{x}\}, \{x,y\}]$ forces y to be ground, and this is not possible.

The abstract operators do behave exactly as the concrete ones on $\top_{\mathit{Sh}}$ and $\bot_{\mathit{Sh}}$, while the other cases are the following:

$$\pi_{\mathit{Sh}}([A_1, U_1], U_2) = [\{B \cap U_1 \mid B \in A_1\}, U_1 \cap U_2] \ , \tag{22}$$

$$\rho([A,U]) = [\rho(A), \rho(U)] \ . \tag{23}$$

The definition of the abstract versions of matching and unification is the main argument of the rest of this paper. Here we show some properties of completeness for projection and renaming. Since the concrete and abstract operators behave in the same way on top and bottom elements, here and in the following proofs we only consider the case when all the arguments are different from $\bot_{\mathrm{Ps}}/\bot_{\mathit{Sh}}$ and $\top_{\mathrm{Ps}}/\top_{\mathit{Sh}}$.

Theorem 5.3. *π_{Sh} is correct and complete w.r.t. π_{Ps}.*

Proof. Given $[\Phi, V] \in \mathtt{Psub}$, we need to prove that $\alpha_{\mathrm{Sh}}(\pi_{\mathrm{Ps}}([\Phi,V],U)) = \pi_{\mathit{Sh}}(\alpha_{\mathrm{Sh}}([\Phi,V]),U)$. We first prove that, for each $[\phi]_V \in \mathit{ISubst}_\sim$, it holds that $\pi_{\mathit{Sh}}(\alpha_{\mathrm{Sh}}([\phi]_V),U) = \alpha_{\mathrm{Sh}}([\phi]_{V\cap U})$. Actually

$$\begin{aligned} \alpha_{\mathrm{Sh}}([\phi]_{V\cap U}) &= [\{\mathit{occ}(\phi,z) \cap V \cap U \mid z \in \mathcal{V}\}, V \cap U] \\ &= \pi_{\mathit{Sh}}([\{\mathit{occ}(\phi,z) \cap V \mid z \in \mathcal{V}\}, V], U) \\ &= \pi_{\mathit{Sh}}(\alpha_{\mathrm{Sh}}([\phi]_V), U) \ . \end{aligned}$$

The result for the lifted α_{Sh} follows trivially. $\square$

Theorem 5.4. *Abstract renaming is correct, complete and γ-complete w.r.t. concrete renaming.*

Proof. First of all, given $\rho \in \mathit{Ren}, y \in \mathcal{V}$ and $\phi \in \mathit{Subst}$, we prove that $\mathit{occ}(\rho(\phi), \rho(y)) = \rho(\mathit{occ}(\phi,y))$. Actually:

$$\begin{aligned} \mathit{occ}(\rho(\phi), \rho(y)) &= \{z \mid z \in \mathcal{V}, \rho(y) \in \mathrm{vars}(\rho(\phi(\rho^{-1}(z))))\} \\ &= \{z \mid z \in \mathcal{V}, y \in \mathrm{vars}(\phi(\rho^{-1}(z)))\} \\ &= \{\rho(k) \mid k \in \mathcal{V}, y \in \mathrm{vars}(\phi(k))\} \quad [\text{by letting } k = \rho^{-1}(z)] \\ &= \rho(\mathit{occ}(\phi), y) \ . \end{aligned}$$

Then we prove that, given $[\phi]_V \in \texttt{Psub}$ and $\rho \in \mathit{Ren}$, $\alpha_{\mathrm{Sh}}(\rho([\phi]_V)) = \rho(\alpha_{\mathrm{Sh}}([\phi]_V))$. Using the fact that ρ as an operation over $\mathit{ISubst}_\sim$ is bijective, we have:

$$\begin{aligned}
\alpha_{\mathrm{Sh}}(\rho([\phi]_V)) &= [\{occ(\rho(\phi), z) \cap \rho(V) \mid z \in \mathcal{V}\}, \rho(V)] \\
&= [\{\rho(occ(\phi, \rho^{-1}(z)) \cap \rho(V) \mid z \in \mathcal{V}\}, \rho(V)] \\
&= \rho([occ(\phi, k) \cap V \mid k \in \mathcal{V}\}, V]) \quad \text{[by letting } z = \rho(k)] \\
&= \rho(\alpha_{\mathrm{Sh}}([\phi]_V)) \enspace .
\end{aligned}$$

This property, lifted to Psub, gives the completeness of abstract renaming. Finally, we need to prove that renaming is γ-complete, i.e., that $\gamma_{\mathrm{Sh}} \circ \rho = \rho \circ \gamma_{\mathrm{Sh}}$.

$$\begin{aligned}
\gamma_{\mathrm{Sh}}(\rho([S, V])) &= \gamma_{\mathrm{Sh}}([\rho(S), \rho(V)]) \\
&= \big[\{[\theta]_V \mid \alpha_{\mathrm{Sh}}([\theta]_V) \sqsubseteq_{Sh} \rho(S)\}, \rho(V)\big] \\
&= \big[\{\rho([\theta]_V) \mid \alpha_{\mathrm{Sh}}(\rho([\theta]_V)) \sqsubseteq_{Sh} \rho(S)\}, \rho(V)\big] \\
&= \big[\{\rho([\theta]_V) \mid \rho(\alpha_{\mathrm{Sh}}([\theta]_V)) \sqsubseteq_{Sh} \rho(S)\}, \rho(V)\big] \\
&= \big[\{\rho([\theta]_V) \mid \alpha_{\mathrm{Sh}}([\theta]_V) \sqsubseteq_{Sh} S\}, \rho(V)\big] \\
&= \rho(\gamma_{\mathrm{Sh}}([S, V])) \enspace .
\end{aligned}$$

which concludes the proof of the theorem. □

6. FORWARD UNIFICATION

We briefly recall from [Cortesi and Filè 1999; Bagnara et al. 2002] the definition of the standard operator unif'_{Sh} for abstract unification on Sharing. The abstract unification is performed between a set of sharing groups A and a single substitution δ, under the assumption that $\mathrm{vars}(\delta) \subseteq U$, and it is defined as follows:

$$\mathsf{unif}'_{Sh}([A, U], \delta) = [\mathbf{u}_{Sh}(A, \delta), U] \tag{24}$$

where $\mathbf{u}_{Sh} : \wp(\wp_f(\mathcal{V})) \times \mathit{ISubst} \to \wp(\wp_f(\mathcal{V}))$ is defined by induction as follows:

$$\begin{aligned}
\mathbf{u}_{Sh}(A, \epsilon) &= A \\
\mathbf{u}_{Sh}(A, \{x/t\} \uplus \theta) &= \mathbf{u}_{Sh}(A \setminus (\mathbf{rel}(A, \{x\}) \cup \mathbf{rel}(A, \mathrm{vars}(t))) \\
&\qquad \cup \mathbf{bin}(\mathbf{rel}(A, \{x\})^*, \mathbf{rel}(A, \mathrm{vars}(t))^*), \theta).
\end{aligned} \tag{25}$$

The auxiliary operators used in the definition of $\mathbf{u}_{Sh}$ are given by:

—the *closure under union* (or *star union*) $(.)^* : \wp(\wp_f(\mathcal{V})) \to \wp(\wp_f(\mathcal{V}))$

$$A^* = \left\{ \bigcup T \mid \emptyset \neq T \in \wp_f(A) \right\}^2 \enspace ; \tag{26}$$

—the *extraction of relevant components* $\mathbf{rel} : \wp(\wp_f(\mathcal{V})) \times \wp_f(\mathcal{V}) \to \wp(\wp_f(\mathcal{V}))$:

$$\mathbf{rel}(A, V) = \{T \in A \mid T \cap V \neq \emptyset\} \enspace ; \tag{27}$$

—the *binary union* $\mathbf{bin} : \wp(\wp_f(\mathcal{V})) \times \wp(\wp_f(\mathcal{V})) \to \wp(\wp_f(\mathcal{V}))$:

$$\mathbf{bin}(A, B) = \{T_1 \cup T_2 \mid T_1 \in A, T_2 \in B\} \enspace . \tag{28}$$

[2]Note that, due to the condition $T \neq \emptyset$, the notation A^+ would be more appropriate. However, we retain the notation A^* for historical reasons.

We recall that we will often abuse the notation and write $\mathbf{rel}(A, o)$ for $\mathbf{rel}(A, \mathrm{vars}(o))$ and $x \in o$ for $x \in \mathrm{vars}(o)$ where o is any syntactic object.

Example 6.1. Take $A = \{\mathtt{xy}, \mathtt{xz}, \mathtt{y}\}$, $U = \{w, x, y, z\}$ and $\delta = \{x/t(y, z), w/t(y)\}$. Note that, since w does not appear in A, then w is always bound to a ground term in $\gamma_{\mathrm{Sh}}([A, U])$. We have $\mathbf{rel}(A, x) = \{\mathtt{xy}, \mathtt{xz}\}$, $\mathbf{rel}(A, y) = \{\mathtt{xy}, \mathtt{y}\}$, $\mathbf{rel}(A, z) = \{\mathtt{xz}\}$ and therefore

$$\begin{aligned}\mathbf{u}_{Sh}(A, \{x/t(y, z)\}) =& A \setminus \{\mathtt{xy}, \mathtt{xz}, \mathtt{y}\} \cup \mathbf{bin}(\{\mathtt{xy}, \mathtt{xz}\}^*, \{\mathtt{xy}, \mathtt{xz}, \mathtt{y}\}^*)\\ =& \mathbf{bin}(\{\mathtt{xy}, \mathtt{xz}, \mathtt{xyz}\}, \{\mathtt{xy}, \mathtt{xz}, \mathtt{xyz}, \mathtt{y}\})\\ =& \{\mathtt{xy}, \mathtt{xz}, \mathtt{xyz}\} \ .\end{aligned}$$

If we take $B = \{\mathtt{xy}, \mathtt{xz}, \mathtt{xyz}\}$, we obtain $\mathbf{rel}(B, w) = \emptyset$, $\mathbf{rel}(B, y) = \{\mathtt{xy}, \mathtt{xyz}\}$ and therefore

$$\begin{aligned}\mathbf{u}_{Sh}(A, \delta) =& \mathbf{u}_{Sh}(B, \{w/t(y)\})\\ =& B \setminus \{\mathtt{xy}, \mathtt{xyz}\} \cup \mathbf{bin}(\emptyset, \{\mathtt{xy}, \mathtt{xyz}\}^*)\\ =& B \setminus \{\mathtt{xy}, \mathtt{xyz}\}\\ =& \{\mathtt{xz}\} \ .\end{aligned}$$

It is worth noting that unif'_{Sh} is not the abstract counterpart of $\mathsf{unif}_{\mathrm{Ps}}$, because $\mathsf{unif}'_{Sh}([S, U], \theta)$ is defined only under the condition that $\mathrm{vars}(\delta) \subseteq U$. Since this is not enough to define a goal-dependent semantics, when this solution is adopted, there is the need of an operator to expand the set of variables of interest in a substitution. Let us introduce the following concrete operator:

$$\iota_{\mathrm{Ps}}([\Sigma, U], V) = [\{\mathrm{mgu}([\sigma]_U, [\epsilon]_V) \mid [\sigma]_U \in \Sigma\}, U \cup V] \ , \tag{29}$$

whose optimal abstract counterpart is simply given by:

$$\iota_{Sh}([\Sigma, U], V) = [\Sigma \cup \{\{x\} \mid x \in V \setminus U\}, U \cup V)] \ . \tag{30}$$

By using ι_{Ps}, the operator $\mathsf{unif}_{\mathrm{Ps}}$ can be equivalently rewritten as:

$$\mathsf{unif}_{\mathrm{Ps}}([\Sigma, U], \theta) = \mathsf{unif}_{\mathrm{Ps}}(\iota_{\mathrm{Ps}}([\Sigma, U], \mathrm{vars}(\theta)), \theta) \ , \tag{31}$$

and now, in the right hand side, $\iota_{\mathrm{Ps}}([\Sigma, U], \mathrm{vars}(\theta))$ is an object of the kind $[\Theta, U \cup \mathrm{vars}(\theta)]$. Therefore, a correct abstract forward unification operator for $\mathbf{U}^f_{\mathrm{Ps}}$ may be obtained as

$$\begin{aligned}\mathbf{U}'^f_{Sh}([\Sigma, U], A_1, A_2) = \pi_{Sh}(&\mathsf{unif}'_{Sh}(\iota_{Sh}(\rho([\Sigma, U]), \mathrm{vars}(\rho(A_1)) \cup \mathrm{vars}(A_2)),\\ &\mathrm{mgu}(\rho(A_1) = A_2)), \mathrm{vars}(A_2)) \ ,\end{aligned} \tag{32}$$

provided that ρ is a renaming such that $\rho(U \cup \mathrm{vars}(A_1)) \cap \mathrm{vars}(A_2) = \emptyset$. However, $\mathbf{U}'^f_{Sh}$ is not optimal w.r.t. $\mathbf{U}^f_{\mathrm{Ps}}$.

Example 6.2. We have that

$$\begin{aligned}&\mathbf{U}'^f_{Sh}([\{\mathtt{xy}, \mathtt{yz}\}, \{x, y, z\}], p(x, y, z), p(u, v, w)) =\\ &\quad \pi_{Sh}([\mathbf{u}_{Sh}(\{\mathtt{xy}, \mathtt{yz}, \mathtt{u}, \mathtt{v}, \mathtt{w}\}, \{x/u, y/v, z/w\}), \{x, y, z, u, v, w\}], \{u, v, w\}) =\\ &\quad \pi_{Sh}([\{\mathtt{xyuv}, \mathtt{yzvw}, \mathtt{xyzuvw}\}, \{x, y, z, u, v, w\}], \{u, v, w\}) =\\ &\quad [\{\mathtt{uv}, \mathtt{vw}, \mathtt{uvw}\}, \{u, v, w\}] \ .\end{aligned} \tag{33}$$

There exists a sharing group $\mathtt{uvw}$ computed by the forward unification. However, when computing $\mathsf{unif}_{\mathrm{Ps}}(\gamma_{\mathrm{Sh}}([\{\mathtt{xy},\mathtt{yz}\},\{x,y,z\}]),\{x/u,y/v,z/w\})$ we know that u,v and w are free in $\gamma_{\mathrm{Sh}}([\{\mathtt{xy},\mathtt{yz}\},\{x,y,z\}]$. Following [Hans and Winkler 1992], we can avoid to compute the star unions when considering the binding y/v in $\mathbf{u}_{Sh}$, obtaining the smaller result $[\{\mathtt{xyuv},\mathtt{yzvw}\},\{x,y,z,u,v,w\}]$. If we now compute the projection on the variables $\{u,v,w\}$ we obtain the entry substitution $[\{\mathtt{uv},\mathtt{vw}\},\{u,v,w\}]$, with an obvious gain of precision.

Example 6.3. Let us consider the following unification.

$$\begin{aligned}&\mathbf{U'}^{f}_{Sh}([\{\mathtt{xy},\mathtt{xz}\},\{x,y,z\}],p(x,y,z),p(t(u,v),h,k))=\\&\qquad\pi_{Sh}([\mathbf{bin}(\{\mathtt{xyh},\mathtt{xzk},\mathtt{xyzhk}\},\{\mathtt{u},\mathtt{v},\mathtt{uv}\}),\{x,y,z,h,k,u\}],\{u,v,h,k\})\ .\end{aligned}\tag{34}$$

Since the term $t(u,v)$ is linear and independent from x, following [Hans and Winkler 1992] we can avoid to compute the star union over $\{\mathtt{xy},\mathtt{xz}\}$, obtaining the abstract object $[\mathbf{bin}(\{\mathtt{xyh},\mathtt{xzk}\},\{\mathtt{u},\mathtt{v},\mathtt{uv}\}),\{x,y,z,h,k,u\}]$. If we project on $\{h,k,u,v\}$ we obtain $\mathbf{bin}(\{\mathtt{h},\mathtt{k}\},\{\mathtt{u},\mathtt{v},\mathtt{uv}\})$ against $\mathbf{bin}(\{\mathtt{h},\mathtt{k},\mathtt{hk}\},\{\mathtt{u},\mathtt{v},\mathtt{uv}\})$. In this way, we are able to prove the independence of h from k.

These examples show that, when computing forward abstract unification by first enlarging the domain of variables of interest, there is a loss of precision. In fact, such a forward abstract unification operator is not optimal. We now show that it is possible to design an optimal operator for forward unification which is able to exploit the information of linearity and freeness coming from the fact that the variables which appear in the third argument of $\mathbf{U}^{f}_{\mathrm{Ps}}$ are fresh. Note that we are not proposing to embed freeness and linearity information inside the domain, but only to use all the information coming from the syntax of the clauses.

6.1 The Refined Forward Unification

We are going to define an abstract operator unif_{Sh} which is correct and optimal w.r.t. $\mathsf{unif}_{\mathrm{Ps}}$.

Definition 6.4. The *abstract unification* $\mathsf{unif}_{Sh}:\mathtt{Sharing}\times \mathit{ISubst}\rightarrow\mathtt{Sharing}$ is defined as

$$\mathsf{unif}_{Sh}([S,U],\theta)=[\mathbf{u}^{f}_{Sh}(S\cup\{\{x\}\mid x\in\mathrm{vars}(\theta)\setminus U\},\mathrm{vars}(\theta)\setminus U,\theta),U\cup\mathrm{vars}(\theta)]$$

where $\mathbf{u}^{f}_{Sh}:\wp(\wp_f(\mathcal{V}))\times\wp_f(\mathcal{V})\times \mathit{ISubst}\rightarrow\wp(\wp_f(\mathcal{V}))$ is defined as:

$$\begin{aligned}\mathbf{u}^{f}_{Sh}(S,U,\epsilon)&=S\\ \mathbf{u}^{f}_{Sh}(S,U,\{x/t\}\uplus\delta)&=\mathbf{u}^{f}_{Sh}((S\setminus(\mathbf{rel}(S,t)\cup\mathbf{rel}(S,x)))\cup\\&\qquad\mathbf{bin}(\mathbf{rel}(S,x),\mathbf{rel}(S,t)),U\setminus\{x\},\delta)\end{aligned}$$

if $x\in U$

$$\begin{aligned}\mathbf{u}^{f}_{Sh}(S,U,\{x/t\}\uplus\delta)&=\mathbf{u}^{f}_{Sh}((S\setminus(\mathbf{rel}(S,t)\cup\mathbf{rel}(S,x)))\cup\\&\qquad\mathbf{bin}(\mathbf{rel}(S,x),\mathbf{rel}(S,Y)^{*})\cup\\&\qquad\mathbf{bin}(\mathbf{rel}(S,x)^{*},\mathbf{rel}(S,Z)^{*})\cup\\&\qquad\mathbf{bin}(\mathbf{bin}(\mathbf{rel}(S,x)^{*},\mathbf{rel}(S,Z)^{*}),\mathbf{rel}(S,Y)^{*}),\end{aligned}$$

$$U \setminus \mathrm{vars}(\{x/t\}), \delta)$$

$$\text{if } x \notin U,\ Y = \mathrm{uvars}(t) \cap U,\ Z = \mathrm{vars}(t) \setminus Y.$$

The idea is simply to carry on, in the second argument of $\mathbf{u}_{Sh}$, the set of variables which are definitively free and to apply the optimizations for the abstract unification with linear terms and free variables (see [Hans and Winkler 1992]). Actually, while the case for $x \in U$ is standard, the case for $x \notin U$ exploits some optimizations which are not found in the literature. When $Z = \emptyset$, we obtain:

$$(S \setminus (\mathbf{rel}(S,t) \cup \mathbf{rel}(S,x))) \cup \mathbf{bin}(\mathbf{rel}(S,x), \mathbf{rel}(S,Y)^*) \ ,$$

which is the standard result when the term t is linear and independent from x. However, when $Z \neq \emptyset$, the standard optimizations which appear, e.g., in [Hans and Winkler 1992] do not apply, since t cannot be proved to be linear and independent from x, and we should obtain the following standard result:

$$(S \setminus (\mathbf{rel}(S,t) \cup \mathbf{rel}(S,x))) \cup \mathbf{bin}(\mathbf{rel}(S,x)^*, \mathbf{rel}(S,t)^*) \ .$$

However, we are able to avoid some star unions by distinguishing the variables in t which are "linear and independent" (the set Y) from the others (the set Z), and observing that two sharing groups in $\mathbf{rel}(S,x)$ may be merged together only under the effect of the unification with some variable in Z. We will come back later to this topic.

We can now define the forward abstract unification $\mathbf{U}^f_{Sh} : \mathtt{Sharing} \times \wp_f(\mathcal{V}) \times \mathsf{Atoms} \times \mathsf{Atoms} \rightarrow \mathtt{Sharing}$. We only need to introduce the necessary renamings and projections, as done for the concrete case:

$$\mathbf{U}^f_{Sh}([S_1,U_1], A_1, A_2) = \pi_{Sh}(\mathsf{unif}_{Sh}(\rho([S_1,U_1]), \mathrm{mgu}(\rho(A_1) = A_2)), \mathrm{vars}(A_2)) \quad (35)$$

with ρ a renaming such that $\rho(U_1 \cup \mathrm{vars}(A_1)) \cap \mathrm{vars}(A_2) = \emptyset$.

6.2 Correctness of Forward Unification

We are going to prove that the unification operator unif_{Sh} is correct w.r.t. the concrete operator $\mathsf{unif}_{\mathrm{Ps}}$. We first need two technical lemmas.

LEMMA 6.5. *Given $\delta, \sigma \in Subst$, $v \in \mathcal{V}$, it is the case that $occ(\delta \circ \sigma, v) = occ(\sigma, occ(\delta, v))$.*

PROOF. By definition, $x \in occ(\delta \circ \sigma, v)$ iff $v \in \delta(\sigma(x))$, i.e., there exists $w \in \mathcal{V}$ such that $w \in \sigma(x)$ and $v \in \delta(w)$. In other words, $x \in occ(\delta \circ \sigma, v)$ iff there exists $w \in \mathcal{V}$ s.t. $w \in occ(\delta, v)$ and $x \in occ(\sigma, w)$ iff $x \in occ(\sigma, occ(\delta, v))$. $\square$

PROPOSITION 6.6. *Let $t \in \mathsf{Terms}$, $\sigma \in Subst$ and $U \in \wp_f(\mathcal{V})$ such that $\mathrm{vars}(t) \subseteq U$. Let $\alpha_{\mathrm{Sh}}([\sigma]_U) \sqsubseteq_{Sh} [S,U]$. Then the following property holds:*

$$\forall v \in \mathcal{V}. v \in \mathrm{vars}(t\sigma) \iff occ(\sigma, v) \cap U \in \mathbf{rel}(S,t) \ .$$

PROOF. Note that $v \in \mathrm{vars}(t\sigma)$ iff $\exists u \in t$ such that $v \in \sigma(u)$. In turn, this holds iff $\exists u \in t$ s.t. $u \in occ(\sigma, v)$ iff $occ(\sigma, v) \cap \mathrm{vars}(t) \neq \emptyset$ iff $(occ(\sigma, v) \cap U) \cap \mathrm{vars}(t) \neq \emptyset$. Note that $X = occ(\sigma, v) \cap U \subseteq S$ and therefore $X \cap \mathrm{vars}(t) \neq \emptyset$ iff $X \in \mathbf{rel}(S,t)$ by definition of $\mathbf{rel}$. $\square$

Now, we begin to analyze the abstract behavior of unification when the second argument is a substitution with only one binding.

