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Planning and Verifying Service Composition

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Abstract

A static approach is proposed to study secure composition of services. We extend the λ-calculus with primitives for selecting and invoking services that respect given security requirements. Security-critical code is enclosed in policy framings with a possibly nested, local scope. Policy framings enforce safety and liveness properties. The actual run-time behaviour of services is over-approximated by a type and effect system. Types are standard, and effects include the actions with possible security concerns — as well as information about which services may be invoked at run-time. An approximation is model-checked to verify policy framings within their scopes. This allows for removing any run-time execution monitor, and for determining the plans driving the selection of those services that match the security requirements on demand.

1 Introduction

Service-oriented computing (SOC) is an emerging paradigm to design distributed applications [39, 38, 20]. In this paradigm, applications are built by assembling together independent computational units, called *services*. A service is a standalone component distributed over a network, and made available through standard interaction mechanisms. An important aspect is that services are *open*, in that they are built with little or no knowledge about their operating environment, their clients, and further services therein invoked. Composition of services may require peculiar mechanisms to handle complex interaction patterns (e.g. to implement transactions), while enforcing non-functional requirements on the system behaviour (e.g. security and service level agreement). Web Services [1, 43, 47] built upon XML technologies are possibly the most illustrative and well developed example of the SOC paradigm. Indeed, a variety of XML-based technologies already exists for describing, discovering and invoking web services [14, 16, 12, 48]. There are also several standards for defining and enforcing non-functional requirements of services, e.g. WS-Security [3], WS-Trust [2] and WS-Policy [46] among the others. *Orchestration* of services consists of their composition and coordination. Languages for that have been recently proposed, e.g. WS-BPEL [12, 32].

Service composition heavily depends on which information about a service is made public, on how to choose those services that match the user's requirements, and on their actual run-time behaviour. Security makes service composition even harder. Services may be offered by different providers, which only partially trust each other. On the one hand, providers have to guarantee the delivered service to respect a given security policy, in any interaction with the operational environment, and regardless of who actually called the service. On the other hand, clients may want to protect their sensible data from the services invoked.

In this paper, we tackle the problem of modelling composition of services in the presence of security constraints. Our main result is a semantic-based method to *plan* which services an application has to choose in order to complete the original task, while guaranteeing security.

Our first technical contribution is on a foundational calculus for service orchestration, called λ^{req}. We model services as expressions of a typed extension of the λ-calculus with primitive constructs to describe and compose services, being selected to enforce the given requirements. Indeed, our selection mechanism matches (suitable abstractions of) service *behaviour*, rather than syntactic signatures, as it happens in the standard discovery mechanisms.

We are interested in enforcing *safety* and *liveness* properties over the abstract behaviour of services. This kind of properties have shown effective to reason about security. For example, history-based access control can be handled as safety properties [4, 42], while liveness properties can be exploited to formalize denial-of-service and brute-force attacks on cryptographic keys [24]. Here, we assume as given a set of primitive *access events*, that abstract from activities with possible security concerns. The security policies are *regular* properties of *execution histories* (i.e. sequences of access events) and have a possibly nested, local scope. Given an expression e, a *safety framing* $\varphi[e]$ enforces the policy φ at each step of the execution of e. A *liveness framing* $\psi\langle e\rangle$ prescribes that the evaluation of e must eventually respect the policy ψ. Our results are independent of the logic chosen for expressing regular properties of sequences, so we do not fix any logic here.

We shall exploit a static analysis technique to determine plans that drive service executions enjoying both safety and liveness properties. Note that, while safety properties can be enforced by an execution monitor, liveness properties cannot [40]. Also, liveness cannot be reduced to safety in general. Moreover, here we cannot predict any bound on the time a service needs to be completed, hence e.g. bounded liveness – a safety property – is inadequate. The considerations above further support the use of static techniques.

We introduce a type and effect system [25, 37, 44] for our calculus. Types are standard, while effects, called *history expressions*, represent all the possible behaviour of services. A service is modelled as a λ^{req} expressions with a functional type of the form $\tau_1 \xrightarrow{H} \tau_2$. Intuitively, when supplied with an argument of type τ_1, the service evaluates to a value of type τ_2, and the side effect of the invocation is an execution history belonging to the history expression H. A service request is modelled by an expression $\mathtt{req}_r\tau$, where r uniquely identifies the request, and τ is the type of the requested service, including the safety and liveness properties on demand. The safety constraint says how the caller protects itself from the service. Instead, the liveness constraint can be seen as the duties the invoked service must fulfill.

For simplicity, here we assume that services are published in a global trusted repository, i.e. a set of typed expressions $\{e_1 : \tau_1 \xrightarrow{H_1} \tau'_1 \cdots e_k : \tau_k \xrightarrow{H_k} \tau'_k\}$. Types are the semantic information made visible about services. Operationally, a service request $\mathtt{req}_r\tau$ results in a sort of "call-by-contract": the repository is searched for a service with a functional type matching the request type τ; additionally, its effect H should respect the safety and liveness constraints in τ. The effect of a service invocation $\mathtt{req}_r\tau$ has the form $\{r[\ell_1] \rhd H_1 \cdots r[\ell_k] \rhd H_k\}$, where $r[\ell_i]$ resolves the request r with the service e_i in the repository. We say that H_i is *valid* when it respects the safety and liveness constraints in τ, as well as all the security framings within H_i itself.

The effect H of a service composition e is obtained by suitably assembling the effects of the component services, and of those services they may invoke in a nested fashion. Note however that composing the effects of the selected services, yet valid, may result in a *non-valid* overall effect. This is because the effect of selecting a given service for a request is not confined to the execution of that service, but it spans over the whole execution. The validity of the effect H of e depends thus on the plan that selects a service for each request. It is convenient to lift all the service choices $r[\ell]$ to the top-level of H, collecting them in a set π, called *plan*. We define then a semantic-preserving transformation that results in effects of the form $\{\pi_1 \rhd H_1 \cdots \pi_n \rhd H_n\}$, where each H_i is free of further choices. Its intuitive meaning is that, under the plan π_i, the effect of the overall service composition e is H_i. If some H_i is valid, then the plan π_i will safely drive the execution of e, without resorting to any run-time monitor, and guaranteeing all the safety and liveness properties required.

Validity of history expressions is still to be ascertained. We do that by model checking (a suitable version of) Basic Process Algebras (BPAs) with finite state automata. A history expressions H is naturally rendered as a BPA process, while a finite state automaton models the validity of H. Because of the possible nesting of framings, validity of history expressions is a non-regular property, so standard model checking techniques cannot be directly applied. We transform then history expressions so to make model checking feasible through specially-tailored finite state automata.

2 Motivating examples

To illustrate our approach, consider a simple certification service that wants to attest a contract between two external parties, while enforcing its own privacy policy. Since specifying a privacy policy can be difficult and error-prone, this task is delegated to a trusted *policy provider*.

Policy providers return a safety policy φ in the form of a closure $\lambda x.\, \varphi[x]$ (indeed, policies are not first class objects in our model). The policy φ has to be enforced in the subsequent execution of the certification service, i.e. $(\lambda x.\, \varphi[x])e$ will evolve to $\varphi[e]$. This paradigm can be seen as a form of *dynamic sandboxing*. The published interface of policy providers guarantees that the distinguished event α_p (modelling the return of a policy) will eventually occur.

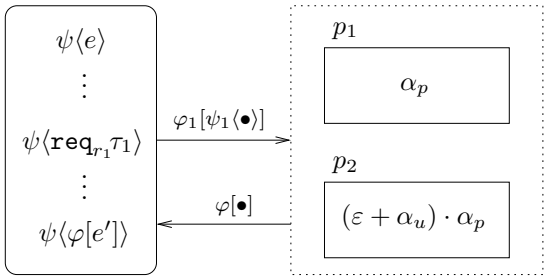
To assert its willingness to provide clients with a signed and non-repudiable copy of the contract, the certification service encloses its code into a liveness framing $\psi\langle\cdots\rangle$. The property ψ states that eventually there will be a signature (modelled by an event α_{sgn}) with no subsequent revocation (the event α_{rvk}).

The certification service requests a policy provider through the expression:

$$\mathtt{req}_{r_1}\tau_1 \qquad\qquad \text{where } \tau_1 = \tau \to (\tau \xrightarrow{\varphi_1[\psi_1\langle\bullet\rangle]} \tau)$$

(here, τ is a base type, whose specification is irrelevant). The request asks for a service that takes an argument of type τ, and returns a policy as a closure of type $\tau \to \tau$. The annotation $\varphi_1[\psi_1\langle\bullet\rangle]$ in τ_1 indicates that, when evaluated, the selected service must respect the safety policy φ_1 and the liveness policy ψ_1. The policy ψ_1 requires that eventually α_p occurs, while φ_1 says that the service cannot visit untrusted sites, modelled by the event α_u.

The leftmost part in the following diagram highlights the evolution of the certification service. The rightmost part shows the two policy providers p_1 and p_2 available in the repository. For simplicity, the boxes only represent the *effect* of the evaluation of the service; the arrows display the service invocation and the delivered safety policy.