PROPOSITION 6.7. *Let* $[\sigma]_U \in \mathit{ISubst}_\sim$, $\{x/t\} \in \mathit{ISubst}$ *such that* $\mathrm{vars}(\{x/t\}) \subseteq U$ *and* σ *and* $\{x/t\}$ *unify. If* $\alpha_{\mathrm{Sh}}([\sigma]_U]) \sqsubseteq_{\mathit{Sh}} [S, U]$ *and* $\delta = \mathrm{mgu}(x\sigma = t\sigma)$, *we obtain:*

$$\begin{aligned}\alpha_{\mathrm{Sh}}(\mathrm{mgu}([\sigma]_U, [x/t]_U)) \sqsubseteq_{\mathit{Sh}} &[(S \setminus (\mathbf{rel}(S,x) \cup \mathbf{rel}(S,t)))\\ &\cup \{\mathit{occ}(\sigma, \mathit{occ}(\delta, v)) \cap U \mid v \in \mathrm{vars}(x\sigma = t\sigma)\}, U] \ .\end{aligned}$$

PROOF. Since $\mathrm{vars}(\{x/t\}) \subseteq U$, we have $\mathrm{mgu}([\sigma]_U, [x/t]_U) = [\mathrm{mgu}(\sigma, \{x/t\})]_U$. Then, by definition of δ, it holds that $\mathrm{mgu}(\sigma, x = t) = \mathrm{mgu}(\mathrm{Eq}(\sigma) \cup x\sigma = t\sigma) = \mathrm{mgu}(x\sigma = t\sigma) \circ \sigma = \delta \circ \sigma$ (see [Palamidessi 1990, Proposition 6.1]). Therefore, we only need to show that:

$$\begin{aligned}\alpha_{\mathrm{Sh}}([\delta \circ \sigma]_U) \sqsubseteq_{\mathit{Sh}} &[(S \setminus (\mathbf{rel}(S,x) \cup \mathbf{rel}(S,t)))\\ &\cup \{\mathit{occ}(\sigma, \mathit{occ}(\delta, v)) \cap U \mid v \in \mathrm{vars}(x\sigma = t\sigma)\}, U] \ .\end{aligned} \tag{36}$$

By definition of α_{Sh}, we have to show that, for all $v \in \mathcal{V}$, $\mathit{occ}(\delta \circ \sigma, v) \cap U \in (S \setminus (\mathbf{rel}(S,x) \cup \mathbf{rel}(S,t))) \cup \{\mathit{occ}(\sigma, \mathit{occ}(\delta, v)) \cap U \mid v \in \mathrm{vars}(x\sigma = t\sigma)\}$. Let $v \in \mathcal{V}$. We have the following cases:

—$v \in \mathrm{vars}(x\sigma = t\sigma)$: by Lemma 6.5, $\{\mathit{occ}(\delta \circ \sigma, v) \cap U \mid v \in \mathrm{vars}(x\sigma = t\sigma)\} = \{\mathit{occ}(\sigma, \mathit{occ}(\delta, v)) \cap U \mid v \in \mathrm{vars}(x\sigma = t\sigma)\}$.

—$v \notin \mathrm{vars}(x\sigma = t\sigma)$: thus $v \notin \mathrm{vars}(\delta)$ and $\mathit{occ}(\delta \circ \sigma, v) = \mathit{occ}(\sigma, v)$. We know that $\mathit{occ}(\sigma, v) \cap U \in S$, by definition of S. Moreover, we show that $\mathit{occ}(\sigma, v) \cap U \notin \mathbf{rel}(S,x) \cup \mathbf{rel}(S,t)$. Since $v \notin \mathrm{vars}(x\sigma = t\sigma)$, we can apply Proposition 6.6 twice to the terms x and t, and obtain $\mathit{occ}(\sigma, v) \cap U \notin \mathbf{rel}(S,x) \cup \mathbf{rel}(S,t)$.

By collecting the results of the two cases, Equation (36) is proved. □

This result may be refined by introducing further hypotheses. We have anticipated that our abstract algorithm uses the fact that some variables are known to be free in the corresponding concrete algorithm to produce better results than standard unification. We may be more formal:

Definition 6.8. We say that a variable $x \in \mathcal{V}$ is *free* in $[\theta]_V$ when $\theta_{|V}(x) \in \mathcal{V}$.

Note that this definition does not depend on the choice of representative for $[\theta]_V$. Moreover, if x is free in and independent from V in $[\theta]_V$, there exists a representative $\theta' \sim_V \theta$ such that $x \notin \mathrm{vars}(\theta)$. It is enough to take $\theta' = \theta''_{|\{-x\}}$ where θ'' is a canonical representative.

Now, we consider again Prop. 6.7, but we assume x to be free and independent from U in $[\sigma]_U$. The following proposition has been proved several times in literature (see, for example, [Hans and Winkler 1992]). We just present it again for the sake of completeness.

PROPOSITION 6.9. *Let* $[\sigma]_U \in \mathit{ISubst}_\sim$, $\{x/t\} \in \mathit{ISubst}$ *such that* $\mathrm{vars}(\{x/t\}) \subseteq U$ *and* σ *and* $\{x/t\}$ *unify. If* $\alpha_{\mathrm{Sh}}([\sigma]_U) \sqsubseteq_{\mathit{Sh}} [S, U]$ *and* x *is free and independent from* U *in* $[\sigma]_U$, *then:*

$$\begin{aligned}&\alpha_{\mathrm{Sh}}(\mathrm{mgu}([\sigma]_U, [x/t]_U))\\ &\qquad \sqsubseteq_{\mathit{Sh}} [(S \setminus (\mathbf{rel}(S,x) \cup \mathbf{rel}(S,t))) \cup \mathbf{bin}(\mathbf{rel}(S,x), \mathbf{rel}(S,t)), U] \ .\end{aligned}$$

PROOF. First of all note that, without loss of generality, we may assume $x \notin \mathrm{vars}(\sigma)$. Then, by Prop.6.7, we have that:

$$\begin{aligned}\alpha_{\mathrm{Sh}}(\mathrm{mgu}([\sigma]_U, [x/t]_U)) \sqsubseteq_{Sh} &[(S \setminus (\mathbf{rel}(S,x) \cup \mathbf{rel}(S,t))) \\ &\cup \{ occ(\sigma, occ(\delta, v)) \cap U \mid \in \mathrm{vars}(x\sigma = t\sigma)\}, U] \ ,\end{aligned}$$

where $\delta = \mathrm{mgu}(x\sigma = t\sigma)$. Since $x \notin \mathrm{vars}(\sigma)$, we have that $x\sigma = t\sigma$ is equal to $x = t\sigma$. Moreover, $x \notin \mathrm{vars}(t\sigma)$ since $x \notin \mathrm{vars}(t)$ and $x \notin \mathrm{vars}(\sigma)$ by hypothesis. Thus $\delta = \mathrm{mgu}(x = t\sigma) = \{x/t\sigma\}$. It follows that $\mathrm{vars}(x\sigma = t\sigma) = \{x\} \cup \mathrm{vars}(t\sigma)$. Therefore, the following equalities hold:

$$\begin{aligned}
&\{ occ(\sigma, occ(\delta, v)) \cap U \mid v \in \mathrm{vars}(x\sigma = t\sigma)\} \\
= \ &\{ occ(\sigma, occ(\delta, v)) \cap U \mid v \in \{x\} \cup \mathrm{vars}(t\sigma)\} \\
= \ &\{ occ(\sigma, occ(\delta, v)) \cap U \mid v \in \mathrm{vars}(t\sigma)\} \qquad [\text{since } x \in \mathrm{dom}(\delta) \text{ and } occ(\delta, x) = \emptyset] \\
= \ &\{ occ(\sigma, \{x, v\}) \cap U \mid v \in \mathrm{vars}(t\sigma)\} \qquad [\text{since } \delta = \{x/t\sigma\}] \\
= \ &\{ (occ(\sigma, x) \cup occ(\sigma, v)) \cap U \mid v \in \mathrm{vars}(t\sigma)\} \\
= \ &\{ (\{x\} \cup occ(\sigma, v)) \cap U \mid v \in \mathrm{vars}(t\sigma)\} \qquad [\text{since } x \notin \mathrm{vars}(\sigma)]
\end{aligned}$$

Moreover, for each $v \in \mathrm{vars}(t\sigma)$, by Proposition 6.6 it holds that $occ(\sigma, v) \cap U \in \mathbf{rel}(S,t)$. Therefore, $\{(\{x\} \cup occ(\sigma, v)) \cap U \mid v \in \mathrm{vars}(t\sigma)\} \subseteq \mathbf{bin}(\{x\}, \mathbf{rel}(S,t))$. Since $x \notin \mathrm{vars}(\sigma)$ and $x \in U$, it follows that $occ(\sigma, x) = \{x\}$ and thus $\{x\} \in \mathbf{rel}(S,x)$ being $\alpha_{\mathrm{Sh}}([\sigma]_U) \sqsubseteq_{Sh} [S,U]$. As a consequence $\mathbf{bin}(\{x\}, \mathbf{rel}(S,t)) \subseteq \mathbf{bin}(\mathbf{rel}(S,x), \mathbf{rel}(S,t))$ from which it follows that $\alpha_{\mathrm{Sh}}([\mathrm{mgu}(\mathrm{Eq}(\sigma) \cup x = t)]_U) \sqsubseteq_{Sh} [(S \setminus (\mathbf{rel}(S,x) \cup \mathbf{rel}(S,t))) \cup \mathbf{bin}(\mathbf{rel}(S,x), \mathbf{rel}(S,t)), U]$. □

Now we want to analyze the case when x is not guaranteed to be free and independent from U in $[\sigma]_U$. In this case, we want to distinguish between those variables which occur uniquely in $x\sigma = t\sigma$ and those who do not. Variables of the first kind are special, in the sense which is formalized by the following proposition.

PROPOSITION 6.10. *Given* $s, t \in \mathsf{Terms}$ *and* $W, Y \in \wp_f(\mathcal{V})$ *such that* s *and* t *unify,* $\mathrm{vars}(s = t) \subseteq W$ *and* $Y \subseteq \mathrm{vars}(t) \cap \mathrm{uvars}(s = t)$*, then* $\delta = \mathrm{mgu}(s = t)$ *enjoys the following properties:*

(1) $\forall v \in \mathrm{vars}(s).\ occ(\delta, v) \cap \mathrm{vars}(s) \neq \emptyset \Rightarrow occ(\delta, v) \cap \mathrm{vars}(t) \neq \emptyset$,

(2) $\forall v \in \mathrm{vars}(s).\ occ(\delta, v) \cap \mathrm{vars}(s) = \{x_1, x_2\} \wedge x_1 \neq x_2 \Rightarrow occ(\delta, v) \cap Z \neq \emptyset$.

where $Z = \mathrm{vars}(t) \setminus Y$*.*

PROOF. We proof the two points separately.

(1) If $occ(\delta, v) \cap \mathrm{vars}(s) \neq \emptyset$ then $v \notin \mathrm{dom}(\delta)$ and therefore $v \in \delta(s)$. Since δ is an unifier for s and t, it should be $v \in \delta(t)$, and therefore there exists $y \in t$ such that $y \in occ(\delta, v)$.

(2) First of all, note that, given two terms s and t in a given signature Σ, the result of $\mathrm{mgu}(s = t)$ does not change if we enlarge Σ with a new constant symbol. Therefore, assume without loss of generality that there is a constant symbol a in the signature. The proof proceeds by contradiction.
Assume that there exist $x_1, x_2 \in \mathrm{vars}(s)$, $v \in W$ such that $x_1, x_2 \in occ(\delta, v)$ and $occ(\delta, v) \cap Z = \emptyset$. Let $\sigma = \{x = a \mid x \in W\}$ and consider the substitution $\delta' = \{z/(\delta(z))\sigma \mid z \in Z\}$. Note that this is an idempotent substitution since it is ground. Now consider $\delta'' = \mathrm{mgu}(\mathrm{Eq}(\delta) \cup \mathrm{Eq}(\delta'))$, which clearly exists and, by

definition of δ', is $\delta'' = \{x/a \mid x \in \mathrm{vars}(\delta(Z))\} \circ \delta$. Therefore, $occ(\delta'', v) = occ(\delta, v)$ because $v \notin \mathrm{vars}(\delta(Z))$ being $occ(\delta, v) \cap Z = \emptyset$.
Moreover, $\delta'' = \mathrm{mgu}(\mathrm{Eq}(\delta) \cup \mathrm{Eq}(\delta')) = \mathrm{mgu}(\{s = t\} \cup \mathrm{Eq}(\delta')) = \mathrm{mgu}(s\delta' = t\delta') \circ \delta' = \delta' \uplus \mathrm{mgu}(s\delta' = t\delta')$. By definition of δ', it holds that $\mathrm{vars}(t\delta') \cap Z = \emptyset$, and thus $\mathrm{vars}(t\delta') \subseteq Y$. From the definition of Y it follows that $\mathrm{vars}(t\delta') \subseteq \mathrm{uvars}(s = t)$, and thus $\mathrm{vars}(t\delta') \subseteq \mathrm{uvars}(s\delta' = t\delta')$, since $\mathrm{rng}(\delta') = \emptyset$. Therefore the term $t\delta'$ is linear and independent from $s\delta'$ and $occ(\mathrm{mgu}(s\delta' = t\delta'), v) = occ(\mathrm{mgu}(s\delta' = t\delta') \uplus \delta', v) = occ(\delta, v)$.
If we apply the result for linear and independent terms (see for example [King 2000, Proposition 3.1] we obtain an absurd, since it is not possible that both x_1 and x_2 are elements of $occ(\mathrm{mgu}(s\delta' = t\delta'), v)$.

This concludes the proof. □

We can use the previous proposition to compute the abstract unification in the case when x is not free and independent in $[\sigma]_U$.

Proposition 6.11. *Let* $[\sigma]_U \in \mathit{ISubst}_\sim$, $\{x/t\} \in \mathit{ISubst}$ *such that* $\mathrm{vars}(\{x/t\}) \subseteq U$ *and* σ *and* $\{x/t\}$ *unify. Given* $Y \in \wp_f(\mathcal{V})$ *such that, for all* $y \in Y$, $\mathrm{vars}(\sigma(y)) \subseteq \mathrm{uvars}(x\sigma = t\sigma)$*, if* $\alpha_{\mathrm{Sh}}([\sigma]_U) \sqsubseteq_{Sh} [S, U]$ *then*

$$\begin{aligned}\alpha_{\mathrm{Sh}}(\mathrm{mgu}([\sigma]_U, [x/t]_U)) \sqsubseteq_{Sh} [&(S \setminus (\mathbf{rel}(S,t) \cup \mathbf{rel}(S,x)))\\ &\cup \mathbf{bin}(\mathbf{rel}(S,x), \mathbf{rel}(S,Y)^*) \cup \mathbf{bin}(\mathbf{rel}(S,x)^*, \mathbf{rel}(S,Z)^*)\\ &\cup \mathbf{bin}(\mathbf{bin}(\mathbf{rel}(S,x)^*, \mathbf{rel}(S,Z)^*), \mathbf{rel}(S,Y)^*), U] \ ,\end{aligned}$$

where $Z = \mathrm{vars}(t) \setminus Y$.

Proof. By Prop. 6.7, we have that

$$\begin{aligned}\alpha_{\mathrm{Sh}}(\mathrm{mgu}([\sigma]_U, [x/t]_U)) \sqsubseteq_{Sh} [&(S \setminus (\mathbf{rel}(S,x) \cup \mathbf{rel}(S,t)))\\ &\cup \{occ(\sigma, occ(\delta, v)) \cap U \mid v \in \mathrm{vars}(x\sigma = t\sigma)\}, U] \ ,\end{aligned}$$

where $\delta = \mathrm{mgu}(x\sigma = t\sigma)$. We show that

$$\begin{aligned}&\{occ(\sigma, occ(\delta, v)) \cap U \mid v \in \mathrm{vars}(x\sigma = t\sigma)\}\\ &\quad\subseteq \mathbf{bin}(\mathbf{rel}(S,x), \mathbf{rel}(S,Y)^*) \cup \mathbf{bin}(\mathbf{rel}(S,x)^*, \mathbf{rel}(S,Z)^*)\\ &\qquad \cup \mathbf{bin}(\mathbf{bin}(\mathbf{rel}(S,x)^*, \mathbf{rel}(S,Z)^*), \mathbf{rel}(S,Y)^*) \cup \{\emptyset\} \ ,\end{aligned}$$

from which the thesis follows. The following equalities hold, for all $v \in \mathrm{vars}(x\sigma = t\sigma)$.

$$\begin{aligned}&occ(\sigma, occ(\delta, v)) \cap U\\ =\ &\bigcup\{occ(\sigma, w) \cap U \mid w \in occ(\delta, v)\}\\ =\ &\bigcup\{occ(\sigma, w) \cap U \mid w \in occ(\delta, v) \cap \mathrm{vars}(x\sigma)\}\\ &\cup \bigcup\{occ(\sigma, w) \cap U \mid w \in occ(\delta, v) \cap \mathrm{vars}(t\sigma)\}\\ &\qquad [\text{since } occ(\delta, v) \subseteq \mathrm{vars}(\delta) \cup \{v\} \subseteq \mathrm{vars}(x\sigma = t\sigma)]\end{aligned}$$

By applying Prop. 6.10 (1) to the equation $x\sigma = t\sigma$ we get $occ(\delta, v) \cap \mathrm{vars}(x\sigma) \neq \emptyset$ iff $occ(\delta, v) \cap \mathrm{vars}(t\sigma) \neq \emptyset$. Since the case $occ(\delta, v) = \emptyset$ is trivial, it only remain to consider the case $occ(\delta, v) \neq \emptyset$ which implies $occ(\delta, v) \cap \mathrm{vars}(t\sigma) \neq \emptyset \neq occ(\delta, v) \cap \mathrm{vars}(x\sigma)$. In the following, we call $A = \bigcup\{occ(\sigma, w) \cap U \mid w \in occ(\delta, v) \cap \mathrm{vars}(x\sigma)\}$ and $B = \bigcup\{occ(\sigma, w) \cap U \mid w \in occ(\delta, v) \cap \mathrm{vars}(t\sigma)\}$. Note that, by

Prop. 6.6, $occ(\sigma, w) \cap U \in \mathbf{rel}(S, \{x\})$ if $w \in \mathrm{vars}(x\sigma)$ and $x \in U$, which implies $A \in \mathbf{rel}(S, \{x\})^*$. For the same reason, $B \in \mathbf{rel}(S, \mathrm{vars}(t))^*$, i.e.,

$$occ(\sigma, occ(\delta, v)) \cap U \in \mathbf{bin}(\mathbf{rel}(S, \{x\})^*, \mathbf{rel}(S, \mathrm{vars}(t))^*) \ ,$$

which is the standard result for abstract unification without considering freeness or linearity. We can do better if we proceed by cases on $occ(\delta, v) \cap \mathrm{vars}(t\sigma)$.

$\boldsymbol{occ(\delta, v) \cap \mathbf{vars}(t\sigma) \subseteq \mathbf{vars}(\sigma(Y))}$. Let $Z' = \mathrm{vars}(t\sigma) \setminus \mathrm{vars}(\sigma(Y))$ it follows that $occ(\delta, v) \cap Z' = \emptyset$. Therefore, by Prop. 6.10(2) applied to the terms $x\sigma$ and $t\sigma$, we have that $\nexists x_1, x_2 \in \mathrm{vars}(x\sigma)$ such that $x_1, x_2 \in occ(\delta, v)$. Since $occ(\delta, v) \cap \mathrm{vars}(x\sigma) \neq \emptyset$, it follows that there exists $x' \in \mathrm{vars}(x\sigma)$ such that $occ(\delta, v) \cap \mathrm{vars}(x\sigma) = \{x'\}$. This implies that $A \in \mathbf{rel}(S, \{x\})$. Moreover, by Prop. 6.6 applied to the set of variables Y, $B \in \mathbf{rel}(S, Y)^*$ and this proves

$$occ(\sigma, occ(\delta, v)) \cap U \in \mathbf{bin}(\mathbf{rel}(S, \{x\}), \mathbf{rel}(S, Y)^*) \ .$$

otherwise. We are in the case that $occ(\delta, v) \cap \mathrm{vars}(t\sigma) \nsubseteq \mathrm{vars}(\sigma(Y))$, i.e., $occ(\delta, v) \cap \mathrm{vars}(\sigma(Z)) \neq \emptyset$. Therefore, there exists $w \in occ(\delta, v) \cap \mathrm{vars}(\sigma(Z))$ and by Proposition 6.6 $occ(\sigma, w) \cap U \in \mathbf{rel}(S, Z)$. This implies that $B \in \{B_1 \cup \ldots B_n \cup C_1 \cup \ldots C_p \mid B_i \in \mathbf{rel}(S, Y), n \geq 0, C_i \in \mathbf{rel}(S, Z), p \geq 1\} = \mathbf{rel}(S, Z)^* \cup \mathbf{bin}(\mathbf{rel}(S, Y)^*, \mathbf{rel}(S, Z)^*)$. As a final result we have that:

$$\begin{aligned} occ(\sigma, occ(\delta, v)) \cap U &\in \mathbf{bin}(\mathbf{rel}(S, \{x\})^*, \mathbf{rel}(S, Z)^* \cup \mathbf{bin}(\mathbf{rel}(S, Y)^*, \mathbf{rel}(S, Z)^*)) \\ &= \mathbf{bin}(\mathbf{rel}(S, \{x\})^*, \mathbf{rel}(S, Z)^*) \cup \\ &\quad\ \mathbf{bin}(\mathbf{bin}(\mathbf{rel}(S, \{x\})^*, \mathbf{rel}(S, Z)^*), \mathbf{rel}(S, Y)^*) \ , \end{aligned}$$

which proves the theorem. □

Now we want to use together Propositions 6.9 and 6.11 to prove correctness of $\mathbf{u}_{Sh}$, and thus to prove correctness of unif_{Sh}.

LEMMA 6.12. *Let* $[\sigma]_V \in ISubst_\sim$, $\theta \in ISubst$ *such that* $\mathrm{vars}(\theta) \subseteq V$ *and* σ *and* θ *unify. Assume given* $U \subseteq V$ *such that, for each* $x \in U$,

(1) x *is free in* $[\sigma]_V$;

(2) x *is independent from* $\mathrm{vars}(\theta)$ *in* $[\sigma]_V$;

(3) *if* $x \in \mathrm{dom}(\theta)$ *then* x *is independent in* $[\sigma]_V$.

If $\alpha_{\mathrm{Sh}}([\sigma]_V) \sqsubseteq_{Sh} [S, V]$ *then* $\alpha_{\mathrm{Sh}}(\mathrm{mgu}([\sigma]_V, [\theta]_V)) \sqsubseteq_{Sh} [\mathbf{u}^f_{Sh}(S, U, \theta), V]$.

PROOF. The proof is by induction on $|\mathrm{dom}(\theta)|$. Assume $|\mathrm{dom}(\theta)| = 0$, then $\theta = \epsilon$ and $\alpha_{\mathrm{Sh}}(\mathrm{mgu}([\sigma]_V, [\theta]_V)) = \alpha_{\mathrm{Sh}}([\sigma]_V) \sqsubseteq_{Sh} [S, V] = [\mathbf{u}^f_{Sh}(S, U, \epsilon), V] = [\mathbf{u}^f_{Sh}(S, U, \theta), V]$.

Now assume that it holds for $|\mathrm{dom}(\theta)| \leq n$ and we show it holds for $|\mathrm{dom}(\theta)| = n + 1$, too. Let θ be $\theta' \uplus \{x/t\}$. We distinguish two cases: either $x \in U$ or $x \notin U$.

(1) ($x \in U$) By definition of $\mathbf{u}^f_{Sh}$ we have that

$$\begin{aligned} &\mathbf{u}^f_{Sh}(S, U, \{x/t\} \uplus \theta') \\ = \ &\mathbf{u}^f_{Sh}((S \setminus (\mathbf{rel}(S, x) \cup \mathbf{rel}(S, t))) \cup \mathbf{bin}(\mathbf{rel}(S, x), \mathbf{rel}(S, t)), U \setminus \{x\}, \theta') \ . \end{aligned}$$

Since $x \in U \cap \mathrm{dom}(\theta)$, by hypothesis x is free and independent in $[\sigma]_V$. Thus we can apply Prop. 6.9, from which we obtain that:

$$\begin{aligned}&\alpha_{\mathrm{Sh}}(\mathrm{mgu}([\sigma]_V, [x/t]_V))\\ \sqsubseteq_{Sh}\ & [S \setminus (\mathbf{rel}(S, x) \cup \mathbf{rel}(S, t)) \cup \mathbf{bin}(\mathbf{rel}(S, x), \mathbf{rel}(S, t)), V]\ .\end{aligned}$$

Let $[\sigma']_V = \mathrm{mgu}([\sigma]_V, [x/t]_V)$ and $U' = U \setminus \{x\}$. We may assume without loss of generality that $\mathrm{vars}(\sigma) \cap U = \emptyset$ and we obtain $\sigma' = \mathrm{mgu}(\mathrm{Eq}(\sigma) \cup \{x = t\}) = \sigma \uplus \{x/t\sigma\}$. Given $u \in U'$, we have $\sigma'(u) = \sigma(u) = u \in \mathcal{V}$, hence u is free in $[\sigma]_V$. If $u \neq v \in \mathrm{vars}(\theta')$, then $v \neq x$ and therefore $u \notin \sigma'(v) = \sigma(v)$. Thus u is independent from $\mathrm{vars}(\theta')$ in $[\sigma']_V$. Moreover, if $u \in \mathrm{dom}(\theta')$, then $u \neq x$, $u \notin t$ and $u \notin vars(\sigma)$, and therefore $u \notin \mathrm{vars}(\sigma') \subseteq \mathrm{vars}(\sigma) \cup \mathrm{vars}(x = t)$. This means that u is independent in $[\sigma']_U$. Therefore, by inductive hypothesis,

$$\begin{aligned}\alpha_{\mathrm{Sh}}(\mathrm{mgu}([\sigma]_V, [\theta]_V)) &= \alpha_{\mathrm{Sh}}(\mathrm{mgu}([\sigma']_V, [\theta']_V))\\ &\sqsubseteq_{Sh} [\mathbf{u}^f_{Sh}(S', U', \theta'), V] = [\mathbf{u}^f_{Sh}(S, U, \theta)]\ ,\end{aligned}$$

which concludes this part of the proof.