The service p_1 exposes a history expression α_p, to mean that p_1 will generate exactly the event α_p and no other security-relevant operations. The service p_2 possibly connects to an untrusted site (thus generating the event α_u), and then provides the client with a privacy policy. This behaviour is modelled by the history expression $(\varepsilon + \alpha_u)\cdot\alpha_p$, where ε stands for the empty history, $\cdot$ for concatenation of histories, and $+$ for non-deterministic choice. The effect associated with the request r_1 is then a *planned selection* of the form:

$$\{r_1[p_1] \rhd \varphi_1[\psi_1\langle\alpha_p\rangle],\ r_1[p_2] \rhd \varphi_1[\psi_1\langle(\varepsilon + \alpha_u)\cdot\alpha_p\rangle\}$$

In the first element, the plan $r_1[p_1]$ indicates that the request r_1 is served by the provider p_1. In this case, the resulting history expression is $\varphi_1[\psi_1\langle\alpha_p\rangle]$. Similarly, when the plan is $r_1[p_2]$. Note that p_1 successfully serves the request r_1, because α_p satisfies both φ_1 and ψ_1. Instead, p_2 can generate the histories α_p and $\alpha_u\alpha_p$. Since the second history does not respect the safety constraint φ_1, then p_2 will be rejected by our static machinery.

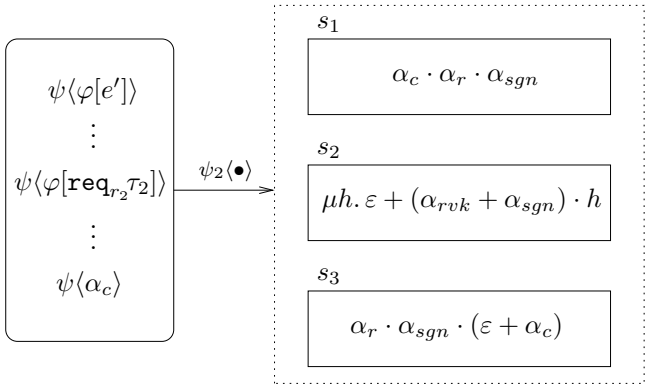
Assume now that the privacy policy φ delivered by p_1 states that connecting to the network (α_c) is prevented after a read (α_r) of local data. The diagram below depicts the invocation of the contracting parties, modelled by the request

$$\mathtt{req}_{r_2}\tau_2 \qquad\qquad \text{where } \tau_2 = \tau \xrightarrow{\psi_2\langle\bullet\rangle} \tau$$

The required guarantee is expressed by the liveness property ψ_2, saying that the certification service will eventually receive back a signed contract (note that

here a bounded liveness in place of ψ_2 would work as well, just because the code of services is visible; however, this is *not* the case in general).

There are three contracting parties. The service s_1 opens a network connection (α_c), reads the contract file (α_r) and then signs it (α_{sgn}); the service s_2 is a loop of either action of revoking or signing (recursion is written through the μ operator); the service s_3 reads the contract file (α_r), signs it (α_{sgn}), and then can possibly open a network connection (α_c).



The planned selection associated with r_2 is then:

$$\begin{aligned}&\{r_2[s_1] \rhd \psi_2\langle\alpha_c \cdot \alpha_r \cdot \alpha_{sgn}\rangle,\\ &\ r_2[s_2] \rhd \psi_2\langle\mu h.(\varepsilon + \alpha_{rvk} + \alpha_{sgn}) \cdot h\rangle,\\ &\ r_2[s_3] \rhd \psi_2\langle\alpha_r \cdot \alpha_{sgn} \cdot (\varepsilon + \alpha_c)\rangle\}\end{aligned}$$

The history expressions exposed by s_1 and s_3 clearly satisfy the requested agreement ψ_2, while that of s_2 does not, because, e.g. the history $(\alpha_{rvk})^n$ is possible, for all n. Therefore, the service request can only be served successfully by either s_1 or s_3. After completion of the service, the safety framing $\varphi[\cdots]$ is left, so the certification service can eventually connect to the network.

Considering only the services matching the requests (i.e. p_1 for r_1 and s_1, s_3 for r_2) we associate the following overall history expression H with the whole certification service:

$$\begin{aligned}H \ = \ &\psi\langle\{r_1[p_1] \rhd \varphi_1[\psi_1\langle\alpha_p\rangle]\} \cdot\\ &\varphi[\{r_2[s_1] \rhd \psi_2\langle\alpha_c \cdot \alpha_r \cdot \alpha_{sgn}\rangle,\ r_2[s_3] \rhd \psi_2\langle\alpha_r \cdot \alpha_{sgn} \cdot (\varepsilon + \alpha_c)\rangle\}] \cdot \alpha_c\rangle\end{aligned}$$

To determine the correct service compositions, we first "linearize" H, by moving all the associations of requests with services at the top-level. Here we obtain:

$$\begin{aligned}\{r_1[p_1] \mid r_2[s_1] \ &\rhd\ \psi\langle\varphi_1[\psi_1\langle\alpha_p\rangle] \cdot \varphi[\psi_2\langle\alpha_c \cdot \alpha_r \cdot \alpha_{sgn}\rangle] \cdot \alpha_c\rangle,\\ r_1[p_1] \mid r_2[s_3] \ &\rhd\ \psi\langle\varphi_1[\psi_1\langle\alpha_p\rangle] \cdot \varphi[\psi_2\langle\alpha_r \cdot \alpha_{sgn} \cdot (\varepsilon + \alpha_c)\rangle] \cdot \alpha_c\rangle\}\end{aligned}$$

where $\mid$ composes independent plans. The first element is valid, because choosing p_1 for r_1 and s_1 for r_2 drives a successful computation. Indeed, the liveness framing ψ is satisfied by the signature α_{sgn} which is never revoked afterwards. Also, the safety framing φ is obeyed, since within its scope there is no connection

α_c after a read α_r. Instead, the history expression associated with the plan $r_1[p_1] \mid r_2[s_3]$ is *not* valid, because α_c may occur after an α_r within the scope of φ. This verification phase is mechanizable and determines all and only the valid elements.

In the next sections we shall describe a way of extracting the relevant information from services (through a type and effect system) and for verifying when their composition is valid (by model checking), so providing us at static time with a winning planning strategy – in our example the plan $r_1[p_1] \mid r_2[s_1]$. The paramount point is that we can execute an expression and resolve its requests according to a winning plan, with no run-time monitoring and with the guarantee that the overall behaviour will always comply with the safety and liveness policies on demand.

3 Programming model

To study secure service composition in a pure framework, we consider λ^{req}, a call-by-value λ-calculus enriched with local security policies and service requests. An *access event* $\alpha \in \mathsf{Ev}$ abstracts from a security critical operation (e.g. writing a file, opening a socket connection). A *history* η is a sequence of access events. A *security policy* $\varphi \in \mathsf{Pol}$ is a regular property of histories. A *safety framing* $\varphi[e]$ enforces the policy φ at each step of the evaluation of e. A *liveness framing* $\psi\langle e\rangle$ requires that the policy ψ will eventually be satisfied while evaluating e. Services $e : \tau$ are typed λ^{req} expressions, collected in a trusted, finite, and global repository Srv, abstracting from the distributed service directories (e.g. UDDI [48]). The types τ are annotated with *history expressions* that over-approximate the possible run-time histories. E.g., when a function with type $\tau \xrightarrow{H} \tau'$ is applied to a value, it will generate one of the histories denoted by H. The repository Srv guarantees that H represents all the possible histories of e.

A *service request* has the form $\mathtt{req}_r\tau$. The label r uniquely identifies the request in an expression, and the request type τ is defined as:

$$\tau \quad ::= \quad \mathit{unit} \mid \tau \xrightarrow{(\varphi,\psi)} \tau$$

where *unit* is the standard singleton type. The annotations (φ, ψ) on the arrow are the safety/liveness constraints imposed on the service. Operationally, this drives a search in the repository Srv for a service with a functional type τ' "compatible" with τ (defined later on in Section 5) and such that τ' respects the constraints imposed by τ. For clarity, we omit the null constraint (tt, tt), we write $\varphi[\bullet]$ for (φ, tt), $\psi\langle\bullet\rangle$ for (tt, ψ) and $\varphi[\psi\langle\bullet\rangle]$ in the general case. Intuitively, a constraint can be seen as a context that wraps the behaviour H of a service, obtaining $\varphi[\psi\langle H\rangle]$, and meaning that the histories denoted by H must satisfy "always φ" and "eventually ψ".

We put some restrictions on the types a programmer can use in a request. First, only functional types are allowed: this models services being considered as remote procedures. Higher-order return values are allowed: if the type of a returned value is functional, then the request can be seen as a code download, e.g. an applet. Second, no constraints should be imposed over the type τ_0 of a request type $\tau_0 \xrightarrow{(\varphi,\psi)} \tau_1$, i.e. in τ_0 there are no annotations. This is because the constraints on the selected service should not affect its argument.

3.1 Syntax

The syntax of our calculus follows. We assume as given the languages for (regular) policies φ, ψ and for guards b. We omit their definition here, as they are not relevant for the subsequent technical development. To enhance readability, our calculus comprises conditional expressions and named abstractions (the variable z in $e' = \lambda_z x.\, e$ stands for e' itself within e).