(2) $(x \notin U)$ By definition of $\mathbf{u}^f_{Sh}$ we have that:

$$\begin{aligned}\mathbf{u}^f_{Sh}(S, U, \{x/t\} \uplus \theta) = \mathbf{u}^f_{Sh}(&(S \setminus (\mathbf{rel}(S, x) \cup \mathbf{rel}(S, t)))\\ &\cup \mathbf{bin}(\mathbf{rel}(S, x), \mathbf{rel}(S, Y)^*) \cup \mathbf{bin}(\mathbf{rel}(S, x)^*, \mathbf{rel}(S, Z)^*)\\ &\cup \mathbf{bin}(\mathbf{bin}(\mathbf{rel}(S, x)^*, \mathbf{rel}(S, Y)^*), \mathbf{rel}(S, Z)^*)), U \setminus \mathrm{vars}(\{x/t\}), \delta)\ ,\end{aligned}$$

where $Y = \mathrm{uvars}(t) \cap U$ and $Z = \mathrm{vars}(t) \setminus Y$. Since $Y \subseteq U$, then for all $u \in Y$ and for all $v \in \mathrm{vars}(x = t)$ with $v \neq u$, it is the case that v and u do not share variables, i.e., $v \neq u \Rightarrow \sigma(u) \notin \sigma(v)$. Therefore $\sigma(u) \in \mathrm{uvars}(x\sigma = t\sigma)$. Then we can apply Prop. 6.11 to obtain

$$\begin{aligned}\alpha_{\mathrm{Sh}}([\sigma]_V, [x/t]_V]) \sqsubseteq_{Sh} &(S \setminus (\mathbf{rel}(S, t) \cup \mathbf{rel}(S, x)))\\ &\cup \mathbf{bin}(\mathbf{rel}(S, x), \mathbf{rel}(S, Y)^*) \cup \mathbf{bin}(\mathbf{rel}(S, x)^*, \mathbf{rel}(S, Z)^*)\\ &\cup \mathbf{bin}(\mathbf{bin}(\mathbf{rel}(S, x)^*, \mathbf{rel}(S, Z)^*), \mathbf{rel}(S, Y)^*), V]\ .\end{aligned}$$

Again, assume $\mathrm{vars}(\sigma) \cap U = \emptyset$, $\sigma' = \mathrm{mgu}(\mathrm{Eq}(\sigma) \cup \{x = t\}) = \mathrm{mgu}(x\sigma = t\sigma) \circ \sigma$ and $U' = U \setminus \mathrm{vars}(\{x/t\})$. Given $u \in U'$, $u \notin \mathrm{vars}(x = t)$ and since u by hypothesis does not share with any variable in $x = t$, we have $u \notin \mathrm{vars}(\{x\sigma/t\sigma\})$. As a result $\sigma'(u) = \sigma(u) = u \in \mathcal{V}$. Moreover, for each variable v, $u \in \sigma'(v)$ iff $u \in \sigma(v)$. Therefore, if $v \in \mathrm{vars}(\theta')$ and $v \neq u$, v and u are independent in $[\sigma']_V$. Finally, if $u \in \mathrm{dom}(\theta')$, then $u \notin \mathrm{vars}(\sigma)$ which implies $u \notin \mathrm{vars}(\sigma')$. By inductive hypothesis we have

$$\begin{aligned}\alpha_{\mathrm{Sh}}(\mathrm{mgu}([\sigma]_V, [\theta]_V)) &= \alpha_{\mathrm{Sh}}(\mathrm{mgu}([\sigma']_V, [\theta']_V))\\ &\sqsubseteq_{Sh} [\mathbf{u}^f_{Sh}(S', U', \theta'), V] = [\mathbf{u}^f_{Sh}(S, U, \theta), V]\ ,\end{aligned}$$

which proves the lemma. □

THEOREM 6.13. (CORRECTNESS OF unif_{Sh}) *The unification operator* unif_{Sh} *is correct w.r.t.* $\mathsf{unif}_{\mathrm{Ps}}$.

PROOF. Given $[\Theta, V] = \mathsf{unif}_{\mathrm{Ps}}([\Theta_1, V_1], \delta)$, we know that, if $[\theta]_V \in \Theta$, then

$$[\theta]_V = \mathrm{mgu}([\theta_1]_{V_1}, [\delta]_{\mathrm{vars}(\delta)}) = \mathrm{mgu}([\theta_1]_{V_1}, [\epsilon]_{V_1 \cup \mathrm{vars}(\delta)}, [\delta]_{\mathrm{vars}(\delta)}) \ .$$

Note that

$$\alpha_{\mathrm{Sh}}(\mathrm{mgu}([\theta_1]_{V_1}, [\epsilon]_{V_1 \cup \mathrm{vars}(\delta)})) \sqsubseteq_{Sh} [\{S \cup \{\{x\} \mid x \in \mathrm{vars}(\delta) \setminus V_1]\}, V_1 \cup \mathrm{vars}(\delta)]$$

and each variable in $\mathrm{vars}(\delta)$ is free and independent in $\mathrm{mgu}([\theta_1]_{V_1}, [\epsilon]_{V_1 \cup \mathrm{vars}(\delta)})$. Therefore, by applying Lemma 6.12, we obtain

$$\alpha_{\mathrm{Sh}}([\theta]_V) \sqsubseteq_{Sh} \mathsf{unif}_{Sh}([S_1, V_1], \delta) \ .$$

The theorem follows by the pointwise extension of α_{Sh} to elements of `Psub`. $\square$

6.3 Optimality of Forward Unification

In this section we prove that the unification operator unif_{Sh} is optimal w.r.t. $\mathsf{unif}_{\mathrm{Ps}}$, that is to say that, given $[S_1, U_1] \in$ `Sharing` and $\theta \in \mathit{ISubst}$, it holds:

$$\alpha_{Sh}(\mathsf{unif}_{\mathrm{Ps}}(\gamma_{Sh}([S_1, U_1]), \theta)) \sqsupseteq_{Sh} \mathsf{unif}_{Sh}([S_1, U_1], \theta)$$

Let $\mathsf{unif}_{Sh}([S_1, U_1], \theta) = [S, U_1 \cup \mathrm{vars}(\theta)]$. To prove optimality, we only need to show that, for each $X \in S$, there exists $[\delta]_{U_1}$ such that $\alpha_{\mathrm{Sh}}([\delta]_{U_1}) \sqsubseteq_{Sh} [S_1, U_1]$ and $\alpha_{Sh}(\mathsf{unif}_{\mathrm{Ps}}([\{[\delta]\}, U_1], \theta)) \sqsupseteq_{Sh} [\{X\}, U_1 \cup \mathrm{vars}(\theta)]$. To ease notation, let us define $U_2 = \mathrm{vars}(\theta) \setminus U_1$, $S_2 = \{\{x\} \mid x \in U_2\}$, $U = U_1 \cup U_2$, $X_1 = X \cap U_1$ and $X_2 = X \cap U_2$.

We begin by checking some properties of the unification algorithm in $\mathbf{u}^f_{Sh}$. To simplify the notation, in the rest of this section we will use a slightly modified version of the operator $\mathbf{u}^f_{Sh}$ which uses the rule $\mathbf{u}^f_{Sh}(T, V, \epsilon) = (T, V)$ (instead of the original rule $\mathbf{u}^f_{Sh}(T, V, \epsilon) = T$). The only consequence of this modification is that the new operator returns a pair whose first argument is the same as in the original operator and whose second argument is a set of variables guaranteed to be free after the unification.

Remark 6.14. Given $(T', V') = \mathbf{u}^f_{Sh}(T, V, \theta)$ the following properties are easily checked from the definition:

(1) $V' \subseteq V$;

(2) if $x \in V' \cap \mathrm{rng}(\theta)$ and $x \in \theta(v)$, then $v \in V$.

(3) $\mathbf{u}^f_{Sh}(T, V, \theta \uplus \theta') = \mathbf{u}^f_{Sh}(T', V', \theta')$

Let $[H, U] = \alpha_{Sh}([\theta]_U)$. We want to prove that each $X \in S$ is obtained as union of a number of sharing groups in H. However, these sharing groups cannot be joined freely but only according to some conditions.

LEMMA 6.15. *For each $X \in S$, either $X \in H$ or there are $B_1, \ldots, B_k \in H$ s.t. $\cup_{i \leq k} B_i = X$ and for each $i \leq k$, $B_i \cap U_1 \neq \emptyset$.*

PROOF. The proof proceeds by induction on the number of bindings n in θ. If $n = 0$, then $\theta = \epsilon$, $S = S_1 \cup S_2$ and $H = \{\{x\} \mid x \in U_1 \cup U_2\}$. If $X \in S_2$ then $X = \{x\}$ for some $x \in U_2$, i.e., $X \in H$. Otherwise, if $X \in S_1$, then $X = \bigcup\{\{x\} \mid x \in X\}$. Since $x \in \mathrm{vars}(S_1)$ entails $x \in U_1$, we may take as B_i's the singletons $\{x\}$ for each $x \in X$ and we have the required result.

If $n \neq 0$ then $\theta = \theta' \uplus \{x/t\}$ and $\mathbf{u}^f_{Sh}(S_1 \cup S_2, U_2, \theta) = \mathbf{u}^f_{Sh}(T, V, \{x/t\})$ where $(T, V) = \mathbf{u}^f_{Sh}(S_1 \cup S_2, U_2, \theta')$. Let $[H', U] = \alpha_{\mathrm{Sh}}([\theta']_U)$. We distinguish the case $x \in V$ and $x \notin V$.

Assume $x \in V$. If $X \in T \setminus (\mathbf{rel}(T, t) \cup \mathbf{rel}(T, x))$ then $X \cap \mathrm{vars}(\{x/t\}) = \emptyset$. By inductive hypothesis, $X = B_1 \cup \ldots \cup B_h$ where each $B_j \in H'$. Since $B_j \cap \mathrm{vars}(\{x/t\}) = \emptyset$ then $B_j \in H$ and therefore the property is satisfied. Otherwise, $X = A_1 \cup A_2$ where $A_1 \in \mathbf{rel}(T, x)$ and $A_2 \in \mathbf{rel}(T, t)$. Note that since $x \notin \mathrm{vars}(\theta')$, then $\mathbf{rel}(H', x) = \{\{x\}\}$. Since $\{x\} \cap U_1 = \emptyset$, it is not possible to join $\{x\}$ with any other sharing group in H', and therefore $\mathbf{rel}(T, x) = \{\{x\}\}$ and $A_1 = \{x\}$. Now assume, without loss of generality, $A_2 \in \mathbf{rel}(T, y)$, with $y \in \mathrm{vars}(t)$. By inductive hypothesis $A_2 = C_1 \cup \ldots \cup C_h$ with each $C_j \in H'$. First of all, note that, for each j, either $C_j \cap \mathrm{vars}(\{x/t\}) = \emptyset$ which entails $C_j \in H$, or $C_j = \mathit{occ}(\theta', w)$ for some $w \in \mathrm{vars}(t)$, which entails $\{x\} \cup C_j = \mathit{occ}(\theta, w) \in H$. Therefore, it is possible to take $k = h$ and B_j equals either to C_j or $C_j \cup \{x\}$ so that $B_j \in H$. Since there is at least one index l such that $y \in C_l$, then $C_l = \mathit{occ}(\theta', y)$ and $x \in B_l$. Therefore $\cup_j B_j = X$. Moreover, either $h = 1$ or $h > 1$ and $C_j \cap U_1 \neq \emptyset$ for each $j \leq h$.

Now assume $x \notin V$. If $X \in T \setminus (\mathbf{rel}(T, t) \cup \mathbf{rel}(T, x))$ then $X \cap \mathrm{vars}(\{x/t\}) = \emptyset$ and everything is as for the case $x \in V$. Otherwise, the three cases in the definition of $\mathbf{u}^f_{Sh}$ may be subsumed saying that $X = A_1 \cup A_2$ where $A_1 \in \mathbf{rel}(S, x)^*$ and $A_2 \in \mathbf{rel}(S, t)^*$. Assume, by inductive hypothesis, that $A_1 = C^1_1 \cup \ldots \cup C^1_h$ where each $C^1_j \in H'$ and $A_2 = C^2_1 \cup \ldots \cup C^2_l$ where each $C^2_j \in H'$. Since $x \notin \mathrm{vars}(\theta')$ then $\mathbf{rel}(H', x) = \{\{x\}\}$. Therefore there exists C^1_j such that $C^1_j = \{x\}$. We assume without loss of generality that $C^1_1 = \{x\}$. As for the case with $x \in V$, we may define B^2_j equals either to C^2_j or $C^2_j \cup \{x\}$ so that $B^2_j \in H$. The same holds for all the elements of the kind C^1_j for $j > 1$. Moreover, there is at least one j such that $C^2_j = \mathit{occ}(\theta', y)$ for some $y \in \mathrm{vars}(t)$, i.e., such that $x \in B^2_j$. Then, we have a collection of elements B^1_j and B^2_j such that each $B^1_j, B^2_j \in H$ and whose union gives X. We only need to prove that $B^1_j \cap U_1 \neq \emptyset$ and $B^2_j \cap U_1 \neq \emptyset$ for each j. Note that if $C^2_j \cap U_1 \neq \emptyset$, then $B^2_j \cap U_1 \neq \emptyset$. Assume $C^2_j \cap U_1 = \emptyset$. By inductive hypothesis, this happens if $C^2_j \in \mathbf{rel}(S, t)$ (otherwise C^2_j is obtained by joining more than one element in H', and therefore it must contains some variable in U_1) . Thus, there exists $y \in \mathrm{vars}(t)$ such that $y \in C^2_j$, and therefore $B^2_j = C^2_j \cup \{x\}$ and $B^2_j \cap U_1 \neq \emptyset$. In the same way, if $C^1_j \cap U_1 \neq \emptyset$ the same holds for B^1_j. Note that, given C^1_j, by inductive hypothesis either $C^1_j \notin \mathbf{rel}(S, x)$ and therefore $C^1_j \cap U_1 \neq \emptyset$, or $C^1_j \in \mathbf{rel}(S, x)$, and therefore $x \in C^1_j$ which entails again $C^1_j \cap U_1 \neq \emptyset$. □

Corollary 6.16. $X = \{x \mid \mathrm{vars}(\theta(x)) \cap X \neq \emptyset\}$.

Proof. By Lemma 6.15 we know $X = B_1 \cup \cdots \cup B_N$ with $B_i \in H$. If $x \in X$ then $x \in B_i$ for some $i \leq N$. Assume $B_i = \mathit{occ}(\theta, w)$. Then $w \in B_i \subseteq X$ and $w \in \mathrm{vars}(\theta(x)) \cap X$. On the opposite direction, assume $z \in \mathrm{vars}(\theta(x)) \cap X$. Since there is only one sharing group B in H such that $z \in B$, namely $B = \mathit{occ}(\theta, z)$, it must be the case that $B = B_j$ for some $j \in \{1, \ldots, N\}$ and therefore $x \in B_i \subseteq X$. □

While Lemma 6.15 proves that X is obtained by joining sharing groups in H, by the definition of $\mathbf{u}^f_{Sh}$ it is obvious that X may be also obtained by joining sharing groups in $S_1 \cup S_2$. Note that two sharing groups in S_1 may be joined only if some conditions are met. In particular, given $X \in \wp_f(\mathcal{V})$ and $\theta \in \mathit{ISubst}$, we define a relation $R \subseteq S_1 \times S_1$ as follows:

$$B_1 R B_2 \iff \exists x_1 \in B_1, x_2 \in B_2, y \in \mathrm{vars}(\theta(x_1)) \cap \mathrm{vars}(\theta(x_2)) \cap X. \\ (x_1 = x_2 \implies y \notin \mathrm{uvars}(\theta(x_1))) \ . \quad (37)$$

We say that X is θ-connected when there exist $B_1, \ldots, B_k \in S_1$ s.t. $\cup_{j \leq k} B_j = X_1$ and $B_1 R^* B_2 \ldots R^* B_k$. Note that, if X is θ-connected and $Y \subseteq U_2$, then given $\theta' = \theta \uplus \theta''$, it holds that $X \cup Y$ is θ'-connected.

Lemma 6.17. *For each $X \in S$, X is θ-connected.*

Proof. The proof is by induction on the number of bindings in θ. If $\theta = \epsilon$ there is nothing to prove since $X \in S_1 \cup S_2$, and thus $X_1 \in S_1$.

Let $\theta = \theta' \uplus \{x/t\}$, $[H', U] = \alpha_{\mathrm{Sh}}([\theta']_U)$, and $(S, V') = \mathbf{u}^f_{Sh}(T, V, \{x/t\})$ where $\mathbf{u}^f_{Sh}(S_1 \cup S_2, U_2, \theta') = (T, V)$.

We distinguish two cases according to the fact that $x \in V$ or not. Consider the case $x \in V$, which implies $x \in U_2$. By hypothesis $x \notin \mathrm{vars}(\theta')$ therefore, by Lemma 6.15, $\mathbf{rel}(T, x) = \{\{x\}\}$. Therefore S is obtained by joining to each $Q \in \mathbf{rel}(T, t)$ the new sharing group $\{x\}$ and removing $\{x\}$ from T. It happens that each $Q \in S$ is θ-connected since: 1) either $Q \in T$ is θ'-connected, from which the thesis; 2) or $Q = Q' \cup \{x\}$ for $Q' \in T$ and $x \in U_2$. In this latter case, Q' is θ'-connected, and thus $Q' \cup \{x\}$ is θ-connected since $x \in U_2$.

The other case is when $x \notin V$. If we take $Q \in S$ and assume $Q \in T \setminus (\mathbf{rel}(T, x) \cup \mathbf{rel}(T, t))$, then it is θ'-connected by inductive hypothesis, and thus it is θ-connected. Otherwise, take $Q = Q_1 \cup Q_2$ with $Q_1 \in \mathbf{rel}(T, x)$ and $Q_2 \in \mathbf{rel}(T, Y)^*$ where $Y = \mathrm{uvars}(t) \cap V$. Given $y \in Y$, since $y \in V$, then for each binding x'/t' in θ', if $y \in \mathrm{vars}(t')$ then $x' \in U_2$ (see Remark 6.14). Therefore $\mathbf{rel}(H, y) = \{K\}$ with $K \subseteq U_2$, and by Lemma 6.15, the same holds for $\mathbf{rel}(T, y)$. This means $Q_2 \subseteq U_2$. Thus $Q \cap U_1 = Q_1 \cap U_1$. Since Q_1 is θ'-connected by inductive hypothesis, it follows that Q_1 is θ-connected.

Now, take $Q_1 \in \mathbf{rel}(T, x)^*$ and $Q_2 \in \mathbf{rel}(T, Z)^*$, where $Z = \mathrm{vars}(t) \setminus Y$. Thus $Q_1 = A_1 \cup \ldots \cup A_k$ with $A_i \in \mathbf{rel}(T, x)$. By inductive hypothesis, A_i is θ'-connected, and therefore it is θ-connected. It follows that for each $i \leq k$ there exist $B^i_1, \ldots, B^i_{k_i} \in S_1$ such that $\cup_{j \leq k_i} B^i_j = A_i \cap U_1$ and $B^i_{j_1} R^* B^i_{j_2}$ for $j_1, j_2 \leq k_i$. The same holds for $Q_2 = C_1 \cup \ldots \cup C_h$ with $C_i \in \mathbf{rel}(T, Z)$: for any $C_i \cap U_1 \neq \emptyset$ we have that $C_i \cap U_1 = \cup_{j \leq h_i} D^i_j$ with $D^i_{j_1} R^* D^i_{j_2}$ for all $j_1, j_2 \leq h_i$.

We need to show that given any B^i_m, D^j_n then $B^i_m R^* D^j_n$. Actually, it is enough to show that for each $i \leq k, j \leq h$ such that $C_j \cap U_1 \neq \emptyset$, there are m, n s.t. $B^i_m R D^j_n$.

Since $x \in A_i$ and $x \in U_1$, without loss of generality we may assume that $x \in B^i_1$. In the other hand, although $\mathrm{vars}(t) \cap C_j \neq \emptyset$, we cannot infer that there exists any D^j_n s.t. $\mathrm{vars}(t) \cap D^j_n \neq \emptyset$ since it may well happen that $\mathrm{vars}(t) \cap C_j \subseteq U_2$ although $U_1 \cap U_j \neq \emptyset$.

Assume $C_j \in \mathbf{rel}(T, z)$ for some $z \in Z \cap U_1$. Then, we may assume without loss of generality that $z \in D^j_1$, and $B^i_1 R D^j_1$ follows from the definition of R, being $z \in X$. Otherwise, $C_j \in \mathbf{rel}(T, z)$ for some $z \in Z \cap U_2$. By applying Lemma 6.15, we have $C_j = E_1 \cup \cdots \cup E_p$ with $E_i \in H'$ and $E_i \cap U_1 \neq \emptyset$ (this holds even if $p = 1$ since $C_j \cap U_1 \neq \emptyset$). Since $\mathbf{rel}(H', z) = \{occ(\theta', z)\}$, then $occ(\theta', z) \cap U_1 \neq \emptyset$, i.e., there exists $z' \in U_1$ such that $z \in \mathrm{vars}(\theta'(z'))$. Then $z' \in C_j$ and we may assume, without loss of generality, that $z' \in D^j_1$. Again, we have $B^i_1 R D^j_1$ by definition of R.

Observe that, if $Q_2 \cap U_1 \neq \emptyset$, by symmetry and transitivity, this alone proves that $B^i_m R^* B^{i'}_{m'}$ and $D^j_n R^* D^{j'}_{n'}$ for each i, m, i', m' and j, n, j', n'. Otherwise, there is no D^j_n and we need to prove in other ways that $B^i_m R^* B^{i'}_{m'}$. Since $Q_2 \cap U_1 = \emptyset$, then $C_i \subseteq U_2$ for each i. This means $C_i = occ(\theta', y)$ for some $y \in U_2$ and since $C_i \subseteq U_2$ it follows immediately that $y \in V$. Then, since $y \in Z$, it must be the case that $y \notin \mathrm{uvars}(t)$ and therefore $B^i_1 R B^{i'}_1$ by definition of R.

It remains the case $Q = Q_1 \cup Q_2 \cup Q_3$ with $Q_1 \in \mathbf{rel}(T, x)^*$, $Q_2 \in \mathbf{rel}(T, Y)^*$ and $Q_3 \in \mathbf{rel}(T, Z)^*$. However, this is a trivial corollary of the previous two cases, since we know that $Q_1 \cup Q_3$ is θ-connected and $Q_2 \subseteq U_2$. $\square$

Fixed $X \in S$, our aim is to provide a substitution δ with $\alpha_{\mathrm{Sh}}([\delta]_{U_1}) \sqsubseteq [S_1, U_1]$ and $\alpha_{\mathrm{Sh}}(\mathrm{mgu}([\delta]_{U_1}, [\theta]_U)) \sqsupseteq [\{X\}, U]$. By Lemma 6.17, $X_1 = B_1 \cup \ldots \cup B_n$ with $B_i \in S_1$ and $B_i R^* B_j$ for each $i, j \leq n$. We let $K_1 = \{B_1, \ldots, B_n\}$. We now want to define a substitution δ such that $\alpha_{\mathrm{Sh}}([\delta]_{U_1}) = [K_1, U_1]$. For each sharing group $B \in K_1$, let us consider a fresh variable w_B. Let $W = \{w_B \mid B \in K_1\}$. For each variable x, let $B_x = \{B^1_x, \ldots, B^k_x\}$ be the set $\mathbf{rel}(K_1, x)$. Let N be the maximum cardinality of all the B_x for $x \in X_1$ i.e., $N = \max_{x \in X_1} |B_x|$. For each $x \in X_1$, we define two terms:

$$s_x = t(\underbrace{c(w_{B^1_x}, w_{B^1_x}), c(w_{B^2_x}, w_{B^2_x}), \ldots, c(w_{B^k_x}, w_{B^k_x})}_{k = |B_x| \text{ times}}, \underbrace{c(w_{B^1_x}, w_{B^1_x}), \ldots, c(w_{B^1_x}, w_{B^1_x})}_{N - |B_x| \text{ times}}) \tag{38}$$

$$s'_x = t(\underbrace{c(w_{B^1_x}, w_{B^2_x}), c(w_{B^2_x}, w_{B^3_x}), \ldots, c(w_{B^k_x}, w_{B^1_x})}_{k = |B_x| \text{ times}}, \underbrace{c(w_{B^1_x}, w_{B^1_x}), \ldots, c(w_{B^1_x}, w_{B^1_x})}_{N - |B_x| \text{ times}}) \tag{39}$$

Note that if $N = 0$ then $X_1 = \emptyset$ and s_x, s'_x are undefined for any variable x.

We introduce the following notation: given a term t we distinguish different occurrences of the same variable by calling (y, n) the n-th occurrence of a variable y in t, where the order is lexicographic. For instance, a term $f(x, g(y, y, x))$ can be seen as the term $f((x, 1), f((y, 1), (y, 2), (x, 2)))$. For each $y \in \mathrm{vars}(\theta(U_1)) \cap X$, we choose a variable $x_y \in U_1$ such that $y \in \theta(x_y)$. Let a be a constant. We are now ready to define the substitution δ in the following way: for each variable $x \in U_1$, $\delta(x)$ is the same as $\theta(x)$ with the difference that each occurrence (y, i) of a variable $y \in \theta(x)$ is replaced by $t_{x,y,i}$ defined as

—$t_{x,y,i} = a$ if $y \notin X$,

—$t_{x,y,i} = s_x$ otherwise, if $x = x_y$ and $i = 1$;

—$t_{x,y,i} = s'_x$ otherwise.

Note that, by Corollary 6.16, if $x \in X_1$, then $\theta(x)$ is not ground. Therefore, by construction, $\mathrm{dom}(\delta) = U_1$ and $\mathrm{rng}(\delta) = W$. It is easy to check that $\alpha_{\mathrm{Sh}}([\delta]_{U_1}) = [K_1, U_1]$ since given a variable w_B, it appears in $\delta(x)$ iff $x \in B$ and therefore $occ(\delta, w_B) \cap U_1 = B$. For all the other variables $occ(\delta, v) = \emptyset$ if $v \in U_1$ and $occ(\delta, v) = \{v\} \nsubseteq U_1$ otherwise. Let us compute the value of $\mathrm{mgu}([\delta]_{U_1}, [\theta]_U)$.

LEMMA 6.18.

$$\mathrm{mgu}(\delta,\theta) = \mathrm{mgu}\{w_1 = w_2 \mid w_1, w_2 \in W\} \circ \rho \circ \theta$$

where $\rho = \{v/s_{x_v} \mid v \in \mathrm{vars}(\theta(U_1)) \cap X\} \cup \{v/a \mid v \in \mathrm{vars}(\theta(U_1)) \setminus X\}$.

PROOF. Since $t_{x_v,v,1} = s_{x_v}$, by using the properties of equation sets it follows that:

$$\begin{aligned}\mathrm{mgu}(\delta,\theta) &= \mathrm{mgu}(\{v = t_{x,v,i} \mid x \in U_1, (v,i) \text{ is an occurrence of } v \text{ in } \theta(x)\}\} \circ \theta \\ &= \mathrm{mgu}(E) \circ \rho \circ \theta \ .\end{aligned}$$

where $E = \{t_{x_v,v,1} = t_{x',v,j} \mid x' \in U_1, (v,j) \text{ is an occurrence of } v \text{ in } \theta(x')\}$. Let us define a relation between variables:

$$vR'u \iff \exists y \in \mathrm{vars}(\theta(v)) \cap X.\ u = x_y \wedge (u = v \Rightarrow y \notin \mathrm{uvars}(\theta(v)))\} \ .$$

Note that R' is not a symmetric relationship. By exploiting the above definition, we can rewrite $\mathrm{mgu}(E)$ as follows:

$$\mathrm{mgu}(E) = \mathrm{mgu}(\{s'_v = s_u \mid v,u \in X_1, vR'u\}) \ . \tag{40}$$

The above characterization shows that $\mathrm{Eq}(\delta) \cup \mathrm{Eq}(\theta)$ is solvable, since s_u and s'_v are terms which unify by construction. Moreover, note that

$$\mathrm{mgu}\{s_u = s'_v\} = \mathrm{mgu}\{w_B = w_{B'} \mid B \in B_u \wedge B' \in B_v\} \ . \tag{41}$$

We want to prove that $\mathrm{mgu}\{s'_v = s_u \mid v,u \in X_1, vR'u\} = \mathrm{mgu}\{w_1 = w_2 \mid w_1, w_2 \in W\}$. It is obvious that $\mathrm{mgu}\{s'_v = s_u \mid v,u \in X_1, vR'u\} = \mathrm{mgu}\{w_B = w_{B'} \mid x,y \in X_1.\ B \in B_x, B' \in B_y, xR'y\} = \mathrm{mgu}\{w_B = w_{B'} \mid B\hat{R}B'\}$ where $\hat{R}$ is the relation on $K_1 \times K_1$ given by

$$B\hat{R}B' \iff \exists x,y \in X_1.\ B \in B_x \wedge B' \in B_y \wedge xR'y \ . \tag{42}$$

By the properties of equality, we know that

$$\mathrm{mgu}\{w_B = w_{B'} \mid B\hat{R}B'\} = \mathrm{mgu}\{w_B = w_{B'} \mid B\hat{R}^*B'\} \ . \tag{43}$$

We now prove that $\hat{R} \subseteq R \subseteq \hat{R}^*$, from which the thesis follows by Lemma 6.17.

If $B\hat{R}B'$ there are $x,y \in X_1$ s.t. $B \in B_x \wedge B' \in B_y \wedge xR'y$. However $B \in B_x$ iff $x \in B \in S_1$ and $B' \in B_y$ iff $y \in B' \in S_1$. Now, assume $z \in \mathrm{vars}(\theta(x)) \cap X$ and $y = x_z$. Then $z \in \mathrm{vars}(\theta(x)) \cap \mathrm{vars}(\theta(y)) \cap X$ and this proves that BRB'. On the other side, assume BRB', i.e., there are $x \in B$, $y \in B'$, $z \in \mathrm{vars}(\theta(x)) \cap \mathrm{vars}(\theta(y)) \cap X$ s.t. $x = y \implies z \notin \mathrm{uvars}(\theta(x))$. Since $x \in B$ and $y \in B'$, then $B \in B_x$ and $B' \in B_y$. Since $z \in \mathrm{vars}(\theta(U_1)) \cap X$ then x_z is defined and $B_{x_z} \neq \emptyset$. Assume that $x = y = x_z$. Then $z \in \mathrm{uvars}(\theta(x))$ and thus $xR'y$ and $B\hat{R}B'$. Otherwise, we may assume without loss of generality that $x \neq x_z$. If $y = x_z$ then $xR'y$ and thus $B\hat{R}B'$. If $y \neq x_z$ we can choose any $B'' \in B_{x_z}$. We know that $xR'x_z$, $yR'x_z$ and thus it holds that $B\hat{R}B''$ and $B'\hat{R}B''$, from which $B\hat{R}^*B'$. The case $y \neq x_z$ is symmetric. □

We can now prove that $\alpha_{\mathrm{Sh}}(\mathrm{mgu}([\delta]_{U_1}, [\theta]_U)) \sqsupseteq_{Sh} [\{X\}, U]$. This immediately implies that $\alpha_{Sh}(\mathsf{unif}_{\mathrm{Ps}}([\{[\delta]\}, U_1], \theta)) \sqsupseteq_{Sh} [\{X\}, U]$, which concludes the optimality proof.

Proposition 6.19.

$$\alpha_{\mathrm{Sh}}(\mathrm{mgu}([\delta]_{U_1}, [\theta]_U)) \sqsupseteq_{Sh} [\{X\}, U] \enspace .$$

Proof. First of all, note that $\mathrm{mgu}([\delta]_{U_1}, [\theta]_U) = [\mathrm{mgu}(\delta, \theta)]_U$ since $\mathrm{vars}(\theta) \subseteq U$. We proceed with two different proofs when $W = \emptyset$ and $W \neq \emptyset$. If $W \neq \emptyset$ then, according to Lemma 6.18, we can choose $\bar{w} \in W$ and define the substitution $\sigma = \{w'/\bar{w} \mid \bar{w} \neq w' \in W\} = \mathrm{mgu}(E)$. It only remains to prove that $occ(\sigma \circ \rho \circ \theta, \bar{w}) \cap U = X$.

It follows easily that $occ(\sigma \circ \rho \circ \theta, \bar{w}) = occ(\rho \circ \theta, W) = occ(\theta, \mathrm{vars}(\theta(U_1)) \cap X) \cup W) = occ(\theta, \mathrm{vars}(\theta(U_1)) \cap X) \cup W$. Since $U \cap W = \emptyset$ it follows that $occ(\sigma \circ \rho \circ \theta, \bar{w}) \cap U = occ(\theta, \mathrm{vars}(\theta(U_1)) \cap X)$.

By definition, $occ(\theta, \mathrm{vars}(\theta(U_1)) \cap X) = \{y \mid \mathrm{vars}(\theta(y)) \cap \mathrm{vars}(\theta(U_1)) \cap X \neq \emptyset\}$. Thus, for any of such y, we have that $\mathrm{vars}(\theta(y)) \cap X \neq \emptyset$ and thus, by Corollary 6.16, $y \in X$. It follows that $occ(\theta, \mathrm{vars}(\theta(U_1)) \cap X) \subseteq X$. For the opposite direction, by Lemma 6.15 there exist $B_1, \ldots, B_k \in H$ such that $\cup B_i = X$ and $B_i \cap U_1 \neq \emptyset$ for each i. Since $B_i \in H$, then there exists v s.t. $B_i = occ(\theta, v)$. Moreover, $v \in X$ since $v \in B_i$ by definition of occ and $\theta(v) = v$. Since $B_i \cap U_1 \neq \emptyset$ it follows that there exists $y \in B_i \cap U_1$ such that $v \in \theta(y) \subseteq \theta(U_1)$ and thus $B_i \subseteq occ(\theta, \mathrm{vars}(\theta(U_1)) \cap X)$. Thus $X \subseteq occ(\theta, \mathrm{vars}(\theta(U_1)) \cap X)$.

When $W = \emptyset$, $\mathrm{mgu}(E) = \epsilon$ and $X = X_2$. In this case, by Lemma 6.15, $X_2 = occ(\theta, x)$ for some $x \in U_2$. Since $X_2 \cap U_1 = \emptyset$, then $x \notin \mathrm{vars}(\theta(U_1))$, i.e., $x \notin \mathrm{dom}(\rho)$ and therefore $occ(\rho \circ \theta, x) = occ(\theta, x) = X_2$. ☐

The optimality result for unif_{Sh} w.r.t. $\mathsf{unif}_{\mathrm{Ps}}$ immediately follows from the above proposition.

Theorem 6.20. (Optimality of unif_{Sh}) unif_{Sh} *is optimal w.r.t.* $\mathsf{unif}_{\mathrm{Ps}}$ *(when there is at least a constant symbol and a symbol of arity equal or greater than two).*

Optimality of unif_{Sh} also implies the following corollary:

Corollary 6.21. *Under the same condition of Theorem 6.20, the result of* unif_{Sh} *does not depend from the order of the bindings in its second argument.*

Note that, in this proof, we worked with a signature endowed with a constant a and term symbols c and t of arity two and N respectively. Actually, it is evident that the proof may be easily rewritten as soon as the signature has a constant and a symbol of arity at least two. Given s of arity n, we may replace in δ a term $t(t_1, \ldots, t_N)$ with $c(t_1, c(t_2, c(\ldots, t_N)))$. Later, we replace $c(t_1, t_2)$ with $s(t_1, t_2, a, a, \ldots, a)$ where a is repeated $n-2$ times.

However, if we do not have a term s with arity equal or greater than two, unif_{Sh} is not optimal. A counterexample is given by the following unification:

$$\mathsf{unif}_{Sh}([\{\mathtt{xy}, \mathtt{xz}\}, \{x, y, z\}], \{y/z\}) = [\{\mathtt{xyz}\}, \{x, y, z\}] \enspace . \tag{44}$$

Observe that, for any substitution θ in the concretization of $[\{\mathtt{xy}, \mathtt{xz}\}, \{x, y, z\}]$, it is the case that either y is ground or z is ground. It is impossible for both variables to be non ground, since it would imply that $\theta(x)$ has at least two different variables, corresponding to the two sharing groups $\mathtt{xy}$ and $\mathtt{xz}$. Obviously, this is not possible if we only have constants and unary term symbols. Therefore, after unifying θ with

y/z, all variables become ground, i.e.,

$$\alpha_{\mathrm{Sh}}(\mathsf{unif}_{\mathrm{Ps}}(\gamma_{\mathrm{Sh}}([\{\mathtt{xy},\mathtt{xz}\},\{x,y,z\}]),\{y/z\})) = [\emptyset,\{x,y,z\}] \ . \tag{45}$$

The latter equation also holds if we do not have constant symbols, since in this case $\gamma_{\mathrm{Sh}}([\{\mathtt{xy},\mathtt{xz}\},\{x,y,z\}]) = [\emptyset,\{x,y,z\}]$.

Finally, in the case when we have symbols of arity two or greater but no constant symbols, we believe that our operator unif_{Sh} is still optimal. However the proof needs to be refined, since we have to replace the constant a in the definition of $t_{x,y,i}$ with some well chosen variable, in order to satisfy the property $\alpha_{\mathrm{Sh}}([\delta]_U) \sqsubseteq [S,U]$. We have not sorted out all the details, since we do not think the problem is particularly relevant.

6.4 Summing Up

We may put together all the results of correctness, optimality and completeness shown so far to prove the main theorem of this section:

THEOREM 6.22. $\mathbf{U}^f_{Sh}$ *is well defined, correct and optimal w.r.t.* $\mathbf{U}^f_{\mathrm{Ps}}$ *(under the same hypothesis of Theorem 6.20)*

PROOF. By Equation (35), we need to prove that:

$$\begin{aligned}&\pi_{Sh}(\mathsf{unif}_{Sh}(\rho([S_1,U_1],\mathrm{mgu}(\rho(A_1)=A_2)),\mathrm{vars}(A_2)) = \\ &\qquad \alpha_{\mathrm{Sh}}(\pi_{\mathrm{Ps}}(\mathsf{unif}_{\mathrm{Ps}}(\rho(\gamma_{\mathrm{Ps}}([S_1,U_1]),\mathrm{mgu}(\rho(A_1)=A_2)),\mathrm{vars}(A_2)))) \ .\end{aligned}$$

By Theorems 5.3 and 5.4, we know that π_{Sh} is correct and complete and that abstract renaming is correct and γ-complete. Moreover, by Theorem 6.20, abstract unification unif_{Sh} is optimal. We have the following equalities.

$$\begin{aligned}&\alpha_{\mathrm{Sh}}(\pi_{\mathrm{Ps}}(\mathsf{unif}_{\mathrm{Ps}}(\rho(\gamma_{\mathrm{Ps}}([S_1,U_1])),\mathrm{mgu}(\rho(A_1)=A_2)),\mathrm{vars}(A_2)))) \\ =\ &\pi_{Sh}(\alpha_{\mathrm{Sh}}(\mathsf{unif}_{\mathrm{Ps}}(\rho(\gamma_{\mathrm{Ps}}([S_1,U_1])),\mathrm{mgu}(\rho(A_1)=A_2)),\mathrm{vars}(A_2)))) && \text{[by Th. 5.3]} \\ =\ &\pi_{Sh}(\alpha_{\mathrm{Sh}}(\mathsf{unif}_{\mathrm{Ps}}(\gamma_{\mathrm{Ps}}(\rho([S_1,U_1])),\mathrm{mgu}(\rho(A_1)=A_2))),\mathrm{vars}(A_2)))) && \text{[by Th. 5.4]} \\ =\ &\pi_{Sh}(\mathsf{unif}_{Sh}(\rho([S_1,U_1]),\mathrm{mgu}(\rho(A_1)=A_2))),\mathrm{vars}(A_2)))) \ . && \text{[by Th. 6.20]}\end{aligned}$$

Thus $\mathbf{U}^f_{Sh}$ is correct and optimal w.r.t. $\mathbf{U}^f_{\mathrm{Ps}}$. The fact that it is well defined (i.e., it does not depend on the choice of the renaming ρ) is a direct consequence of optimality. ☐

Generally speaking, in order to obtain optimality, it is always a better choice to abstract a concrete operator "as a whole", instead of abstracting each component and then composing the abstract operators. According to this rule, we could think that a better approximation may be reached by abstracting $\mathbf{U}^f_{\mathrm{Ps}}$ as a whole. However, since abstract projection/renaming is complete and γ-complete, this does not happen, as shown by the previous theorem. Studying the direct abstraction of this composition would still be useful to find a direct implementation which is more efficient than computing unif_{Sh} and projecting later, but we do not consider this problem here.

Since $\mathbf{U}^f_{Sh}$ generates less sharing groups then $\mathbf{U'}^f_{Sh}$ and since checking whether a variable is in U is easy, we can expect an improvement in the efficiency of the analysis by replacing $\mathbf{U'}^f_{Sh}$ with $\mathbf{U}^f_{Sh}$ in the computation of the entry substitution. If computing Y and Z at each step of $\mathbf{u}^f_{Sh}$ seems difficult, it is always possible to precompute these values before the actual analysis begins, since they depend on the syntax of the program only. Moreover, in the definition of $\mathbf{u}^f_{Sh}$, when $x \in U$ we know that $\mathbf{rel}(S,x) = \{\{x\}\}$, since θ is an idempotent substitution and $x \notin U_1$.

We said before that this operator introduces new optimizations which, up to our knowledge, are not used even in more complex domains for sharing analysis which include linearity and freeness information. We give here one example which shows their effects.

Example 6.23. Let us consider the following unification.

$$\mathbf{U}^f_{Sh}([\{\mathtt{xw},\mathtt{xz},\mathtt{yw},\mathtt{yz}\},\{x,y,w,z\}],p(x,y,w,z),p(f(u,h),f(u,k),s,t)) \ . \tag{46}$$

By applying the optimizations suggested from the unification algorithm in presence of linearity and freeness information in [Hans and Winkler 1992], we may start from the abstract object $S = \{\mathtt{xw},\mathtt{xz},\mathtt{yw},\mathtt{yz},\mathtt{u},\mathtt{h},\mathtt{k},\mathtt{s},\mathtt{t}\}$ and process the bindings one at a time, keeping in mind that u,h,k,s,t are initially free. This means that in the binding $x/f(u,h)$, the term $f(u,h)$ is linear, and therefore we can avoid to compute the star union in $\mathbf{rel}(S,x)$, thus obtaining:

$$\begin{aligned}&\{\mathtt{k},\mathtt{s},\mathtt{t},\mathtt{yw},\mathtt{yz}\} \cup \mathbf{bin}(\{\mathtt{xw},\mathtt{xz}\},\{\mathtt{u},\mathtt{h},\mathtt{uh}\}) = \\ &\qquad\qquad \{\mathtt{k},\mathtt{s},\mathtt{t},\mathtt{yw},\mathtt{yz},\mathtt{xwu},\mathtt{xwh},\mathtt{xzu},\mathtt{xzh},\mathtt{xwuh},\mathtt{xzuh}\} \ .\end{aligned}$$

However, after this unification, the variable u can be bound to a non-linear term. Therefore, when we consider the next binding $y/f(u,k)$, according to [Hans and Winkler 1992], we are forced to compute all the star unions, obtaining:

$$\{\mathtt{s},\mathtt{t}\} \cup \mathbf{bin}(\{\mathtt{yw},\mathtt{yz}\}^*,(\{\mathtt{k}\} \cup \mathbf{bin}(\{\mathtt{xw},\mathtt{xz}\},\{\mathtt{u},\mathtt{uh}\}))^*) \cup \{\mathtt{xwh},\mathtt{xzh}\} \ .$$

Finally, in the bindings w/s and z/t we may omit all the star unions since t and s are free, and we get the final result:

$$\mathbf{bin}(\{\mathtt{yws},\mathtt{yzt}\}^*,(\{\mathtt{k}\} \cup \mathbf{bin}(\{\mathtt{xws},\mathtt{xzt}\},\{\mathtt{u},\mathtt{uh}\}))^*) \cup \{\mathtt{xwsh},\mathtt{xzth}\} \ .$$

Observe that we obtain the sharing group ywsztk, and thus, after projecting over $\{u,h,k,s,t\}$, we obtain the sharing group stk. However, when we consider the second binding, we know that k is free and independent from y, and this is enough to apply a new optimization. In fact, k can share with more than one sharing group related to y only if k shares with u. If we compute the abstract unification with our algorithm we obtain:

$$\begin{aligned}&\{\mathtt{ywsk},\mathtt{yztk}\} \cup \mathbf{bin}(\{\mathtt{yws},\mathtt{yzt}\}^*,\mathbf{bin}(\{\mathtt{xws},\mathtt{xzt}\},\{\mathtt{u},\mathtt{uh}\})^*) \\ &\quad \cup \mathbf{bin}(\mathbf{bin}(\{\mathtt{yws},\mathtt{yzt}\}^*,\mathbf{bin}(\{\mathtt{xws},\mathtt{xzt}\},\{\mathtt{u},\mathtt{uh}\})^*),\{k\}) \cup \{\mathtt{xwsh},\mathtt{xzth}\}\end{aligned} \tag{47}$$

and when we project over $\{u,h,k,s,t\}$, the sharing group stk does not appear. The result does not change by permuting the order of the bindings. If we consider the binding $y/f(u,k)$ before $x/f(u,h)$, with the standard operators we get:

$$\mathbf{bin}(\{\mathtt{xws},\mathtt{xzt}\}^*,(\{\mathtt{h}\} \cup \mathbf{bin}(\{\mathtt{yws},\mathtt{yzt}\},\{\mathtt{u},\mathtt{uk}\}))^*) \cup \{\mathtt{ywsk},\mathtt{yztk}\}$$

and, when we project over $\{u,h,k,s,t\}$, we obtain the sharing group sth, which does not appear in our result.