Expressions

e, e'	$::=$	
	$*$	unit
	x	variable
	α	access event
	$\texttt{if}\, b\, \texttt{then}\, e\, \texttt{else}\, e$	conditional
	$\lambda_z x.\, e$	abstraction
	$e\, e'$	application
	$\varphi[e]$	safety framing
	$\psi\langle e \rangle$	liveness framing
	$\texttt{req}_r \tau$	service request

The values v of our calculus are the variables, the abstractions, the requests, and the distinguished element $*$. The following abbreviation is standard: $e; e' = (\lambda.\, e')\, e$. Without loss of generality, we assume that each framing has an opening event, e.g. for all $\varphi[e]$, the expression e has the form $\alpha; e'$, for some α and e'. The opening event can be dummy, with no influence on security.

3.2 Operational semantics

The evaluation of a λ^{req} expression requires to check all the policies within their framings, and to serve requests. In this section, we do not have yet an operational mechanism to resolve requests, and so we resort to an *oracle*. We assume that, upon a request $\texttt{req}_r \tau$, the oracle selects from Srv a service (if any) that respects the type and the constraints expressed by τ. The oracle guarantees that the execution of the selected service satisfies the safety/liveness policies in τ (although, e.g. it might violate a policy that was already active before the request). In the following sections, we will develop a static machinery that will enable us to efficiently implement the oracle, guaranteeing that an expression will never go wrong. Consequently, we will safely discard all the security framings, and avoid to check them dynamically.

Here, we are not interested in characterizing what the oracle guarantees, and we simply model it as a set of service choices, called *plan*. A plan formalises how a request is resolved into an actual service, and takes the form of a *finite* injective mapping from request labels to services. Plans have the following syntax:

Plans

π, π'	$::=$	
	0	empty
	$r[\ell]$	service choice
	$\pi \mid \pi'$	composition

The empty plan 0 has no choices; the plan $r[\ell]$ associates the service $e_\ell : \tau_\ell$ with the request labelled r. The composition operator $\mid$ on plans is associative, commutative and idempotent, and its identity is the empty plan 0.

Now we define the behaviour of expressions through the following small-step operational semantics. The configurations are pairs η, e and a transition $\eta, e \to_\pi \eta', e'$ means that, starting from a history η, the plan π allows the expression e to evolve to e' and to extend η to η'. An expression is initially evaluated starting from the empty history ε. We write $\eta \models \varphi$ when the history η obeys the (safety/liveness) policy φ. We assume as given a total function $\mathcal{B}$ that evaluates the guards in conditionals.

Operational semantics of λ^{req}

$$\frac{\eta, e_1 \to_\pi \eta', e_1'}{\eta, e_1 e_2 \to_\pi \eta', e_1' e_2} \quad \text{(E-App1)} \qquad \frac{\eta, e_2 \to_\pi \eta', e_2'}{\eta, v e_2 \to_\pi \eta', v e_2'} \quad \text{(E-App2)}$$

$$\eta, (\lambda_z x.\, e) v \to_\pi \eta, e\{v/x, \lambda_z x.\, e/z\} \quad \text{(E-AbsApp)}$$

$$\eta, \alpha \to_\pi \eta\alpha, * \quad \text{(E-Ev)} \qquad \eta, \mathtt{if}\ b\ \mathtt{then}\ e_{tt}\ \mathtt{else}\ e_{ff} \to_\pi \eta, e_{\mathcal{B}(b)} \quad \text{(E-If)}$$

$$\frac{\eta, e \to_\pi \eta', e' \quad \eta' \models \varphi}{\eta, \varphi[e] \to_\pi \eta', \varphi[e']} \quad \text{(E-SF1)} \qquad \frac{\eta \models \varphi}{\eta, \varphi[v] \to_\pi \eta, v} \quad \text{(E-SF2)}$$

$$\frac{\eta, e \to_\pi \eta', e' \quad \eta \not\models \psi}{\eta, \psi\langle e \rangle \to_\pi \eta', \psi\langle e' \rangle} \quad \text{(E-LF1)} \qquad \frac{\eta \models \psi}{\eta, \psi\langle e \rangle \to_\pi \eta, e} \quad \text{(E-LF2)}$$

$$\frac{e_\ell : \tau_\ell \in \mathsf{Srv} \quad \pi = r[\ell] \mid \pi'}{\eta, (\mathtt{req}_r \tau) v \to_\pi \eta, e_\ell\, v} \quad \text{(E-Req)}$$

The first two rules implement call-by-value evaluation; as usual, functions are not reduced within their bodies. The third rule implements β-reduction. Notice that the whole function body $\lambda_z x.\, e$ replaces the self variable z after the substitution, so giving an explicit copy-rule semantics to recursive functions. The evaluation of an event α consists in appending α to the current history, and producing the no-operation value $*$. A conditional $\mathtt{if}\ b\ \mathtt{then}\ e_{tt}\ \mathtt{else}\ e_{ff}$ evaluates to e_{tt} (resp. e_{ff}) if b evaluates to true (resp. false).