7. MATCHING AND BACKWARD UNIFICATION

Up to our knowledge, in all the collecting denotational semantics for logic programs, backward unification is performed by using unification instead of matching. In this case, instead of $\mathbf{U}^b_{\mathrm{Ps}}$, the concrete semantics uses a backward unification operator which unifies two concrete objects in `Psub` with a substitution:

$$\mathbf{U'}^b_{\mathrm{Ps}}([\Theta_1, U_1], [\Theta_2, U_2], A_1, A_2) = \\ \pi_{\mathrm{Ps}}(\mathsf{unif}''_{\mathrm{Ps}}(\rho([\Theta_1, U_1]), [\Theta_2, U_2], \mathrm{mgu}(\rho(A_1) = A_2)), U_2 \cup \mathrm{vars}(A_2)) \ , \quad (48)$$

where ρ is a renaming such that $\rho(U_1 \cup \mathrm{vars}(A_1)) \cap (U_2 \cup \mathrm{vars}(A_2)) = \emptyset$ and

$$\mathsf{unif}''_{\mathrm{Ps}}([\Theta_1, U_1], [\Theta_2, U_2], \delta) = \\ [\{\mathrm{mgu}([\theta_1]_{U_1}, [\theta_2]_{U_2}, [\delta]_{\mathrm{vars}(\delta)}) \mid [\theta_1]_{U_1} \in \Theta_1, [\theta_2]_{U_2} \in \Theta_2\}, U_1 \cup U_2] \quad (49)$$

is simply the pointwise extension of mgu over `Psub`. It is worth noting that $\mathsf{unif}''_{\mathrm{Ps}}(\rho([\Theta_1, U_1]), [\Theta_2, U_2], \delta)$ is a very specific kind of unification, since $\rho(U_1)$ and U_2 are disjoint. The optimal abstract operator $\mathbf{U'}^b_{Sh}$ w.r.t. $\mathbf{U'}^b_{\mathrm{Ps}}$ is very similar to that proposed in [Cortesi and Filè 1999] (see Section 8.2 for further details), and it is given by:

$$\mathbf{U'}^b_{Sh}([S_1, U_1], [S_2, U_2], A_1, A_2) = \\ \pi_{Sh}(\mathsf{unif}_{Sh}([\rho(S_1) \cup S_2, \rho(U_1) \cup U_2], \mathrm{mgu}(\rho(A_1) = A_2)), U_2 \cup \mathrm{vars}(A_2)) \ . \quad (50)$$

As said before, this choice gives a loss of precision already at the concrete level, which leads to a loss of precision in the abstract counterpart. When we compute $\mathbf{U'}^b_{\mathrm{Ps}}([\Theta_1, U_1], [\Theta_2, U_2], A_1, A_2)$, we essentially unify all pairs θ_1 and θ_2, elements of Θ_1 and Θ_2, with $\delta = \mathrm{mgu}(A_1 = A_2)$ (assuming we do not need renamings). However, it could be possible to consider only the pairs in which θ_1 is an instance of $\mathrm{mgu}(\theta_2, \delta)$ w.r.t. the variables of interest in $U_1 \cap U_2$. If this does not hold, then θ_1 cannot be a success substitution corresponding to the call substitution θ_2, and therefore we are unifying two objects which pertain to different computational paths, with an obvious loss of precision, already at the concrete level. This problem has been pointed out by Marriott et al. [1994, Section 5.5].

We now want to define the optimal abstract operator $\mathbf{U}^b_{Sh}$ corresponding to $\mathbf{U}^b_{\mathrm{Ps}}$. This is accomplished by composing the forward unification operator unif_{Sh} with a new operator match_{Sh}, which is the abstract counterpart of $\mathsf{match}_{\mathrm{Ps}}$.

Definition 7.1. Given $[S_1, U_1], [S_2, U_2] \in$ `Sharing`, we define

$$\mathsf{match}_{Sh}([S_1, U_1], [S_2, U_2]) = \\ [S'_1 \cup S'_2 \cup \{X_1 \cup X_2 \mid X_1 \in S''_1, X_2 \in (S''_2)^*, X_1 \cap U_2 = X_2 \cap U_1\}, U_1 \cup U_2]$$

where $S'_1 = \{B \in S_1 \mid B \cap U_2 = \emptyset\}$ and $S''_1 = S_1 \setminus S'_1$, $S'_2 = \{B \in S_2 \mid B \cap U_1 = \emptyset\}$ and $S''_2 = S_2 \setminus S'_2$

The idea is that we may freely combine sharing groups in S_2 in case they have some variable in common with U_1 and the projection of the result is a sharing group in S_1. This means that new aliasings between variables may arise in the concrete counterpart of S_2 (the entry substitution), as long as they do not affect the variables of the exit substitution.

Definition 7.2. The abstract backward unification may be defined as

$$\mathbf{U}^b_{Sh}([S_1,U_1],[S_2,U_2],A_1,A_2) = \pi_{Sh}(\mathsf{match}_{Sh}(\rho([S_1,U_1]), \\ \mathsf{unif}_{Sh}([S_2,U_2],\mathrm{mgu}(\rho(A_1)=A_2))), U_2 \cup \mathrm{vars}(A_2)) \ . \quad (51)$$

where ρ is a renaming such that $\rho(U_1 \cup \mathrm{vars}(A_1)) \cap (U_2 \cup \mathrm{vars}(A_2)) = \emptyset$.

Example 7.3. Let $U_1 = \{u,v,w\}$, $U_2 = \{x,y,z\}$, $\Sigma_1 = \{[\{v/t(u,w,w)\}]_{U_1}, [\{v/t(u,u,w)\}]_{U_1}\}$. $\Sigma_2 = \{[\{y/t(x,z,z)\}]_{U_2}, [\{y/t(x,x,z)\}]_{U_2}\}$ and $\rho = id$. We have

$$\mathbf{U'}^b_{\mathrm{Ps}}([\Sigma_1,U_1],[\Sigma_2,U_2],p(u,v,w),p(x,y,z)) = \pi_{\mathrm{Ps}}([\Sigma, U_1 \cup U_2], U_2) \ , \quad (52)$$

with $[\theta]_{U_1\cup U_2} = [\{y/t(x,x,x), z/x, u/x, v/t(x,x,x), w/x\}]_{U_1\cup U_2} \in \Sigma$. Let $[S_1,U_1] = \alpha_{\mathrm{Sh}}([\Sigma_1,U_1])$, $[S_2,U_2] = \alpha_{\mathrm{Sh}}([\Sigma_2,U_2])$, $S_1 = \{\mathtt{uv},\mathtt{vw}\}$ and $S_2 = \{\mathtt{xy},\mathtt{yz}\}$. We obtain

$$\mathbf{U'}^b_{Sh}([S_1,U_1],[S_2,U_2],p(u,v,w),p(x,y,z)) = \pi_{Sh}([S, U_1 \cup U_2], U_2) \ , \quad (53)$$

and $\mathtt{xyzuvw} \in S$. So, it seems that u, v and w may share a common variable. Note that θ is obtained by unifying $\sigma_2 = \{y/t(x,z,z)\}$ with $\sigma_1 = \{v/t(u,u,w)\}$ but $\sigma_1(v) = t(u,u,w)$ is not an instance of $\mathrm{mgu}(\sigma_2, \mathrm{mgu}(p(x,y,z) = p(u,v,w)))(v) = t(x,z,z)$. Therefore, σ_1 and σ_2 do pertain to different computational paths. Using the backward unification with matching, we obtain

$$\mathbf{U}^b_{\mathrm{Ps}}([\Sigma_1,U_1],[\Sigma_2,U_2],p(u,v,w),p(x,y,z)) = \\ \pi_{\mathrm{Ps}}([\{[y/t(x,z,z), u/x, v/t(x,z,z), w/z], [y/t(x,x,z), u/x, v/t(x,x,z), w/z]\}, \\ \{x,y,z,u,v,w\}], \{u,v,w\}) \ , \quad (54)$$

which does not contain θ. In the abstract domain, we have:

$$\mathbf{U}^b_{Sh}([S_1,U_1],[S_2,U_2],p(u,v,w),p(x,y,z)) = \\ \pi_{Sh}([\{\mathtt{xyuv},\mathtt{yzvw}\}, U_1 \cup U_2], U_2) \ . \quad (55)$$

After the unification we know that x and z are independent. Note that, although based on the notion of abstract matching, the operators defined in [King and Longley 1995; Hans and Winkler 1992] cannot establish this property. The algorithm in [Muthukumar and Hermenegildo 1992] computes the same result of ours in this particular example, but since their matching is partially performed by first projecting the sharing information on the term positions of the calling atom and of the clause head, this does not hold in general. For example, their algorithm states that x and z may possibly share when the unification is performed between the calling atom $p(t(x,y,z))$ and the head $p(t(u,v,w))$, where t is a function symbol, p a unary predicate and the call substitution is the same as before.

7.1 Correctness

We can prove that $\mathbf{U}^b_{Sh}$ is actually the best correct abstraction of the backward concrete unification $\mathbf{U}^b_{\mathrm{Ps}}$. To prove correctness we only need to show the match_{Sh} is correct w.r.t. $\mathsf{match}_{\mathrm{Ps}}$. Correctness of $\mathbf{U}^b_{Sh}$ will follow from the fact that $\mathbf{U}^b_{Sh}$ is a composition of correct abstract operators.

THEOREM 7.4. (CORRECTNESS OF match_{Sh}) match_{Sh} *is correct w.r.t.* $\mathsf{match}_{\mathrm{Ps}}$.

PROOF. Consider $[\Sigma_i, U_i] \sqsubseteq_{\mathrm{Ps}} \gamma_{Sh}([S_i, U_i])$ for $i \in \{1,2\}$, $v \in \mathcal{V}$ and $[\sigma]_{U_1 \cup U_2} \in \mathsf{match}_{\mathrm{Ps}}([\Sigma_1, U_1], [\Sigma_2, U_2])$. We need to prove that

$$\alpha_{\mathrm{Sh}}([\sigma]_{U_1 \cup U_2}) \in \mathsf{match}_{Sh}([S_1, U_1], [S_2, U_2]) \ . \tag{56}$$

Assume $[\sigma] = \mathrm{mgu}([\sigma_1], [\sigma_2])$ with $[\sigma_1] \in \Sigma_1$ and $[\sigma_2] \in \Sigma_2$. Let σ_1 and σ_2 be two canonical representatives for $[\sigma_1]$ and $[\sigma_2]$ such that $\mathrm{vars}(\sigma_1) \cap \mathrm{vars}(\sigma_2) = U_1 \cap U_2$. If $\sigma_1 \preceq_{U_1 \cap U_2} \sigma_2$, there exists $\delta \in \mathit{Subst}$ such that $\sigma_1(x) = \delta(\sigma_2(x))$ for each $x \in U_1 \cap U_2$. We may assume, without loss of generality, that $\mathrm{dom}(\delta) = \mathrm{vars}(\sigma_2(U_1 \cap U_2))$. Now, the following equalities hold.

$$\begin{aligned}
\sigma =& \mathrm{mgu}(\mathrm{Eq}(\sigma_2), \mathrm{Eq}(\sigma_1)) \\
=& \mathrm{mgu}(\{\sigma_2(x) = \sigma_2(\sigma_1(x)) \mid x \in U_1\}) \circ \sigma_2 \\
=& \mathrm{mgu}(\{x = \sigma_1(x) \mid x \in U_1 \setminus U_2\} \cup \{\sigma_1(x) = \sigma_2(x) \mid x \in U_1 \cap U_2\}) \circ \sigma_2 \\
& \quad [\text{by partitioning } \mathrm{dom}(\sigma_2), \text{ since } \sigma_2(\sigma_1(x)) = \sigma_1(x) \text{ for } x \in U_1] \\
=& \mathrm{mgu}(\{x = \sigma_1(x) \mid x \in U_1 \setminus U_2\}) \circ \delta \circ \sigma_2 \\
& \quad [\text{since } \sigma_1(x) = \delta(\sigma_2(x)) \text{ and } \mathrm{dom}(\delta) = \mathrm{vars}(\sigma_2(U_1 \cap U_2))] \\
=& \sigma_{1|U_1 \setminus U_2} \circ \delta \circ \sigma_2 \\
=& \sigma_{1|U_1 \setminus U_2} \uplus (\delta \circ \sigma_2) \ .
\end{aligned} \tag{57}$$

Now, given a variable v, by Lemma 6.5, $\mathit{occ}(\sigma, v) \cap (U_1 \cup U_2) = (\mathit{occ}(\sigma_{1|U_1 \setminus U_2}, v) \cap U_1) \cup (\mathit{occ}(\sigma_2, \mathit{occ}(\delta, v)) \cap U_2)$. We want to prove that $\mathit{occ}(\sigma, v) \cap (U_1 \cup U_2) \in \mathsf{match}_{Sh}([S_1, U_1], [S_2, U_2])$.

Since $\mathrm{dom}(\sigma) = U_1 \cup U_2$, we may assume that $v \notin U_1 \cup U_2$, otherwise $\mathit{occ}(\sigma, v) \cap (U_1 \cup U_2) = \emptyset$. We distinguish two cases:

— $v \notin \mathrm{rng}(\delta)$, which implies $v \notin \mathrm{rng}(\sigma_{1|U_2})$. Note that, if $v \in \mathrm{dom}(\delta)$ then $\mathit{occ}(\sigma_2, \mathit{occ}(\delta, v)) = \emptyset \in S'_2$, otherwise $\mathit{occ}(\sigma_2, \mathit{occ}(\delta, v)) = \mathit{occ}(\sigma_2, v) \in S'_2$. So, it always holds that $\mathit{occ}(\sigma_2, \mathit{occ}(\delta, v)) \in S'_2$. We now distinguish some subcases. If $v \in \mathrm{rng}(\sigma_1)$ then $\mathit{occ}(\sigma_{1|U_1 \setminus U_2}, v) = \mathit{occ}(\sigma_1, v)$. Moreover, since $v \in \mathrm{rng}(\sigma_1)$, then $v \notin \mathrm{vars}(\sigma_2)$ and thus $\mathit{occ}(\sigma_2, v) = \{v\}$. We have that $\mathit{occ}(\sigma, v) \cap (U_1 \cup U_2) = \mathit{occ}(\sigma_1, v) \in S'_1$. Otherwise, if $v \in \mathrm{rng}(\sigma_2)$, then $v \notin \mathrm{vars}(\sigma_1)$ and $\mathit{occ}(\sigma_1, v) = \{v\}$. Therefore $\mathit{occ}(\sigma, v) \cap (U_1 \cup U_2) = \mathit{occ}(\sigma_2, \mathit{occ}(\delta, v)) \in S'_2$. Otherwise, if $v \notin \mathrm{rng}(\sigma_1) \cup \mathrm{rng}(\sigma_2)$ then $\mathit{occ}(\sigma, v) \cap (U_1 \cup U_2) = \emptyset$.

— $v \in \mathrm{rng}(\delta)$. We want to prove that $\mathit{occ}(\sigma, v) = X_1 \cup X_2$ where $X_1 = \mathit{occ}(\sigma_1, v)$ and $X_2 = \mathit{occ}(\sigma_2, \mathit{occ}(\delta, v))$ enjoy the following properties: $X_1 \in S''_1$, $X_2 \in S''^{*}_2$, $X_1 \cap U_2 = X_2 \cap U_1$. First of all, note that $\mathit{occ}(\sigma_{1|U_1 \setminus U_2}, v) \cap U_1 = X_1 \setminus U_2$. Moreover, $\mathit{occ}(\sigma_2, \mathit{occ}(\delta, v)) \cap U_1 = \mathit{occ}(\sigma_{2|U_1}, \mathit{occ}(\delta, v)) \cap U_1$, which in turn is equal to $\mathit{occ}(\delta \circ \sigma_{2|U_1}, v) \cap U_1 = \mathit{occ}(\sigma_{1|U_2}, v) \cap U_1 = \mathit{occ}(\sigma_1, v) \cap U_1 \cap U_2 \supseteq X_1 \cap U_2$. This proves that $\mathit{occ}(\sigma, v) = X_1 \cup X_2$ and $X_1 \cap U_2 = X_2 \cap U_1$.
While it is obvious that $X_1 \in S_1$ and $X_2 \in S^{*}_2$, we still need to prove that $X_1 \in S''_1$ and $X_2 \in S''^{*}_2$. For each $y \in \mathit{occ}(\delta, v)$, by definition of δ we have that $y \in \sigma_2(U_1 \cap U_2)$ and therefore $\mathit{occ}(\sigma_2, y) \cap U_1 \neq \emptyset$. This proves that $X_2 \in S''^{*}_2$. Moreover, if $v \in \mathrm{rng}(\delta)$ then $v \in \mathrm{rng}(\sigma_{1|U_2})$ and therefore $\mathit{occ}(\sigma_1, v) \in S''_1$. $\square$

7.2 Optimality

In order to prove an optimality result for $\mathbf{U}^b_{Sh}$, we first need to prove optimality for match_{Sh}. However this is not enough, since, generally speaking, composition of optimal operator may fail to be optimal. Actually, we will need to establish for match_{Sh} a stronger result than optimality. We show that, under suitable hypotheses, match_{Sh} is complete in a weak way, which is enough for our goal.

THEOREM 7.5. *Assume we have a signature with at least a constant symbol a. Then the operator match_{Sh} is optimal on the first argument and complete on the second one when $\mathsf{match}_{\mathrm{Ps}}$ is restricted to the case when the second argument contains a single substitution only. In formulas:*

$$\mathsf{match}_{Sh}([S_1, U_1], \alpha_{\mathrm{Sh}}([\{\sigma_2\}, U_2])) = \alpha_{Sh}(\mathsf{match}_{\mathrm{Ps}}(\gamma_{Sh}([S_1, U_1]), [\{[\sigma_2]\}, U_2])) \ .$$

for each $[\{[\sigma_2]\}, U_2] \in$ `Psub` and $[S_1, U_1] \in$ `Sharing`.

PROOF. Since match_{Sh} is correct w.r.t. $\mathsf{match}_{\mathrm{Ps}}$, it follows that:

$$\alpha_{Sh}(\mathsf{match}_{\mathrm{Ps}}(\gamma_{Sh}([S_1, U_1]), [\{[\sigma_2]\}, U_2])) \sqsubseteq_{Sh} \mathsf{match}_{Sh}([S_1, U_1], \alpha_{\mathrm{Sh}}([\{[\sigma_2]\}, U_2])).$$

So, we only need to prove that:

$$\mathsf{match}_{Sh}([S_1, U_1], \alpha_{Sh}([\{[\sigma_2]\}, U_2])) \sqsubseteq_{Sh} \alpha_{Sh}(\mathsf{match}_{\mathrm{Ps}}(\gamma_{Sh}([S_1, U_1]), [\{[\sigma_2]\}, U_2])) \ . \tag{58}$$

Assume, without loss of generality, that σ_2 is a canonical representative of $[\sigma_2]_{U_2}$ and $\mathrm{rng}(\sigma_2) \cap U_1 = \emptyset$. Take $B \in S$, where $[S, U_1 \cup U_2] = \mathsf{match}_{Sh}([S_1, U_1], [S_2, U_2])$, with $[S_2, U_2] = \alpha_{Sh}([\{[\sigma_2]\}, U_2])$. We have three cases.

—If $B \in S_1'$ then $B \in S_1$ and $B \subseteq U_1 \setminus U_2$. Let $\delta = \{x/v \mid x \in B\} \cup \{x/a \mid x \in \mathrm{vars}(\sigma_2(U_1 \setminus B))\}$ and $\sigma_1 = (\delta \circ \sigma_2)_{|U_1}$ where v is a fresh variable. It follows that $\mathrm{dom}(\sigma_1) = U_1$ and $\mathrm{rng}(\sigma_1) = \{v\}$ with $occ(\sigma_1, v) = B$, therefore $[\sigma_1, U_1] \sqsubseteq_{\mathrm{Ps}} \gamma_{Sh}([S_1, U_1])$. Clearly $\sigma_1 \preceq_{U_1 \cap U_2} \sigma_2$ since $U_1 \cap U_2 \subseteq U_1 \setminus B$. Let $\sigma = \mathrm{mgu}(\sigma_1, \sigma_2)$. Since $B \cap \mathrm{dom}(\sigma_2) = \emptyset$ and v is a fresh variable, it follows that $occ(\sigma, v) = B$, and thus $B \in \alpha_{Sh}(\mathsf{match}_{\mathrm{Ps}}(\gamma_{Sh}([S_1, U_1]), [\{[\sigma_2]\}, U_2]))$.

—If $B \in S_2'$, there exists $v \in \mathcal{V}$ such that $occ(\sigma_2, v) \cap U_2 = B$. Let $X = \mathrm{vars}(\sigma_2(U_1)))$ and take $\delta = \{x/a \mid x \in X\}$. Then $\sigma_1 = (\delta \circ \sigma_2)_{|U_1}$ is such that $occ(\sigma_1, v) \cap U_1 = \emptyset$ for each $v \in \mathcal{V}$, therefore $\sigma_1 \in \gamma_{Sh}([S_1, U_1])$. Moreover $\mathrm{mgu}(\sigma_2, \sigma_1) \in \mathsf{match}_{\mathrm{Ps}}(\gamma_{Sh}([S_1, U_1]), [\{[\sigma_2]\}, U_2])$. By the proof of Theorem 7.4, Equation (57), we have $\mathrm{mgu}(\sigma_1, \sigma_2) = \delta \circ \sigma_2$. Since $B \cap U_1 = \emptyset$, then $v \notin X = \mathrm{vars}(\delta)$, and therefore $occ(\delta \circ \sigma_2, v) \cap U_2 = occ(\sigma_2, v) \cap U_2 = B$. Hence $B \in \alpha_{Sh}(\mathsf{match}_{\mathrm{Ps}}(\gamma_{Sh}([S_1, U_1]), [\{[\sigma_2]\}, U_2]))$.

—We now assume $B = X_1 \cup \bigcup X$ with $X \subseteq S_2''$, $X_1 \in S_1''$, $\bigcup X \cap U_1 = X_1 \cap U_2$. Then, for each $H \in X$, there exists $v_H \in \mathcal{V}$ such that $occ(\sigma_2, v_H) \cap U_2 = H$. Since $H \cap U_1 \neq \emptyset$ for each $H \in X$, then $v_H \in Y = \mathrm{vars}(\sigma_2(U_1))$. Consider the substitution

$$\delta = \{v_H/v \mid H \in X\} \uplus \{w/a \mid w \in Y, \forall H \in X. w \neq v_H\}$$

for a fresh variable v and

$$\sigma_1 = (\delta \circ \sigma_2)_{|U_1} \uplus \{x/v \mid x \in X_1 \setminus U_2\} \ .$$

We want to prove $[\{\sigma_1\}, U_1] \in \gamma_{Sh}([S_1, U_1])$. By definition of σ_1 we have that $occ(\sigma_1, v) \cap U_1 = (occ(\sigma_2, \{v_H \mid H \in X\}) \cap U_1) \cup X_1 \setminus U_2 = (\bigcup X \cap U_1) \cup X_1 \setminus U_2 = X_1 \in S_1$. Otherwise, for $w \neq v$ we have that either $occ(\sigma_1, w) = \emptyset$ when $w \in U_1$ or $occ(\sigma_1, w) = occ(\sigma_2, w)$ disjoint from U_1. In both cases, $occ(\sigma_1, w) \cap U_1 =$

$\emptyset \in S_1$. By definition of σ_1, $[\text{mgu}(\sigma_1, \sigma_2)] \in \mathsf{match}_{\text{Ps}}(\gamma_{Sh}([S_1, U_1]), [\{[\sigma_2]\}, U_2])$. Moreover, we know from (57) that

$$\text{mgu}(\sigma_2, \sigma_1) = \delta \circ \sigma_2 \uplus \{x/v \mid x \in X_1 \setminus U_2\} \ .$$

Let $\sigma = \text{mgu}(\sigma_1, \sigma_2)$. Note that $occ(\sigma, v) \cap (U_1 \cup U_2) = X_1 \setminus U_2 \cup occ(\sigma_2, \{v_H \mid H \in X\}) \cap U_2$. By definition of v_H, $occ(\sigma_2, v_H) \cap U_2 = H$, hence $occ(\sigma, v) \cap (U_1 \cup U_2) = (X_1 \setminus U_2) \cup \bigcup X = X_1 \cup \bigcup X = B$.

This proves the theorem. □

Corollary 7.6. (Optimality of match_{Sh}) match_{Sh} *is optimal (when there is at least a constant symbol).*

Proof. Given $[S_1, U_1], [S_2, U_2] \in \texttt{Sharing}$, we have

$$\begin{aligned}
&\alpha_{\text{Sh}}(\mathsf{match}_{\text{Ps}}(\gamma_{\text{Sh}}([S_1, U_1]), \gamma_{\text{Sh}}([S_2, U_2])))\\
&= \alpha_{\text{Sh}}(\sqcup_{\text{Ps}}\{\mathsf{match}_{\text{Ps}}(\gamma_{\text{Sh}}([S_1, U_1]), [\{[\sigma]\}, U_2]) \mid \alpha_{\text{Sh}}([\sigma]_{U_2}) \sqsubseteq_{Sh} [S_2, U_2]\})\\
&\qquad [\text{since } \mathsf{match}_{\text{Ps}} \text{ is additive}]\\
&= \sqcup_{Sh} \{\mathsf{match}_{Sh}([S_1, U_1], [X, U_2]) \mid X = \alpha_{\text{Sh}}([\sigma]_{U_2}) \sqsubseteq_{Sh} [S_2, U_2]\}\\
&\qquad [\text{by completeness of } \sqcup_{Sh} \text{ and Theorem 7.5}]\\
&= \mathsf{match}_{Sh}([S_1, U_1], \sqcup_{Sh}\{[X, U_2] \mid X = \alpha_{\text{Sh}}([\sigma]_{U_2}) \sqsubseteq_{Sh} [S_2, U_2]\}) \ .\\
&\qquad [\text{since } \mathsf{match}_{Sh} \text{ is additive}]
\end{aligned}$$

Since α_{Sh} is a Galois insertion, it is surjective, and therefore $\sqcup_{Sh}\{[X, U_2] \mid X = \alpha_{\text{Sh}}([\sigma]_{U_2}) \sqsubseteq_{Sh} [S_2, U_2]\} = [S_2, U_2]$ and we obtain

$$\alpha_{\text{Sh}}(\mathsf{match}_{\text{Ps}}(\gamma_{\text{Sh}}([S_1, U_1]), \gamma_{\text{Sh}}([S_2, U_2]))) = \mathsf{match}_{Sh}([S_1, U_1], [S_2, U_2]) \ ,$$

which concludes the proof. □

7.3 Summing Up

The particular property enjoyed by match_{Sh} and proved in Theorem 7.5 may be used to prove that $\mathbf{U}^b_{Sh}$ is optimal w.r.t. $\mathbf{U}^b_{\text{Ps}}$. However, we first need to prove another property concerning unif_{Sh}.

Lemma 7.7. *Assume we have a constant symbol and a term symbol of arity two or greater. Given* $[S_1, U_1] \in \texttt{Sharing}$ *and* $\theta \in \mathit{ISubst}$*, there exists a substitution* $\delta \in \mathit{ISubst}$ *such that* $\alpha_{\text{Sh}}([\delta]_{U_1}) \sqsubseteq_{Sh} [S_1, U_1]$ *and*

$$\alpha_{\text{Sh}}(\mathsf{unif}_{\text{Ps}}([\{[\delta]\}, U_1], \theta)) = \mathsf{unif}_{Sh}([S_1, U_1], \theta) \ . \tag{59}$$

Proof. The optimality result proved in Theorem 6.20 shows that there exists $[\Sigma_1, U_1] \sqsubseteq_{\text{Ps}} \gamma_{\text{Sh}}([S_1, U_1])$ such that $\alpha_{\text{Sh}}(\mathsf{unif}_{\text{Ps}}([\Sigma_1, U_1], \theta)) = \mathsf{unif}_{Sh}([S_1, U_1], \delta)$. We need a stronger result which proves that Σ_1 can be chosen a singleton.

Assume $\mathsf{unif}_{Sh}([S_1, U_1], \theta) = [S, U_1 \cup U_2]$ where $U_2 = \text{vars}(\theta) \setminus U_1$ and $S = \{X^1, \ldots, X^n\}$. Following the construction in Section 6.3, for each X^i let us define X^i_1, X^i_2, K^i, K^i_1, K^i_2, W^i, s^i_x, ${s'}^i_x$, U as in the proof of optimality for unif_{Sh}. We choose W^i, W^j such that $W^i \cap W^j = \emptyset$ if $i \neq j$ and we denote by w^i_B the elements of W^i.

For each $y \in \text{vars}(\theta(U_1)) \cap (\cup_{1 \leq i \leq n} X^i)$, we choose a variable $x_y \in U_1$ such that $y \in \theta(x_y)$. Then, we define the substitution δ in the following way: for each variables $x \in U_1$, $\delta(x)$ is the same as $\theta(x)$, with the exception that each occurrence (y, j) of a variable $y \in \theta(x)$ is replaced by $t_{x,y,j} = t(t^1_{x,y,j}, \ldots, t^n_{x,y,j})$, where:

—$t^i_{x,y,j} = a$ if $y \notin X^i$,

—$t^i_{x,y,j} = s^i_x$ otherwise, if $x = x_y$ and $j = 1$;

—$t^i_{x,y,j} = {s'}^i_x$ otherwise.

By construction $\mathrm{dom}(\delta) = U_1$ and $\mathrm{rng}(\delta) = \bigcup_{1 \leq i \leq n} W^i$. It is easy to check that $\alpha_{Sh}([\{\delta\}, U_1]) = [\bigcup_{1 \leq i \leq n} K^i_1, U_1] \sqsubseteq_{Sh} [S_1, U_1]$. Using the properties of the equation sets we can prove that

$$\begin{aligned}
& \mathrm{mgu}(\delta, \theta) \\
= {} & \mathrm{mgu}(\{v = t_{x,v,j} \mid x \in U_1, (v, j) \text{ is an occurrence of } v \text{ in } \theta(x)\}) \circ \theta \\
= {} & \mathrm{mgu}(E) \circ \rho \circ \theta \ ,
\end{aligned}$$

where

$$\rho = \{v/t_{x_v,v,1} \mid v \in \mathrm{vars}(\theta(U_1))\} \ , \tag{60}$$

$$\begin{aligned} E = \{t^i_{x_v,v,1} = t^i_{x',v,j} \mid i \in \{1, \ldots, n\}, v \in X^i, x' \in U_1, \\ (v, j) \text{ is an occurrence of } v \text{ in } \theta(x')\} \ . \end{aligned} \tag{61}$$

Now, each $E^i = \{t^i_{x_v,v,1} = t^i_{x',v,j} \mid x' \in U_1, (v, j)$ is an occurrence of v in $\theta(x'), v \in X^i\}$ is the same equation which appears in (40) for $X = X^i$. Therefore, for each $i \in \{1, \ldots, n\}$ such that $W^i \neq \emptyset$, we choose a single $w^i \in W^i$ and define η^i with $\mathrm{dom}(\eta^i) = W^i \setminus \{w^i\}$ and $\eta^i(w^i_B) = w^i$ for each $w^i_B \in W^i$. If $W^i = \emptyset$, we choose $\eta^i = \epsilon$. We know from the proof of Theorem 6.20 that $\eta^i = \mathrm{mgu}(E^i)$, and $\mathrm{mgu}(E) = \eta = \biguplus_{1 \leq i \leq n} \eta^i$ since $\mathrm{vars}(E^i) \cap \mathrm{vars}(E^j) = \emptyset$ for $i \neq j$. Therefore

$$\mathrm{mgu}(\delta, \theta) = \eta \circ \rho \circ \theta \ . \tag{62}$$

We now want to prove that $\alpha_{\mathrm{Sh}}([\eta \circ \rho \circ \theta]_{U_1 \cup U_2}) \sqsupseteq_{\mathrm{Ps}} [\{X^i\}, U_1 \cup U_2]$ for each $i \in \{1, \ldots, n\}$. If $X^i_1 \neq \emptyset$ then $W^i \neq \emptyset$, and we have $occ(\eta \circ \rho \circ \theta, w^i) = occ(\eta^i \circ \rho \circ \theta, w^i)$. Following the proof of Lemma 6.18 with $X = X^i$, we have that $occ(\eta \circ \rho \circ \theta, w^i) \cap U = X^i$. When $X^i_1 = \emptyset$, we may choose $v^i \in \theta(X^i_2)$. In this case, $occ(\eta \circ \rho \circ \theta, v^i) \cap U = occ(\theta, v^i) \cap U = X^i$ as proved in Proposition 6.19. □

As in the proof of Theorem 6.20, we have proved the previous theorem by assuming that we have term symbols for each arity. However, it is possible to rewrite terms so that a constant symbol and a binary (or more) term symbol suffice.

THEOREM 7.8. $\mathbf{U}^b_{Sh}$ *is correct and optimal w.r.t.* $\mathbf{U}^b_{\mathrm{Ps}}$ *(under the same hypothesis of Theorem 6.20).*

PROOF. Correctness immediately follows by the fact that $\mathbf{U}^b_{\mathrm{Ps}}$ is obtained by tupling and composition of correct semantic functions.

By using Theorem 7.5 and Lemma 7.7, it is possible to prove that

$$\begin{aligned} & \mathsf{match}_{Sh}([S_1, U_1], \mathsf{unif}_{Sh}([S_2, U_2], \theta)) = \\ & \qquad \alpha_{\mathrm{Sh}}(\mathsf{match}_{\mathrm{Ps}}(\gamma_{\mathrm{Sh}}([S_1, U_1]), \mathsf{unif}_{Sh}(\gamma_{\mathrm{Sh}}([S_2, U_2]), \theta))) \ , \end{aligned} \tag{63}$$

i.e., that the composition of match_{Sh} and unif_{Sh}, as used in $\mathbf{U}^b_{Sh}$, is optimal.

Assume given $[S_1, U_1]$ and $[S_2, U_2] \in \texttt{Psub}$ and $\theta \in \mathit{ISubst}$. Consider $[\{[\sigma]\}, U_2] \in \gamma_{\mathrm{Sh}}([S_2, U_2])$ obtained by Lemma 7.7 such that $\mathsf{unif}_{\mathrm{Ps}}([\{[\sigma]\}, U_2]), \theta) = [\{[\delta]\}, U_2 \cup \mathrm{vars}(\theta)]$ and $\alpha_{\mathrm{Sh}}([\{[\delta]\}, U_2 \cup \mathrm{vars}(\theta)]) = \mathsf{unif}_{\mathit{Sh}}([S_2, U_2], \theta)$. Then, we have

$$\begin{aligned} &\mathsf{match}_{\mathit{Sh}}([S_1, U_1], \mathsf{unif}_{\mathit{Sh}}([S_2, U_2], \theta)) \\ =&\mathsf{match}_{\mathit{Sh}}([S_1, U_1], \alpha_{\mathrm{Sh}}(\mathsf{unif}_{\mathrm{Ps}}([\{[\sigma]\}, U_2], \theta)) \\ =&\alpha_{\mathrm{Sh}}(\mathsf{match}_{\mathrm{Ps}}(\gamma_{\mathrm{Sh}}([S_1, U_1]), \mathsf{unif}_{\mathrm{Ps}}([\{\sigma\}, U_1], U_2, \theta))) \end{aligned} \tag{64}$$

by Theorem 7.5, so that, in general

$$\begin{aligned} &\mathsf{match}_{\mathit{Sh}}([S_1, U_1], \mathsf{unif}_{\mathit{Sh}}([S_2, U_2], \theta)) \sqsubseteq_{\mathit{Sh}} \\ &\qquad \alpha_{\mathrm{Sh}}(\mathsf{match}_{\mathrm{Ps}}(\gamma_{\mathrm{Sh}}([S_1, U_1]), \mathsf{unif}_{\mathrm{Ps}}(\gamma_{\mathrm{Sh}}([S_2, U_2]), \theta))) \ . \end{aligned} \tag{65}$$

The proof that $\mathbf{U}^b_{\mathrm{Ps}}$ is optimal follows from this result, completeness of π_{Sh} and γ-completeness of ρ. $\square$

To the best of our knowledge, this is the first abstract matching operator which is optimal for the corresponding concrete operator. Note that, in several paper such as [King and Longley 1995; Hans and Winkler 1992], abstract matching is defined without defining concrete matching.

We now give an example of a program where the use of $\mathbf{U}^f_{\mathit{Sh}}$ and $\mathbf{U}^b_{\mathit{Sh}}$ gives better results than the standard operators $\mathbf{U}'^f_{\mathit{Sh}}$ and $\mathbf{U}'^b_{\mathit{Sh}}$.

Example 7.9. Consider the trivial program with just one clause `p(u,v,w)` $\leftarrow$ and the goal $p(x, y, z)$ with call substitution $\{\mathtt{xy}, \mathtt{yz}\}$. Using our abstract operators, we obtain the entry substitution $\{\mathtt{uv}, \mathtt{vw}\}$ and the success substitution $\{\mathtt{xy}, \mathtt{yz}\}$ (see Ex. 6.2 and 7.3), thus proving that x and z are independent. If we replace either $\mathbf{U}^b_{\mathit{Sh}}$ or $\mathbf{U}^f_{\mathit{Sh}}$ with $\mathbf{U}'^f_{\mathit{Sh}}$ or $\mathbf{U}'^b_{\mathit{Sh}}$, then the success substitution will contain the sharing group `xyz`. In fact, as shown in Ex. 6.2, the entry substitution in the latter case would be $[\{\mathtt{uv}, \mathtt{vw}, \mathtt{uvw}\}, \{u, v, w\}]$. If we compute the success substitution we obtain:

$$\begin{aligned} &\mathbf{U}'^b_{\mathit{Sh}}([\{\mathtt{uv}, \mathtt{vw}, \mathtt{uvw}\}, \{u, v, w\}], [\{\mathtt{xy}, \mathtt{yz}\}, \{x, y, z\}], p(u, v, w), p(x, y, z)), \\ &\qquad\qquad\qquad\qquad \{x, y, z\}) = [\{\mathtt{xy}, \mathtt{yz}, \mathtt{xyz}\}, \{x, y, z\}] \end{aligned}$$

which contains the sharing group `xyz`.

It is worth noting that the improvement in the previous example is obtained with a program in *head normal form.* Usually, when programs are in head normal form, the forward and backward unification may be replaced by renamings, which are complete and do not cause any loss in precision. However, there is the need of an unification operator for the explicit constraints which appear in the body of the clauses. In general, the analyses we obtain in our framework are more precise than those which can be obtained by using the standard domain `Sharing` by translating the same program in head normal form.

Example 7.10. Consider again Ex. 7.9 and the program `p(u,f(s),w)` $\leftarrow$ which is not in head normal form. Using our abstract operators, we obtain the success substitution $\{\mathtt{xy}, \mathtt{yz}\}$, as in Ex. 7.9. If we normalize the program, we obtain the clause `p(u,v,w)` $\leftarrow$ `v=f(s)`. The entry substitution obtained from $\{\mathtt{xy}, \mathtt{yz}\}$ by simply renaming the variables x, y, z in u, v, w and introducing the new variable s

is $\{\mathtt{uv}, \mathtt{vw}, \mathtt{s}\}$. By using the standard operator for unification, when applying the binding $v/f(s)$ we obtain $\{\mathtt{uvs}, \mathtt{vws}, \mathtt{uvws}\}$, and thus the success substitution will contain the sharing group $\mathtt{xyz}$, resulting in a loss of precision.

It is worth noting that it would be possible to use our improved forward abstract unification in a normalized program by enlarging the set of variables of interest only when new variables are effectively met, instead of adding all the variables which appear in the body of a clause once for all when the entry substitution is computed. In the example above, the variable s can be introduced when unifying the abstract object $\{\mathtt{uv}, \mathtt{vw}\}$ with $v/f(s)$. Since $\mathsf{unif}_{Sh}([\{\mathtt{uv}, \mathtt{vw}\}, \{u, v, w\}], \{v/f(s)\}) = [\{\mathtt{uvs}, \mathtt{vws}\}, \{u, v, w, s\}]$, we still obtain as success substitution $\{\mathtt{xy}, \mathtt{yz}\}$, thus proving that x and z are independent.

8. RELATED WORKS

8.1 Relationship with *ESubst*

The domain *ESubst* proposed by Jacobs and Langen [1992] uses a non standard definition of substitution. We may prove that *ESubst* is isomorphic to $\mathit{ISubst}_\sim$. This formalizes the intuition, which has never been proved before, that working with *ESubst* is essentially like working with substitutions.

We now briefly recall the definition of the domain *ESubst*. For the sake of clarity, in the following, we call E-substitution the nonstandard substitution defined in [Jacobs and Langen 1992]. An E-substitution σ is a mapping from a finite set of variables $\mathrm{dom}(\sigma) \subseteq \mathcal{V}$ to Terms. This approach differs from the standard definition of substitutions, which are mappings from $\mathcal{V}$ to Terms that are almost everywhere the identity. The preorder on E-substitutions is defined as follows:

$$\sigma \leq_E \theta \iff \mathrm{dom}(\theta) \subseteq \mathrm{dom}(\sigma) \wedge \big(\forall t \in \mathsf{Terms}.\ \mathrm{vars}(t) \subseteq \mathrm{dom}(\theta) \Rightarrow \\ \exists \delta \text{ an E-substitution s.t. } \sigma t = \delta(\theta(t))\big)\ , \quad (66)$$

where the application of an E-substitution to a term is defined as usual.

Let $\sim_E$ be the equivalence relation on E-substitutions induced by $\leq_E$. The domain *ESubst* is defined as the set of equivalence classes of E-substitutions w.r.t. $\sim_E$, that is $\mathit{ESubst} = \{[\sigma]_{\sim_E} \mid \sigma \text{ is an E-substitution}\}$. The next theorem shows that *ESubst* is isomorphic to $\mathit{ISubst}_\sim$.

THEOREM 8.1. *ESubst and $\mathit{Subst}_\sim$ are isomorphic posets (if there is at least a term symbol of arity strictly greater than one).*

PROOF. To each E-substitution θ we may associate a substitution θ' such that $\theta'(x) = \theta(x)$ if $x \in \mathrm{dom}(\theta)$ and $\theta'(x) = x$ otherwise. Note that, for each term t, $\theta(t) = \theta'(t)$: an E-substitution and the corresponding standard substitution behave in the same way on terms.

We may prove that, if $\theta_1 \leq_E \theta_2$, then $\theta'_1 \preceq_{\mathrm{dom}(\theta_2)} \theta'_2$. By definition, if $\theta_1 \leq_E \theta_2$ then $\mathrm{dom}(\theta_2) \subseteq \mathrm{dom}(\theta_1)$ and $\forall t \in \mathsf{Terms}$ with $\mathrm{vars}(t) \subseteq \mathrm{dom}(\theta_2)$, there exists an E-substitution δ such that $\theta_1(t) = \delta(\theta_2(t))$. Let $\mathrm{dom}(\theta_2) = \{x_1, \ldots, x_n\}$ and consider a term t such that $\mathrm{vars}(t) = \{x_1, \ldots, x_n\}$ (note that t exists iff there is at least a term symbol of arity strictly greater than 1). By definition, there exists an E-substitution δ such that $\theta_1(t) = \delta(\theta_2(t))$, that is, for any $v \in \mathrm{dom}(\theta_2)$ it holds $\theta_1(v) = \delta(\theta_2(v))$. This means that $\theta'_1(v) = \delta'(\theta'_2(v))$ and therefore $\theta'_1 \preceq_{\mathrm{dom}(\theta_2)} \theta'_2$.

On the converse, for each $\theta \in \mathit{Subst}$ and $U \in \wp_f(\mathcal{V})$, we associate a corresponding E-substitution θ^{*U} such that $\mathrm{dom}(\theta^{*U}) = U$ and $\theta^{*U}(v) = \theta(v)$ for each $v \in U$. As for the previous case, we have that if $\theta_1 \preceq_U \theta_2$, then $\theta_1^{*U} \leq_E \theta_2^{*U}$. First of all, note that $\mathrm{dom}(\theta_1^{*U}) = U = \mathrm{dom}(\theta_2^{*U})$. Moreover, by definition of $\preceq_U$, there is $\delta \in \mathit{Subst}$ such that $\theta_1(v) = \delta(\theta_2(v))$ for each $v \in U$. Now, given a term t such that $\mathrm{vars}(t) \subseteq U$, we may check that $\theta_1^{*U}(t) = \delta^{*\mathrm{vars}(\theta_2(U))}(\theta_2^{*U}(t))$ and this proves $\theta_1^{*U} \leq_E \theta_2^{*U}$.

Now, we may lift these operations to equivalence classes to obtain the function $\iota : \mathit{ESubst} \rightarrow \mathit{Subst}_{\sim}$ such that

$$\iota([\theta]_{\sim_E}) = [\theta']_{\mathrm{dom}(\theta)} \ . \tag{67}$$

The map ι is well defined: if $\theta_1 \sim_E \theta_2$ then $\mathrm{dom}(\theta_1) = \mathrm{dom}(\theta_2)$ and, by the above property, $\theta_1' \sim_{\mathrm{dom}(\theta_2)} \theta_2'$. Moreover, there is an inverse ι^{-1} given by

$$\iota^{-1}([\theta]_U) = [\theta^{*U}]_{\sim_E} \ . \tag{68}$$

It is easy to check that ι^{-1} is well defined: if $\theta_1 \preceq_U \theta_2$, then $\theta_1^{*U} \leq_E \theta_2^{*U}$.

It is immediate to check, given the properties above, that ι and ι^{-1} are one the inverse of the other. Moreover, they are both monotonic. If $[\theta_1]_E \leq_E [\theta_2]_E$ then $\mathrm{dom}(\theta_2) \subseteq \mathrm{dom}(\theta_1)$ and $\theta_1' \preceq_{\mathrm{dom}(\theta_2)} \theta_2'$, i.e., $\iota([\theta_1]_{\sim_E}) = [\theta_1']_{\mathrm{dom}(\theta_1)} \preceq [\theta_2']_{\mathrm{dom}(\theta_2)} = \iota([\theta_2]_{\sim_E})$. On the converse, if $[\theta_1]_U \preceq [\theta_2]_V$ then $[\theta_1]_V \preceq [\theta_2]_V$ and therefore $\iota^{-1}([\theta_1]_V) \leq_E \iota^{-1}([\theta_2]_V)$. We only need to prove that $\iota^{-1}([\theta_1]_U) \leq_E \iota^{-1}([\theta_1])_V$. This follows from that fact that, given a term t with $\mathrm{vars}(t) \subseteq V$, $\theta_1^{*U}(t) = \theta_1^{*V}(t)$. $\square$

It is worth noting that the most general unifier as defined in [Jacobs and Langen 1992] corresponds to mgu in $\mathit{ISubst}_{\sim}$. In formulas, given term t_1 and t_2, we have that

$$\iota([\mathrm{mgu}(t_1, t_2)]_{\sim_E}) = [\mathrm{mgu}(\{t_1 = t_2\})]_{\mathrm{vars}(t_1 = t_2)} \ , \tag{69}$$

where mgu on the left is the operator in Definition 1 of [Jacobs and Langen 1992] and $\iota : \mathit{ESubst} \rightarrow \mathit{ISubst}_{\sim}$ is the isomorphism defined in the proof of Theorem 8.1. Up to our knowledge, this is the first proof of the relationship between the mgu in a domain of existential substitutions and the standard mgu for substitutions. Moreover, it is worth noting that by adding a bottom element to $\mathit{ISubst}_{\sim}$ and *ESubst*, they turn out to be isomorphic complete lattices.

8.2 A Case Study

In Section 3 we said that, in order to define a good collecting semantics for correct answer substitutions, there are several possible directions. We may work with a domain of existentially quantified substitutions like $\mathit{Subst}_{\sim}$, or we may work with standard substitutions, being careful to keep enough representatives for each equivalence class. We have already discussed the benefits of using equivalence classes. In order to show the kind of problems which arise from the use of domains of substitutions, without any equivalence relation, we want to show a small flaw of the semantic framework defined in [Cortesi and Filè 1999] for the analysis of sharing, and widely used in several other works on program analysis such as [Bagnara et al. 2002; Zaffanella 2001].

The framework is based upon the domain $\mathtt{Rsub} = (\wp(\mathit{Subst}) \times \wp_f(\mathcal{V})) \cup \{\top_{\mathrm{Rs}}, \bot_{\mathrm{Rs}}\}$ which is a complete lattice, partially ordered as follows: $\top_{\mathrm{Rs}}$ is the top element, $\bot_{\mathrm{Rs}}$ is the bottom element and $[\Sigma_1, U_1] \sqsubseteq_{\mathrm{Rs}} [\Sigma_2, U_2]$ if and only if $U_1 = U_2$ and $\Sigma_1 \subseteq \Sigma_2$. An object $[\Sigma, U]$ is a set of substitution Σ where the set of variables of interest U is explicitly provided.