To evaluate a safety framing $\varphi[e]$, we must consider two cases. If, starting from the current history η, e may evolve to e' and extend the history to η', then the whole framing $\varphi[e]$ may evolve to $\varphi[e']$, provided that η' satisfies φ. Otherwise, if e is a value and the current history satisfies φ, then the scope of the framing is left. In both cases, as soon as a history is found not to respect φ, the evaluation gets stuck, to model a security exception. For simplicity, we do not model here exceptions and exception handling, but extending our language in this direction is straightforward.

Within a liveness framing $\psi\langle e \rangle$, the expression e evolves as long as the property ψ is not satisfied. As soon as the current history obeys ψ, the framing is discarded. Note that we cannot operationally guarantee that ψ will eventually

hold: indeed, this is a liveness property, so it cannot be enforced by execution monitoring alone [40]. It is then particularly relevant that our static analysis is able to discover whether a liveness framing will be eventually discarded or not.

The rule for service invocation enquires the oracle to select from Srv a service that respects the types and the required constraints. If no such service exists, the execution gets stuck.

4 History expressions

To statically predict the histories generated by programs at run-time, as well as the scopes of policies, we introduce *history expressions* with the following abstract syntax. History expressions are a sort of context-free grammars, and include the empty history ε, access events α, sequencing $H \cdot H'$, non-deterministic choice $H + H'$, safety and liveness framings $\varphi[H]$ and $\psi\langle H\rangle$, recursion $\mu h.H$ (μ binds the occurrences of the variable h in H), and planned selection.

History Expressions

H, H'	$::=$	
	ε	empty
	h	variable
	α	access event
	$H \cdot H'$	sequence
	$H + H'$	choice
	$\varphi[H]$	safety framing
	$\psi\langle H\rangle$	liveness framing
	$\mu h.H$	recursion
	$\{\pi_1 \triangleright H_1 \cdots \pi_k \triangleright H_k\}$	planned selection

Safety and liveness framings are the abstract counterparts of the analogous constructs in λ^{req}. Given a plan π, a planned selection $\{\pi_1 \triangleright H_1 \cdots \pi_k \triangleright H_k\}$ chooses those H_i such that π includes π_i. Intuitively, the history expression $H = \{r[\ell_1] \triangleright H_1, r[\ell_2] \triangleright H_2\}$ is associated with a request r that can be resolved into either e_{ℓ_1} or e_{ℓ_2}. The histories denoted by H depend on the given plan π: if π chooses ℓ_1 (resp. ℓ_2) for r, then H denotes one of the histories represented by H_1 (resp. H_2); otherwise, H denotes no histories. Due to our assumption on the form of plans, we always discard the components $\pi_i \triangleright H_i$ from a planned selection, if π_i turns out to be non-injective.

We assume that the operator $\cdot$ has precedence over $+$, that in turn has precedence over μ. We say that a history expression H is *closed* when it has no free variables, i.e. $fv(H) = \emptyset$, where free variables are defined as expected, e.g.:

$$fv(h) = \{h\} \qquad fv(\mu h.\, H) = fv(H) \setminus \{h\}$$

$$fv(\{\pi_1 \triangleright H_1 \cdots \pi_k \triangleright H_k\}) = \textstyle\bigcup_{i \in 1..k} fv(H_i)$$

To define the semantics of history expressions, we enrich histories with a set Frm of special *framing events*, parametrized by policies in Pol. The events $[_\varphi$ and $]_\varphi$ denote the opening and closing of a safety framing $\varphi[\cdots]$, while $\langle_\psi$ and $\rangle_\psi$ play

the same role for liveness framings. Formally, a *history* η is a sequence $\beta_1 \cdots \beta_k$ where $\beta_i \in \mathsf{Ev} \cup \mathsf{Frm}$, $\mathsf{Frm} = \{\, [_\varphi,]_\varphi, \langle_\varphi, \rangle_\varphi \mid \varphi \in \mathsf{Pol} \,\}$, and $\mathsf{Ev} \cap \mathsf{Frm} = \emptyset$.

For example, a history $\alpha[_\varphi \alpha']_\varphi$ represents a computation that (i) generates an event α, (ii) enters the scope of the safety framing $\varphi[\cdots]$, (iii) generates α' within the scope of φ, and (iv) leaves the scope of φ. Note that histories with events in Ev only were enough to give the operational semantics of our calculus, because the role of framing events is played by framed expressions.

Hereafter, a history may end with the truncation marker !. The history $\eta!$ represents a prefix of a possibly non-terminating computation that generates the sequence of events η. We assume that histories are undistinguishable after truncation, i.e. $\eta!$ followed by η' equals to $\eta!$. A history η is *balanced* when either η is empty, or η is an access event, or $\eta =!$, or $\eta = [_\varphi \eta']_\varphi$ with η' balanced, or $\langle_\psi \eta' \rangle_\psi$ with η' balanced, or $\eta = \eta'\eta''$ with both η' and η'' balanced. A history is *well-formed* when it is the prefix of a balanced history. For example, $\alpha[_\varphi \alpha'[_{\varphi'} \alpha'']_{\varphi'}]_\varphi$ is balanced, $\alpha[_\varphi \alpha'$ is well-formed but not balanced, and $\alpha[_\varphi \alpha'[_{\varphi'} \alpha'']_\varphi$ is not well-formed. Hereafter, we will only deal with well-formed histories, because they model exactly the histories that will show up when executing λ^{req} expressions. Let $\mathcal{H}$ range over sets of balanced histories. We define $\mathcal{H}\mathcal{H}'$ as the set of histories $\{\, \eta\eta' \mid \eta \in \mathcal{H}, \eta' \in \mathcal{H}' \,\}$, $\varphi[\mathcal{H}]$ as $\{\, [_\varphi \eta]_\varphi \mid \eta \in \mathcal{H} \,\}$, and $\psi\langle\mathcal{H}\rangle$ as the set $\{\, \langle_\psi \eta \rangle_\psi \mid \eta \in \mathcal{H} \,\}$.

The *denotational semantics* of history expressions is defined over the lifted cpo of sets of balanced histories [49], ordered by (lifted) set inclusion $\subseteq_\perp$, where, for all balanced histories $\mathcal{H}, \mathcal{H}'$, $\perp \subseteq_\perp \mathcal{H}$, and $\mathcal{H} \subseteq_\perp \mathcal{H}'$ if $\mathcal{H} \subseteq \mathcal{H}'$. The least upper bound between two elements of the cpo is standard set union $\cup$, assuming that $\perp \cup \mathcal{H} = \mathcal{H}$. The *strict* least upper bound is denoted by $\cup_\perp$, and it is such that $\perp \cup_\perp \mathcal{H} = \perp$. We stipulate that concatenation of sets of histories is strict, i.e. it returns $\perp$ whenever one of its arguments is such.

The semantics $[\![H]\!]^\pi_\rho$ of a history expression H (in an environment ρ and under a plan π) is defined by the following rules. The environment ρ maps variables to sets of balanced histories. Hereafter, we feel free to omit curly braces when writing singleton sets, and we omit the empty environment for closed expressions.

Semantics of history expressions

$$[\![\varepsilon]\!]^\pi_\rho = \varepsilon \qquad [\![H \cdot H']\!]^\pi_\rho = [\![H]\!]^\pi_\rho \, [\![H']\!]^\pi_\rho \qquad [\![H + H']\!]^\pi_\rho = [\![H]\!]^\pi_\rho \cup_\perp [\![H']\!]^\pi_\rho$$

$$[\![\alpha]\!]^\pi_\rho = \alpha \qquad [\![\varphi[H]]\!]^\pi_\rho = \varphi[[\![H]\!]^\pi_\rho] \qquad [\![\psi\langle H\rangle]\!]^\pi_\rho = \psi\langle[\![H]\!]^\pi_\rho\rangle$$

$$[\![h]\!]^\pi_\rho = \rho(h) \qquad [\![\mu h.H]\!]^\pi_\theta = \bigcup_{n>0} f^n(!) \quad \text{where } f(X) = [\![H]\!]^\pi_{\theta\{X/h\}}$$

$$[\![\{\}]\!]^\pi_\rho = \perp \qquad [\![\{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}]\!]^\pi_\rho = \textstyle\bigcup_{i\in 1..k} [\![\{\pi_i \rhd H_i\}]\!]^\pi_\rho$$

$$[\![\{0 \rhd H\}]\!]^\pi_\rho = [\![H]\!]^\pi_\rho \qquad [\![\{\pi_0 \mid \pi_1 \rhd H\}]\!]^\pi_\rho = [\![\{\pi_0 \rhd H\}]\!]^\pi_\rho \cup_\perp [\![\{\pi_1 \rhd H\}]\!]^\pi_\rho$$

$$[\![\{r[\ell] \rhd H\}]\!]^\pi_\rho = \begin{cases} [\![H]\!]^\pi_\rho & \text{if } \pi = r[\ell] \mid \pi' \\ \perp & \text{otherwise} \end{cases}$$

Example 1. Consider $H = \mu h.\, \alpha + h \cdot h + \varphi[h]$. For all plans π, the semantics of H consists of all the (possibly truncated) histories having an arbitrary number of occurrences of α, and arbitrarily nested, balanced safety framings of φ. For instance, $\varphi[\alpha]\varphi[\alpha\varphi[\alpha]] \in [\![H]\!]^\pi$, for all plans π. □