The main operation in Rsub is the concrete unification $\mathbf{U}_{Rs} : \mathtt{Rsub} \times \mathtt{Rsub} \times \mathit{ISubst} \to \mathtt{Rsub}$ such that:

$$\begin{aligned}
\mathbf{U}_{Rs}(\bot_{\mathrm{Rs}}, \xi, \delta) =& \mathbf{U}_{Rs}(\xi, \bot_{\mathrm{Rs}}, \delta) = \bot_{\mathrm{Rs}} \\
\mathbf{U}_{Rs}(\xi, \top_{\mathrm{Rs}}, \delta) =& \mathbf{U}_{Rs}(\top_{\mathrm{Rs}}, \xi, \delta) = \top_{\mathrm{Rs}} \qquad \text{if } \xi \neq \bot_{\mathrm{Rs}} \\
\mathbf{U}_{Rs}([\Sigma_1, U_1], [\Sigma_2, U_2], \delta) =& [\{\mathrm{mgu}(\sigma_1, \sigma_2, \delta) \mid \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2, \\
& \mathrm{vars}(\sigma_1) \cap \mathrm{vars}(\sigma_2) = \emptyset\}, U_1 \cup U_2] \ .
\end{aligned} \tag{70}$$

Although it is well defined for all the values of the domain, $\mathbf{U}_{Rs}([\Sigma_1, U_1], [\Sigma_2, U_2], \delta)$ may be restricted to those values such that $U_1 \cap U_2 = \emptyset$ and $\mathrm{vars}(\delta) \subseteq U_1 \cup U_2$, since this the only way $\mathbf{U}_{Rs}$ is used in the semantics defined in [Cortesi and Filè 1999].

The abstract domain is the same Sharing we use in our paper, with abstraction map $\alpha_{\mathrm{Sh}} : \mathtt{Rsub} \to \mathtt{Sharing}$ and unification $\mathbf{U}_{Sh} : \mathtt{Sharing} \times \mathtt{Sharing} \times \mathit{ISubst} \to \mathtt{Sharing}$ defined by:

$$\alpha_{\mathrm{Sh}}([\Sigma, U]) = \bigsqcup\nolimits_{Sh} \{\alpha_{\mathrm{Sh}}([\sigma]_U) \mid \sigma \in \Sigma\} \ , \tag{71}$$

$$\mathbf{U}_{Sh}([\Sigma_1, U_1], [\Sigma_2, U_2], \delta) = \mathsf{unif}_{Sh}([\Sigma_1 \cup \Sigma_2, U_1 \cup U_2], \delta) \tag{72}$$

The domain of $\mathbf{U}_{Sh}$ is restricted to the case $U_1 \cap U_2 = \emptyset$ and $\mathrm{vars}(\delta) \subseteq U_1 \cup U_2$.

By looking at the paper, we think that, in the idea of the authors, $[\Sigma, U] \in \mathtt{Rsub}$ should have been treated as $[\{[\sigma]_U \mid \sigma \in \Sigma\}, U] \in \mathtt{Psub}$ is in our framework. However, the condition $\mathrm{vars}(\sigma_1) \cap \mathrm{vars}(\sigma_2) = \emptyset$, introduced in $\mathbf{U}_{Rs}$ in order to avoid variable clashes between the two chosen substitutions, is not enough for this purpose. Actually, $\mathbf{U}_{Rs}$ only checks that σ_1 and σ_2 do not have variables in common, without considering their sets of variables of reference U_1 and U_2. This unification can lead to counterintuitive results.

Example 8.2. Consider the following concrete unification:

$$\mathbf{U}_{Rs}([\{\{x/y\}\}, \{x\}], [\{\epsilon\}, \{y\}], \epsilon) = [\{\{x/y\}\}, \{x, y\}] \ . \tag{73}$$

Being $\mathrm{vars}(\epsilon) = \emptyset$, the concrete unification operator allows us to unify $\{x/y\}$ with ϵ without renaming the variable y, which is not a variable of interest in the first element but it is treated as if it was. This also causes the incorrectness of $\mathbf{U}_{Sh}$. If we consider Eq. (73) and compute the result on the abstract side by using the abstract unification operator $\mathbf{U}_{Sh}$, we have:

$$\begin{aligned}
& \mathbf{U}_{Sh}(\ \alpha_{Sh}([\{\{x/y\}\}, \{x\}]),\ \alpha_{Sh}([\{\epsilon\}, \{y\}]),\ \epsilon) \\
=\ & \mathbf{U}_{Sh}(\qquad [\{\mathtt{x}\}, \{x\}], \qquad\qquad [\{\mathtt{y}\}, \{y\}], \qquad \epsilon) = [\{\mathtt{x}, \mathtt{y}\}, \{x, y\}] \ .
\end{aligned} \tag{74}$$

This is not a correct approximation of the concrete result, since:

$$\alpha_{Sh}([\{\{x/y\}\}, \{x, y\}]) = [\{\mathtt{xy}\}, \{x, y\}] \not\sqsubseteq_{Sh} [\{\mathtt{x}, \mathtt{y}\}, \{x, y\}] \ . \tag{75}$$

This counterexample proves that the abstract unification operator $\mathbf{U}_{Sh}$ is not correct w.r.t. the concrete one $\mathbf{U}_{Rs}$, invalidating the Theorem 6.3 in [Cortesi and

Filè 1999]. The problem can be solved by introducing a stronger check on variable clashes, namely by replacing the condition $\mathrm{vars}(\sigma_1) \cap \mathrm{vars}(\sigma_2) = \emptyset$ with $(\mathrm{vars}(\sigma_1) \cup U_1) \cap (\mathrm{vars}(\sigma_2) \cup U_2) = \emptyset$ in the definition of $\mathbf{U}_{Rs}$, thus obtaining the following operator:

$$\begin{aligned}\mathbf{U}^*_{Rs}([\Sigma_1, U_1], [\Sigma_2, U_2], \delta) = [\{&\mathrm{mgu}(\sigma_1, \sigma_2, \delta) \mid \sigma_1 \in \Sigma_1, \sigma_2 \in \Sigma_2, \\ &(\mathrm{vars}(\sigma_1) \cup U_1) \cap (\mathrm{vars}(\sigma_2) \cup U_2) = \emptyset\}, U_1 \cup U_2] \ . \quad (76)\end{aligned}$$

By using $\mathbf{U}^*_{Rs}$ instead of $\mathbf{U}_{Rs}$, the proof of Theorem 6.3 in [Cortesi and Filè 1999] becomes valid.

THEOREM 8.3. *$\mathbf{U}_{Sh}$ is correct w.r.t. $\mathbf{U}^*_{Rs}$.*

PROOF. If we look at the proof of Theorem 6.3 in [Cortesi and Filè 1999], it appears that the problem is in the base case of the inductive reasoning, when $i = 0$. Here, it is stated that given $[A_1, U_1]$ and $[A_2, U_2]$ in `Sharing` with $U_1 \cap U_2 = \emptyset$, $\sigma_i \in \gamma_{Sh}([A_i, U_i])$ for $i \in \{1, 2\}$ with $\mathrm{vars}(\sigma_1) \cap \mathrm{vars}(\sigma_2) = \emptyset$, then it holds that $[\{\rho_0\}, U_0] \sqsubseteq_{\mathrm{Rs}} \gamma_{\mathrm{Sh}}([R_0, U_0])$ where $\rho_0 = \sigma_1 \uplus \sigma_2$, $U_0 = U_1 \cup U_2$ and $R_0 = A_1 \cup A_2$. However, the substitutions $\sigma_1 = \{x/y\} \in \gamma_{Sh}([\{\mathtt{x}\}, \{x\}])$ and $\sigma_2 = \epsilon \in \gamma_{Sh}([\{\mathtt{y}\}, \{y\}])$ of the previous example make the statement false. On the contrary, when $\mathbf{U}^*_{Rs}$ is used instead of $\mathbf{U}_{Rs}$, then σ_1 and σ_2 are required to satisfy the condition $(\mathrm{vars}(\sigma_1) \cup U_1) \cap (\mathrm{vars}(\sigma_2) \cup U_2) = \emptyset$. From this, it truly follows that $[\{\rho_0\}, U_0] = [\{\sigma_1 \uplus \sigma_2\}, U_0] \sqsubseteq_{\mathrm{Rs}} \gamma_{Sh}([R_0, U_0])$. The inductive case for $i > 0$ is identical to that in [Cortesi and Filè 1999], since for any $A, B \in$ `Rsub` and $\delta \in \mathit{ISubst}$ it holds that $\mathbf{U}^*_{Rs}(A, B, \delta) \sqsubseteq_{\mathrm{Rs}} \mathbf{U}_{Rs}(A, B, \delta)$. □

Observer that, in order to define a real semantics for logic programs, a renaming operation should be introduced in the framework of Cortesi and Filè [1999]. This can be done along the line of Cortesi et al. [1994]. Due to the kind of renamings involved, by replacing $\mathbf{U}_{Rs}$ with $\mathbf{U}^*_{Rs}$, the semantics in [Cortesi et al. 1994] does not change. Therefore this flaw does not affect the real analysis of logic programs.

8.3 Other Related Works

8.3.1 *Backward Unification.* The idea of using a refined operator for computing the answer substitution is not new, and may be traced back to the operational frameworks in [Bruynooghe 1991; Le Charlier and Van Hentenryck 1994]. Working within the first one, Hans and Winkler [1992] propose a correct abstract operator w.r.t. matching for the domain `SFL`. King and Longley [1995] do the same, but in a denotational framework similar to ours. Also Muthukumar and Hermenegildo [1991; 1992] use a refined algorithm for backward unification, although it is not presented in algebraic form. However, up to our knowledge, this is the first paper which formally introduces matching from the point of view of a collecting denotational semantics, deriving the abstract operator from the concrete one, and proving correctness and optimality. Moreover, we are the first one to present the best correct approximation of matching for the domain `Sharing` or one of its derivatives (see Example 7.3).

8.3.2 *Forward/Backward Unification and* `PSD`*.* Although the usual goal of sharing analyses is to discover the pairs of variables which may possibly share, `Sharing` is a domain that keeps track of set-sharing information. Bagnara et al. [2002] propose a new domain, called PSD, which is the complete shell [Giacobazzi et al. 2000] of pair sharing w.r.t. Sharing. They recognize that, in an abstract object $[S, U]$, some sharing groups in S may be *redundant* as far as pair sharing is concerned. Although our forward unification is more precise than the standard unification, it could be the case that they have the same precision in PSD. This would mean that $\mathbf{U}^f_{Sh}([S_1, U_1], A_1, A_2)$ and $\mathbf{U}'^f_{Sh}([S_1, U_1], A_1, A_2)$ only differ for redundant sharing groups. However, this is not the case, and Examples 6.2, 6.3 and 6.23 show improvements which are still significant in PSD. The same holds for backward unification in Example 7.3. It would be interesting to examine in details the behavior of our unification operators in the domain PSD, since it is not clear whether it is still complete w.r.t. pair-sharing when our specialized operators are used.

8.3.3 *Domains with Freeness and Linearity.* Although the use of freeness and linearity information has been pursued in several papers (e.g., [Muthukumar and Hermenegildo 1991; Hans and Winkler 1992]), optimal operators for these domains have never been developed. Actually, the standard mgu in SFL [Muthukumar and Hermenegildo 1992; Hans and Winkler 1992; Bagnara et al. 2000], when unifying with a binding $\{x/t\}$ where neither x nor t are linear, does compute all the star unions. In $\mathbf{u}^f_{Sh}$, however, we apply an optimization which is able to avoid some sharing groups (see e.g., Example 6.23). This optimization could be integrated in a domain which explicitly contains freeness and linearity informations.

Actually, Bagnara et al. [2000] include some optimizations for the standard abstract unification of SFL which are similar to ours, in the case of a binding $\{x/t\}$ with x linear. In addition, in [Zaffanella 2001; Howe and King 2001] it is proposed to remove the check for independence between x and t. We think it should be possible to devise an optimal abstract unification for an enhanced domain including linearity information, by combining these improvements with our results. A first optimality result, for a domain with only set sharing and linearity, is presented in [Amato and Scozzari 2002].

8.3.4 *Another Optimality Proof.* Codish et al. [2000] provide an alternative approach to the analysis of sharing by using set logic programs and ACI1 unification. They define abstract operators which are proved to be correct and optimal, and examine the relationship between set substitutions and Sharing, proving that they are essentially isomorphic. However, they do not extend this correspondence to the abstract operators, so that a proof of optimality of $\mathbf{U}^f_{Sh}$ w.r.t. $\mathbf{U}^f_{\mathrm{Ps}}$ starting from their results should be feasible but it is not immediate. Moreover, since they provide a goal-independent analysis, they do not have different operators for forward and backward unification.

9. CONCLUSIONS

We think that two are the major contributions of this paper:

— We propose a refined framework with specialized operators for forward and backward unification. We provide the corresponding abstract operators for sharing analysis which are proved to be correct and optimal. The obtained analysis is shown to be strictly more precise than the original one.

— Our definition of $\mathbf{U}^f_{Sh}$ sheds new light on the abstract unification in the presence of freeness and linearity information, suggesting new optimizations which can also be used in more powerful domains such as `SFL`. The only domains known to us which are more precise, in every circumstance, of `Sharing` with $\mathbf{U}^f_{\mathrm{Ps}}$ are those defined in [Amato and Scozzari 2002], with the corresponding optimal abstract operators. All the other operators in the literature are not able to exploit the optimization we obtain by distinguishing linear and non-linear variables in a term.

Our idea of specialized operators for forward and backward unification is orthogonal to most of other proposals for improving precision and/or efficiency of the analysis. To the best of our knowledge, this is the first work which optimizes the abstract forward unification for sharing analysis by using a specialized operator. In [Marriott et al. 1994] the concrete *unify* operator is essentially our $\mathbf{U}^f_{\mathrm{Ps}}$, but the abstract operator is given only for groundness analysis, where specializing the forward unification gives no gain in precision. In other works about goal-dependent analysis, such as [Muthukumar and Hermenegildo 1991; Hans and Winkler 1992], the algorithm used for computing the entry substitution is simply the standard unification.

This is also the first work where a specialized backward unification operator is proved to be optimal, although matching has been used in several papers [Hans and Winkler 1992; King and Longley 1995; Muthukumar and Hermenegildo 1992] to improve backward unification. To the best of our knowledge, all the abstract operators proposed so far for `Sharing` were not optimal. Matching, however, does not remove some imprecisions of goal-dependent versus goal-independent analysis which have been pointed out by de La Banda et al. [1998].

As a future work, we think that our results could be easily generalized for designing optimal unification operators for more complex domains possibly including linearity, freeness and structural information. Moreover, the problem of efficiently implementing the backward unification could be addressed.

A. CORRECTNESS OF THE GOAL-DEPENDENT COLLECTING SEMANTICS

In this appendix we provide the tedious proof that the collecting semantics we define is correct w.r.t. computed answers. We begin by formally introducing a notation for SLD-derivations, following [Lloyd 1987; Apt 1990]. Given a goal $G = g_1 \dots g_k$ and a clause $cl = H \leftarrow B$ such that $\mathrm{vars}(G) \cap \mathrm{vars}(cl) = \emptyset$, we write

$$G \xrightarrow[\sigma]{cl} (g_1 \dots g_{i-1} B g_{i+1} \dots g_k B)\sigma \tag{77}$$

when $\sigma = \mathrm{mgu}(g_i, H)$. Given a goal G and a program P, an *SLD-derivation* of G in P is given by a sequence of clauses $cl_1, \dots, cl_n$ and idempotent substitutions $\sigma_1, \dots, \sigma_n$, such that

$$G \xrightarrow[\sigma_1]{cl_1} G_1 \xrightarrow[\sigma_2]{cl_2} \cdots \xrightarrow[\sigma_n]{cl_n} G_n \ , \tag{78}$$

where each cl_i is the renaming of a clause in P apart from $G, cl_1, \dots, cl_{i-1}$. The goal G_n is called the *end-goal*, n is the length of the derivation and $(\sigma_n \circ \sigma_{n-1} \circ \cdots \circ \sigma_2 \circ \sigma_1)_{|\mathrm{vars}(G)}$ is the (partial) computed answer. An SLD-refutation is an SLD-derivations with empty end-goal (denoted by $\Box$). A *leftmost* SLD-derivation is an SLD-derivation where we always rewrite the leftmost atom in the goal (i.e., such that $i = 1$ at every step in (77)).

We write $G \xrightarrow[\sigma]{*} G'$ to denote an SLD-derivation with end-goal G' and partial computed answer σ. We also write $G \xrightarrow[\sigma]{\leq i} G'$ to denote an SLD-derivation with end-goal G', partial computed answer σ and whose length is less or equal then i. A substitution σ is a *computed answer* for G in P if there is an SLD-refutation $G \xrightarrow[\sigma]{*} \Box$.

In this appendix we will prove the relationship between the set of computed answers for P and its collecting semantics $\mathcal{P}[\![P]\!]$.

A.1 Relevant Denotations

We have defined a denotation as a continuous map in $\mathsf{Atoms} \rightarrow \mathtt{Psub} \rightarrow^c \mathtt{Psub}$. We now want to characterize the denotations which may arise as the results of our collecting semantics.

Definition A.1. A denotation $d \in \mathcal{D}en$ is said to be *relevant* when

—d is strict, i.e., $dA\bot_{\mathrm{Ps}} = \bot_{\mathrm{Ps}}$;
—$dA[\Theta, V]$ is either $\bot_{\mathrm{Ps}}$ or $[\Theta', V \cup \mathrm{vars}(A)]$ for some Θ'.

Note that the least denotation $\lambda A.\lambda[\Theta, V].\bot_{\mathrm{Ps}}$ is relevant. A relevant denotation is well-behaved, in the sense that either it does not say anything, or gives informations for all and only the variables which occur in the atom A and the entry substitution $[\Theta, V]$.

PROPOSITION A.2. *If d is relevant, then*

(*1*) $\mathcal{B}[\![B]\!]d\bot_{\mathrm{Ps}} = \bot_{\mathrm{Ps}}$;
(*2*) $\mathcal{B}[\![B]\!]d[\Theta, V]$ *is either* $\bot_{\mathrm{Ps}}$ *or* $[\Theta', V \cup \mathrm{vars}(B)]$ *for some* Θ';
(*3*) $\mathcal{C}[\![H \leftarrow B]\!]d$ *is relevant;*
(*4*) $\mathcal{P}[\![P]\!]$ *is relevant.*

PROOF. The first two points easily follow by induction on the structure of the body B. For the third point, consider the definition of $\mathcal{C}$. Note that $\mathbf{U}^f_{\mathrm{Ps}}(x, A, H) = \pi_{\mathrm{Ps}}(\mathsf{unif}_{\mathrm{Ps}}(\rho(x), \mathrm{mgu}(\rho(A) = H)), \mathrm{vars}(H))$. Since $\mathrm{vars}(\rho(A))$ is disjoint from H by definition of ρ, and since we consider relevant mgus, then either $\mathrm{vars}(\mathrm{mgu}(\rho(A) = H)) = \mathrm{vars}(\rho(A)) \cup \mathrm{vars}(H)$ or $\mathrm{mgu}(\rho(A) = H) = \bot$. In the latter case, $\mathcal{C}[\![H \leftarrow B]\!]dA = \bot_{\mathrm{Ps}}$, otherwise $\mathbf{U}^f_{\mathrm{Ps}}(x, A, H) = [\Theta', \mathrm{vars}(H)]$ for some Θ'. By the previous point, we have that $\mathcal{B}[\![B]\!]d(\mathbf{U}^f_{\mathrm{Ps}}(x, A, H))$ is either $\bot_{\mathrm{Ps}}$ or $[\Theta'', \mathrm{vars}(H) \cup \mathrm{vars}(B)]$ for some Θ''. In the first case, $\mathcal{C}[\![H \leftarrow B]\!]dA = \bot_{\mathrm{Ps}}$, otherwise, assuming $x = [\Sigma, V]$, we have

$$\begin{aligned}\mathcal{C}[\![H \leftarrow B]\!]dAx = \mathbf{U}^b_{\mathrm{Ps}}([\Theta'', \mathrm{vars}(H) \cup \mathrm{vars}(B)], x, H, A) = \\ \pi_{\mathrm{Ps}}\Big(\mathsf{match}_{\mathrm{Ps}}(\rho([\Theta'', \mathrm{vars}(H) \cup \mathrm{vars}(B)]), \\ \mathsf{unif}_{\mathrm{Ps}}([\Sigma, V], \mathrm{mgu}(\rho(H) = A))), V \cup \mathrm{vars}(A)\Big) \ .\end{aligned}$$

For the same reason explained above, and since we can ignore the case in which $\rho(H)$ and A do not unify, we have that $\mathsf{unif}_{\mathrm{Ps}}([\Sigma, V], \mathrm{mgu}(\rho(H) = A)) = [\Sigma', V \cup \mathrm{vars}(A)]$ and therefore

$$\pi_{\mathrm{Ps}}(\mathsf{match}_{\mathrm{Ps}}(\rho([\Theta'', \mathrm{vars}(H) \cup \mathrm{vars}(B)]), [\Sigma', V \cup \mathrm{vars}(A)]), V \cup \mathrm{vars}(A)) = [\Sigma'', V \cup \mathrm{vars}(A)] \ ,$$

which is what we wanted to prove.

The forth point follows by the fact that, given the proof of the third point, $\mathcal{C}[\![cl]\!]d$ is relevant for each clause cl, and that lowest upper bound of relevant denotations are easily seen to be relevant. □

A.2 Unused variables

Definition A.3. Given $[\phi]_V \in \mathit{ISubst}$ and $x \in \mathcal{V}$, we say that x is *unused* in $[\phi]_V$ when $[\phi]_V = \mathrm{mgu}\big(\pi_{V \setminus \{x\}}([\phi]_V), [\epsilon]_{\{x\}}\big)$.

First of all, note that this definition does not depend on the choice of representatives. If a variable x is unused in $[\phi]_V$, it means that $[\phi]_V$ does not constraint in any way its value. In other words, x is free and independent from all the other variables in V. This is made clear by the following characterization:

Proposition A.4. *The variable $x \in V$ is unused in $[\phi]_V$ iff it is free and independent in $[\phi]_V$.*

Proof. If x is free and independent in $[\phi]_V$, we may assume without loss of generality that $x \notin \mathrm{vars}(\phi)$. Let $V' = V \setminus \{x\}$. We have that

$$\mathrm{mgu}(\pi_{V'}([\phi]_V), [\epsilon]_{\{x\}}) = \mathrm{mgu}([\phi]_{V'}, [\epsilon]_{\{x\}}) = [\phi_{|V'}]_V = [\phi]_V \ ,$$

which proves that x is unused. On the other hand, assume ϕ is a canonical representative and $\mathrm{mgu}([\phi]_{V'}, [\epsilon]_{\{x\}}) = [\phi]_V$. Then $\phi_{|V'} \sim_V \phi$. It is obvious that x is free and independent in $[\phi_{|V'}]_V = [\phi]_V$, since $x \notin \mathrm{dom}(\phi_{|V'})$ and $x \notin \mathrm{rng}(\phi)$. □

A.3 $\mathit{ISubst}_\sim$ and composition

The operations described in Section 3.2 are those required to provide a collecting semantics for logic programs over the domain $\mathit{ISubst}_\sim$. Note that we do not define any notion of composition, although it plays a central role with the standard substitutions. Actually, composition cannot be defined in our framework since, given any element of $\mathit{ISubst}_\sim$, variables not of interest are considered up to renaming only, and therefore cannot be named. Nonetheless, in order to prove the equivalence between the standard semantics based on SLD-resolution and our collecting semantics, we will need to relate the composition of substitutions with unification in $\mathit{ISubst}_\sim$.

Lemma A.5. (Composition Lemma) *Let $\sigma_1, \sigma_2, \sigma_3 \in \mathit{Subst}$, $U, V \in \wp_f(\mathcal{V})$. Then it holds that:*

$$\mathrm{mgu}([\sigma_3 \circ \sigma_2]_U, [\sigma_2 \circ \sigma_1]_V) = [\sigma_3 \circ \sigma_2 \circ \sigma_1]_{U \cup V}$$

provided that:

—$\mathrm{dom}(\sigma_1) \cap U = \emptyset$;

—*if* $y \in \sigma_2(\sigma_1(V)) \setminus \sigma_2(\sigma_1(U \cap V))$ *then* $y \notin \mathrm{dom}(\sigma_3) \cup \sigma_3(\sigma_2(U))$.

PROOF. Let $\theta \in [\sigma_3 \circ \sigma_2]_U$, $\eta \in [\sigma_2 \circ \sigma_1]_V$ canonical representatives such that $(\mathrm{vars}(\theta) \cup U) \cap (\mathrm{vars}(\eta) \cup V) \subseteq U \cap V$. By definition, there exist $\rho, \rho' \in \mathit{Ren}$ such that $\theta = (\rho' \circ \sigma_3 \circ \sigma_2)_{|U}$ and $\eta = (\rho \circ \sigma_2 \circ \sigma_1)_{|V}$.

Then $\mathrm{mgu}([\sigma_3 \circ \sigma_2]_U, [\sigma_2 \circ \sigma_1]_V) = [\mathrm{mgu}(\theta, \eta)]_{U \cup V}$. It holds that $\mathrm{mgu}(\theta, \eta) = \mathrm{mgu}(\eta(\mathrm{Eq}(\theta))) \circ \eta$. It follows that $\eta(\mathrm{Eq}(\theta)) = \{\eta(x) = \eta(\theta(x)) \mid x \in U\} = \{\eta(x) = \theta(x) \mid x \in U\}$ since θ is a canonical representative. If $x \in U \cap V$, then $\eta(x) = \theta(x)$ becomes $\rho \circ \sigma_2 \circ \sigma_1(x) = \rho' \circ \sigma_3 \circ \sigma_2(x)$ which is $\rho \circ \sigma_2(x) = \rho' \circ \sigma_3 \circ \sigma_2(x)$ since $\mathrm{dom}(\sigma_1) \cap U = \emptyset$ by hypothesis. Thus $\{\eta(x) = \theta(x) \mid x \in U \cap V\} \sim \{\rho(y) = \rho' \circ \sigma_3(y) \mid y \in \sigma_2(U \cap V)\}$. If $x \notin V$ then $\{\eta(x) = \theta(x) \mid x \in U \setminus V\} = \{x = \theta(x) \mid x \in U \setminus V\}$.

Now $\delta = \{\rho(y)/\rho' \circ \sigma_3(y) \mid y \in \sigma_2(U \cap V)\} \cup \{x/\theta(x) \mid x \in U \setminus V\}$ is an idempotent substitution. Actually, all the $\rho(y)$'s are distinct variables and different from $U \setminus V$ therefore δ is a substitution. Moreover, $\mathrm{dom}(\delta) \subseteq \mathrm{vars}(\eta(V)) \cup (U \setminus V)$ is disjoint from $\mathrm{rng}(\delta) = \mathrm{vars}(\theta(U))$.

Let ρ'' be the substitution

$$\rho''(x) = \begin{cases} \rho'(x) & \text{if } x \in \sigma_3\sigma_2(U) \\ \rho(x) & \text{if } x \in \sigma_2(\sigma_1(V)) \setminus \sigma_2(\sigma_1(U \cap V)) \\ x & \text{otherwise} \end{cases}$$

Note that, thanks to the second hypothesis of the lemma, we are sure that the first and second case in the definition of ρ'' may not occur together. We want to prove that $\delta\eta(x) = \rho''\sigma_3\sigma_2\sigma_1(x)$ for each $x \in U \cup V$. Since ρ'' restricted to $\mathrm{vars}(\sigma_3(\sigma_2(\sigma_1(U \cup V))))$ is an injective map from variables to variables, by Lemma 3.4 this implies $\delta \circ \eta \sim_{U \cup V} \sigma_3 \circ \sigma_2 \circ \sigma_1$, which is the statement of the theorem.

Thus if $x \in U \setminus V$ then $\eta(x) = x$ and $\delta(\eta(x)) = \theta(x) = \rho'(\sigma_3(\sigma_2(x))) = \rho''(\sigma_3(\sigma_2(x))) = \rho''(\sigma_3(\sigma_2(\sigma_1(x))))$ since $\mathrm{dom}(\sigma_1) \cap U = \emptyset$ and by definition of ρ''.

If $x \in U \cap V$ then $\delta(\eta(x)) = \delta(\rho(\sigma_2(x)))$ since $\mathrm{dom}(\sigma_1) \cap U = \emptyset$ and thus $\delta(\eta(x)) = \rho'\sigma_3\sigma_2(x)$, which is equal to $\rho''\sigma_3\sigma_2\sigma_1(x)$ since $\mathrm{dom}(\sigma_1) \cap U = \emptyset$ and by definition of ρ''.

If $x \in V \setminus U$ then $\delta(\eta(x)) = \delta(\rho(\sigma_2\sigma_1(x)))$. Let $y \in \mathrm{vars}(\sigma_2\sigma_1(x))$. If we assume that $y \in \mathrm{vars}(\sigma_2(U \cap V))$, then $\delta(\rho(y)) = \rho'\sigma_3(y) = \rho''(\sigma_3(y))$ by definition of δ and ρ''. If $y \notin \mathrm{vars}(\sigma_2(U \cap V))$ then $\delta(\rho(y)) = \rho(y) = \rho''(y) = \rho''\sigma_3(y)$ by definition of ρ'' and the second condition in the theorem. In both cases we obtain $\delta\rho(y) = \rho''\sigma_3(y)$ for each $y \in \mathrm{vars}(\sigma_2(\sigma_1(x)))$. Therefore, for each $x \in U \cap V$, $\delta\eta(x) = \delta\rho\sigma_2\sigma_1(x) = \rho''\sigma_3\sigma_2\sigma_1(x)$ and this concludes the proof. □

A.4 Proof of Correctness

Let D_P be defined as $\lambda d. \left(\bigsqcup_{cl \in P} \mathcal{C}[\![cl]\!]d\right)$ and let D_P^i be the i-th iteration of D_P with $D_P^0 = \lambda A.\lambda x.\bot_{\mathrm{Ps}}$. Note that $D_P^\omega = \mathcal{P}[\![P]\!]$ and D_P^i is relevant for each i.

LEMMA A.6. (CORRECTNESS LEMMA) *Let* $i \in \mathbb{N}$, $[\phi]_V \in \mathit{ISubst}_\sim$, $G \in \mathsf{Bodies}$ *and* $P \in \mathsf{Progs}$. *If* $[\phi]_{V \cup G} = \mathrm{mgu}([\phi]_V, [\epsilon]_G)$ *and* $G\phi \xrightarrow[\sigma]{\leq i} \square$ *is a leftmost SLD-refutation where all clauses are renamed apart from* V, G, ϕ *and the program* P, *then* $\mathcal{B}[\![G]\!]D^i_P[\{[\phi]\}, V] \sqsupseteq [\{[\sigma \circ \phi]\}, V \cup \text{vars}(G)]$.

Remark A.7. The condition $[\phi]_{V \cup G} = \text{mgu}([\phi]_V, [\epsilon]_G)$ is used to check that the chosen representative ϕ does not bind and variable in $\text{vars}(G) \setminus V$. All the variables in $\text{vars}(G) \setminus V$ are forced to be unused, according to Definition A.3.

Remark A.8. The theorem probably holds under weaker conditions on the variables of the SLD-resolution. However, proving the result in this case would be more difficult. Since the obtained generalization is not very interesting, we valued that it was not worth the effort.

Proof. The proof is by double induction on i and on the structure of the goal G. Assume fixed $\Phi = \{[\phi]_V\}$ such that $[\phi]_{V \cup G} = \text{mgu}([\phi]_V, [\epsilon]_G)$.

We start with the case $i = 0$. The only SLD-refutation of length 0 is the SLD-derivation for the empty goal $\Box$, whose computed answer substitution is ϵ. In the collecting semantics, we have $\mathcal{B}[\![\Box]\!]D^i_P[\{[\phi]\}, V] = [\{[\phi]\}, V] = [\{[\epsilon \circ \phi]\}, V]$ which is the required result.

If $i > 0$, assume the lemma holds for all $j < i$ and we prove it for i, by induction on the structure of goals. The case for the empty goal has been already examined, so we assume $G = A, G'$ where A is an atom. To ease the exposition, we first consider the atomic case where $G' = \Box$ and then we analyze the general one.

Atomic goal.

Given the not-empty SLD-derivation $G\phi \xrightarrow[\sigma]{*} \Box$, we may decompose it as:

$$G\phi \xrightarrow[\sigma_1]{\rho(cl)} (C_1 \dots C_n)\rho\sigma_1 \xrightarrow[\sigma_2]{*} \Box$$

where $cl = H \leftarrow C_1 \dots C_n$ is a program clause, $\sigma_1 = \text{mgu}(G\phi, H\rho)$ and ρ is a renaming of cl apart from G, V, ϕ and the program P. Note that this implies the standard renaming condition for SLD-resolutions, i.e., that $\rho(cl)$ is renamed apart from $G\phi$. Since G is atomic, then

$$\mathcal{B}[\![G]\!]D^i_P[\Phi, V] = D^i_P G[\Phi, V] \sqsupseteq \mathcal{C}[\![H \leftarrow C_1 \dots C_n]\!]D^{i-1}_P G[\Phi, V] \ ,$$

which, in turn, is equal to $\mathbf{U}^b_{\text{Ps}}(\mathcal{B}[\![C_1 \dots C_n]\!]D^{i-1}_P(\mathbf{U}^f_{\text{Ps}}([\Phi, V], G, H)), [\Phi, V], H, G)$. We know that

$$\mathbf{U}^f_{\text{Ps}}([\{[\phi]\}, V], G, H) = \pi_{\text{Ps}}(\text{mgu}(\rho'([\phi]_V), [\text{mgu}(\rho'(G) = H)]_{\rho'(G) \cup H}), \text{vars}(H))$$

where ρ' is any renaming such that $\rho'(\text{vars}(G) \cup V) \cap \text{vars}(H) = \emptyset$. We can choose as ρ' the renaming ρ^{-1} since $\rho(\text{vars}(cl)) \cap \text{vars}(G) = \emptyset$ and $\rho(\text{vars}(cl)) \cap V = \emptyset$ implies that $\rho^{-1}(\text{vars}(G) \cup V) \cap \text{vars}(H) = \emptyset$. In turn, this implies that

$$\begin{aligned}
&\text{mgu}(\rho'([\phi]_V), [\text{mgu}(\rho'(G) = H)]_{\rho'(G) \cup H}) \\
&= \rho^{-1}(\text{mgu}([\phi]_V, [\text{mgu}(G = \rho(H))]_{G \cup \rho(H)}) \\
&= \rho^{-1}(\text{mgu}([\phi]_V, [\text{mgu}(G = \rho(H))]_{G \cup \rho(H)}, [\epsilon]_G)) \\
&= \rho^{-1}(\text{mgu}([\phi]_{V \cup G}, [\text{mgu}(G = \rho(H))]_{G \cup \rho(H)}) \\
&= \rho^{-1}([\text{mgu}(\phi, \text{mgu}(G = \rho(H)))]_{V \cup G \cup \rho(H)}) \ .
\end{aligned}$$

The last pass is only valid when $(V \cup \text{vars}(G) \cup \text{vars}(\phi)) \cap (\text{vars}(G) \cup \text{vars}(\rho(H)) \subseteq (V \cup \text{vars}(G)) \cap (\text{vars}(G) \cup \text{vars}(\rho(H))) = \text{vars}(G)$. This is the case since $\text{vars}(\phi) \cap \rho(\text{vars}(cl)) = \emptyset$, thanks to our choice of ρ.

By standard properties of substitutions, we obtain:

$$\begin{aligned} & \rho^{-1}([\text{mgu}(\phi, \text{mgu}(G = \rho(H)))]_{V \cup G \cup \rho(H)}) \\ = \ & \rho^{-1}([\text{mgu}(G\phi = (\rho(H))\phi) \circ \phi]_{V \cup G \cup \rho(H)}) \\ = \ & \rho^{-1}([\text{mgu}(G\phi = \rho(H)) \circ \phi]_{V \cup G \cup \rho(H)}) \\ = \ & \rho^{-1}([\sigma_1 \circ \phi]_{V \cup G \cup \rho(H)}) \ , \end{aligned}$$

since $\text{vars}(\phi) \cap \text{vars}(\rho(H)) = \emptyset$. For the same reason, $\sigma_1 \circ \phi \sim_{\text{vars}(\rho(H))} \sigma_1$. It follows that

$$\rho^{-1}(\sigma_1 \circ \phi) \sim_{\text{vars}(H)} \rho^{-1}(\sigma_1) = \rho^{-1} \circ \sigma_1 \circ \rho \sim_{\text{vars}(H)} \sigma_1 \circ \rho \ .$$

Therefore $\mathbf{U}^f_{\text{Ps}}([\{[\phi]\}, V], G, H) = [\{[\sigma_1 \circ \rho]\}, \text{vars}(H)]$ and

$$\begin{aligned} & \mathbf{U}^b_{\text{Ps}}(\mathcal{B}[\![C_1 \dots C_n]\!] D_P^{i-1}(\mathbf{U}^f_{\text{Ps}}([\Phi, V], G, H)), [\Phi, V], H, G) \sqsupseteq \\ & \qquad \mathbf{U}^b_{\text{Ps}}(\mathcal{B}[\![C_1 \dots C_n]\!] D_P^{i-1}[\{[\sigma_1 \circ \rho]\}, \text{vars}(H)], [\Phi, V], H, G) \ . \end{aligned}$$

Note that the SLD resolution $(C_1 \dots C_n)\rho\sigma_1 \xrightarrow[\sigma_2]{*} \Box$ can be seen as $(C_1 \dots C_n)(\sigma_1 \circ \rho) \xrightarrow[\sigma_2]{*} \Box$. In order to apply the inductive hypothesis on the latter derivation, we need to verify that $[\sigma_1 \circ \rho]_{\text{vars}(cl)} = \text{mgu}([\sigma_1 \circ \rho]_{\text{vars}(H)}, [\epsilon]_{\text{vars}(C_1 \dots C_n)})$. By definition $\sigma_1 \circ \rho = \text{mgu}(G\phi, H\rho) \circ \rho$. Moreover, since $\rho(\text{vars}(cl)) \cap \text{vars}(G\phi) = \emptyset$ and $\rho(\text{vars}(cl)) \cap \text{vars}(H\rho) = \text{vars}(H\rho)$, it follows that for all $v \in \rho(\text{vars}(cl) \setminus \text{vars}(H))$, $v \notin \text{vars}(\sigma_1)$. Hence, for each $v \in \text{vars}(cl) \setminus \text{vars}(H)$, $\sigma_1(\rho(v)) = \rho(v)$. Moreover, if $\rho(v)$ occurs in $(\sigma_1 \circ \rho)(x)$ for some x, then $\rho(v)$ occurs in $\rho(x)$ and this is only possible if $x = v$. By Prop. A.4, this proves that $\text{mgu}([\sigma_1 \circ \rho]_{\text{vars}(H)}, [\epsilon]_{\text{vars}(C_1 \dots C_n)}) = [\sigma_1 \circ \rho]_{\text{vars}(cl)}$. Thus, by inductive hypothesis, we have that:

$$\begin{aligned} & \mathbf{U}^b_{\text{Ps}}(\mathcal{B}[\![C_1 \dots C_n]\!] D_P^{i-1}[\{[\sigma_1 \circ \rho]\}, \text{vars}(H)], [\Phi, V], H, G) \sqsupseteq \\ & \qquad \mathbf{U}^b_{\text{Ps}}([\{[\sigma_2 \circ \sigma_1 \circ \rho]\}, \text{vars}(cl)], [\Phi, V], H, G) \ . \end{aligned}$$

We know that $\mathsf{unif}_{\text{Ps}}([\{[\phi]\}, V], \text{mgu}(\rho(H) = G)) = [\{[\sigma_1 \circ \phi]\}, V \cup \text{vars}(G) \cup \text{vars}(\rho(H))]$. Therefore, choosing ρ as the renaming for $\mathbf{U}^b_{\text{Ps}}$, we obtain

$$\begin{aligned} & \mathsf{match}_{\text{Ps}}(\rho([\{[\sigma_2 \circ \sigma_1 \circ \rho], \text{vars}(cl)]), [\{[\sigma_1 \circ \phi]\}, V \cup \text{vars}(G) \cup \text{vars}(\rho(H))]) \\ = \ & \mathsf{match}_{\text{Ps}}([\{[\rho \circ \sigma_2 \circ \sigma_1], \text{vars}(\rho(cl))]), [\{[\sigma_1 \circ \phi]\}, V \cup \text{vars}(G) \cup \text{vars}(\rho(H))]) \\ = \ & \mathsf{match}_{\text{Ps}}([\{[\sigma_2 \circ \sigma_1], \text{vars}(\rho(cl))]), [\{[\sigma_1 \circ \phi]\}, V \cup \text{vars}(G) \cup \text{vars}(\rho(H))]) \ . \end{aligned}$$

Since $\text{vars}(\rho(cl)) \cap (V \cup \text{vars}(G) \cup \text{vars}(\rho(H))) = \text{vars}(\rho(H))$ and $\sigma_2 \circ \sigma_1 \preceq_{\text{vars}(\rho(H))} \sigma_1 \circ \phi$ (being $\text{vars}(\phi) \cap \text{vars}(\rho(H)) = \emptyset$) then it holds:

$$\begin{aligned} & \mathsf{match}_{\text{Ps}}([\{[\sigma_2 \circ \sigma_1], \text{vars}(\rho(cl))]), [\{[\sigma_1 \circ \phi]\}, V \cup \text{vars}(G) \cup \text{vars}(\rho(H))]) = \\ & \qquad [\text{mgu}([\sigma_2 \circ \sigma_1]_{\rho(cl)}, [\sigma_1 \circ \phi]_{V \cup G \cup \rho(H)}), V \cup \text{vars}(G) \cup \text{vars}(\rho(H))] \end{aligned}$$

We would like to apply the Composition Lemma (Lemma A.5) to this unification. We need to check that:

—$\mathrm{dom}(\phi) \cap \rho(cl) = \emptyset$;
—$y \in \sigma_1\phi(V \cup \mathrm{vars}(G) \cup \rho(H)) \setminus \sigma_1\phi(\rho(H))$ then $y \notin \mathrm{dom}(\sigma_2) \cup \sigma_2\sigma_1(\rho(cl))$.

The first property trivially follows by the hypothesis that ρ renames cl apart from ϕ. For the second condition, note that, since $\sigma_1 = \mathrm{mgu}(G\phi, H\rho)$, if $y \in \sigma_1\phi(G)$ then $y \in \sigma_1(\rho(H)) = \sigma_1\phi(\rho(H))$. Therefore $y \in \sigma_1\phi(V \cup \mathrm{vars}(G)) \setminus \sigma_1\phi(\rho(H))$ iff $y \in \sigma_1\phi(V \setminus G) = \phi(V \setminus G)$. However, since such a variable does not appear in the initial goal of the SLD-resolution $G\phi$ and since the resolution is renamed apart from ϕ, it happens that it does not appear in $\mathrm{vars}(\sigma_2)$, and thus in $\mathrm{dom}(\sigma_2)$. We now show that $y \notin \sigma_2\sigma_1(\rho(cl))$. By hypothesis, $y \notin \sigma_1\phi(\rho(cl))$, and since $\rho(cl)$ is renamed apart from ϕ, it follows that $y \notin \sigma_1\rho(cl)$. Moreover, as we have seen before, $y \notin \mathrm{vars}(\sigma_2)$, hence $y \notin \mathrm{vars}(\sigma_2\sigma_1\rho(cl))$.

It turns out that we may apply the Composition Lemma (Lemma A.5) and we obtain

$$[\mathrm{mgu}([\sigma_2 \circ \sigma_1]_{\rho(cl)}, [\sigma_1 \circ \phi]_{V \cup G \cup \rho(H)}), V \cup \mathrm{vars}(G) \cup \mathrm{vars}(\rho(H))] = \\ [\{\sigma_2 \circ \sigma_1 \circ \phi\}, \rho(cl) \cup V \cup G] \ .$$

By projecting over $G \cup V$ we obtain

$$\mathcal{B}[\![G]\!] D_P^i[\Phi, V] \sqsupseteq [\{\sigma_2 \circ \sigma_1 \circ \phi]\}, V \cup \mathrm{vars}(G)] \ ,$$

which concludes the proof of the atomic case.

Non-atomic goal.

In this case, decompose the (leftmost) SLD-resolution for $G = A, G'$ in the following way:

$$A\phi, G'\phi \xrightarrow[\sigma_1]{*} G'\phi\sigma_1 \xrightarrow[\sigma_2]{*} \Box \ , \tag{79}$$

where both the sub-derivations have length strictly less than i. Note that, since the complete derivation is renamed apart from V, G, ϕ and the program P, the same holds for the first sub-derivation. Moreover, since $[\phi]_{V \cup G} = \mathrm{mgu}([\phi]_V, [\epsilon]_G)$, each $v \in A$ is free and independent in $[\phi]_{V \cup G}$, i.e., $[\phi]_{V \cup A} = \mathrm{mgu}([\phi]_V, [\epsilon]_A)$. Therefore, we may apply what proved in the atomic case above, obtaining

$$D_P^i A[\Phi, V] \sqsupseteq [\{\sigma_1 \circ \phi\}, V \cup \mathrm{vars}(A)] \ .$$

The second sub-derivation in (79) is renamed apart from

—V since the complete derivation is renamed apart from V;
—A and G' since the complete derivation is renamed apart from G;
—$\sigma_1 \circ \phi$ since the complete derivation is renamed apart from ϕ and the second part is renamed apart from σ_1;
—P, since the complete derivation is renamed apart from P.

Moreover, assume $x \in \mathrm{vars}(G') \setminus \mathrm{vars}(V \cup A)$ and $x \neq y \in \mathrm{vars}(V \cup G)$. By hypothesis, $[\phi]_{V \cup G} = \mathrm{mgu}([\phi]_V, [\epsilon]_G)$, which implies that $\phi(x) \in \mathcal{V}$ and $\phi(x) \notin \mathrm{vars}(\phi(y))$. Since $\mathrm{vars}(\sigma_1) = W \cup X$ where W is a fresh set of variables disjoint from $V \cup G$ and ϕ and $X \subseteq \mathrm{vars}(A\phi)$, it happens that $\phi(x) \notin \mathrm{vars}(\sigma_1)$. Therefore $\sigma_1(\phi(x)) = \phi(x)$ and $\phi(x) \notin \mathrm{vars}(\sigma_1(\phi(y)))$. This implies that $[\sigma_1 \circ \phi]_{V \cup G} =$

$\mathrm{mgu}([\sigma_1 \circ \phi]_{V \cup A}, [\epsilon]_{G'})$ by Prop. A.4. This means that we may apply the inductive hypothesis on the second sub-derivation, obtaining:

$$\mathcal{B}[\![G']\!] D_P^i [\{\sigma_1 \circ \phi\}, V \cup \mathrm{vars}(A)] \sqsupseteq [\{\sigma_2 \circ \sigma_1 \circ \phi\}, V \cup \mathrm{vars}(G)] \enspace .$$

Since $\mathcal{B}[\![A, G']\!] D_P^i [\Phi, V] = \mathcal{B}[\![G']\!] D_P^i (D_P^i A[\Phi, V])$ by the above disequalities and monotonicity of $\mathcal{B}$, we obtain

$$\mathcal{B}[\![A, G']\!] D_P^i [\Phi, V] \sqsupseteq [\{\sigma_2 \circ \sigma_1 \circ \phi\}, V \cup \mathrm{vars}(G)] \enspace .$$

which concludes the proof. $\square$

Now we may use standard properties of SLD-resolution together with Lemma A.6 to prove the required correctness theorem.

THEOREM A.9. (SEMANTIC CORRECTNESS) *Given a program P and an goal G, if θ is a computed answer for the goal G, then*

$$\mathcal{B}[\![G]\!](\mathcal{P}[\![P]\!]) G[\{\epsilon\}, \mathrm{vars}(G)] \sqsupseteq [\{[\theta]\}, \mathrm{vars}(G)] \enspace .$$

PROOF. If θ is a computed answer for a goal G, and ρ is a renaming, then $\theta' = (\rho \circ \theta)_{|\mathrm{vars}(G)}$ is a computed answer too (cf. [Apt 1990]) and $\theta \sim_{\mathrm{vars}(G)} \theta'$. Consider any such θ' with the property that $\mathrm{vars}(\theta') \cap \mathrm{vars}(P) = \emptyset$ and let $G \xrightarrow[\theta']{*} \Box$ be a leftmost SLD-resolution for θ'. Since there exists a leftmost SLD-resolution $G \xrightarrow[\theta']{*} \Box$ which is renamed apart from P, then, by Lemma A.6, the thesis follows. $\square$

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