Example 2. Let $H = \{r[\ell_1] \triangleright \alpha_1 \cdot \{r'[\ell'_1] \triangleright \beta_1, r'[\ell'_2] \triangleright \beta_2\},\ r[\ell_2] \triangleright \alpha_2\}$, and let $\pi = r[\ell_1] \mid r'[\ell'_2]$. Then:

$$\begin{aligned}
[\![H]\!]^\pi &= [\![\{r[\ell_1] \triangleright \alpha_1 \cdot \{r'[\ell'_1] \triangleright \beta_1, r'[\ell'_2] \triangleright \beta_2\}\}]\!]^\pi \cup [\![\{r[\ell_2] \triangleright \alpha_2\}]\!]^\pi \\
&= [\![\alpha_1 \cdot \{r'[\ell'_1] \triangleright \beta_1, r'[\ell'_2] \triangleright \beta_2\}]\!]^\pi \cup \bot \\
&= [\![\alpha_1]\!]^\pi [\![\{r'[\ell'_1] \triangleright \beta_1, r'[\ell'_2] \triangleright \beta_2\}]\!]^\pi \\
&= \{\alpha_1\}([\![\{r'[\ell'_1] \triangleright \beta_1\}]\!]^\pi \cup [\![\{r'[\ell'_2] \triangleright \beta_2\}]\!]^\pi) \\
&= \{\alpha_1\}(\bot \cup [\![\beta_2]\!]^\pi) \ = \ \{\alpha_1\}\{\beta_2\} \ = \ \{\alpha_1\beta_2\}
\end{aligned}$$ □

Example 3. Let $H = \alpha + \{r[\ell] \triangleright \beta\}$, and let $\pi = 0$. Then:

$$[\![H]\!]^\pi \ = \ [\![\alpha]\!]^\pi \cup_\bot [\![\{r[\ell] \triangleright \beta\}]\!]^\pi \ = \ \{\alpha\} \cup_\bot \bot \ = \ \bot$$

This example models a situation where, e.g. H has been extracted from an expression $\texttt{if}\, b\, \texttt{then}\, \alpha\, \texttt{else}\, (\texttt{req}_r \tau)*$, and e_ℓ is the only service whose type is compatible with τ. The semantics of H is undefined (i.e. $\bot$) because, in case b is false, the plan $\pi = 0$ is not able to choose any of the proposed services. □

4.1 Validity

We now define when histories and history expressions are valid. Intuitively, valid histories represent viable computations. Instead, invalid ones happen to violate some security constraints, so they are going to be identified and rejected by our static analysis. For example, consider the history $\eta_0 = \alpha_c\alpha_r\varphi[\alpha_c]$, where φ requires that no α_c occurs after α_r (see Section 2). Then, η_0 is *not* valid according to our intended meaning, because the rightmost α_c occurs within a safety framing enforcing φ, and $\alpha_c\alpha_r\alpha_c$ does not obey φ. Consider now the history $\eta_1 = \alpha\psi\langle\alpha\rangle\alpha_{sgn}$, where ψ requires that eventually α_{sgn}. Then, η_1 is *not* valid, because the event α_{sgn} occurs after the liveness framing has been closed.

Note that our notion of validity ensures that, at each step of execution, the policies enforced by safety and liveness framings can always inspect the *whole* history generated so far. This is motivated by our basic assumption that no events can be hidden. For example, the history $\alpha_0\alpha_1\varphi[\alpha_2]\alpha_3$ is valid when $\alpha_0\alpha_1 \models \varphi$ (to make sure φ is satisfied while entering the framing) and $\alpha_0\alpha_1\alpha_2 \models \varphi$ (even if α_0 and α_1 are outside of the safety framing), while $\alpha_0\alpha_1\alpha_2\alpha_3$ is not required to satisfy φ any longer. However, this is not a limitation, as shown in the discussion preceding Lemma 8 below.

To give a formal definition of validity, we introduce the notion of safe and live sets (S- and L-sets for short), which are sets of histories. For example, the history η_0 above has one S-set $\varphi[\{\alpha_c\alpha_r, \alpha_c\alpha_r\alpha_c\}]$. Intuitively, this means that the scope of the framing $\varphi[\cdots]$ spans over the histories $\alpha_c\alpha_r$ and $\alpha_c\alpha_r\alpha_c$. For each S-set of the form $\varphi[\mathcal{H}]$, validity requires that *all* the histories in $\mathcal{H}$ obey φ. Similarly, η_1 has one L-set, $\psi\langle\{\alpha, \alpha\alpha\}\rangle$. For each L-set $\psi\langle\mathcal{H}\rangle$, validity requires that *at least one* of the histories in $\mathcal{H}$ satisfies ψ.

Some notations are now needed. Let $\eta^\flat$ be the history obtained from η by erasing all the framing events, and let η^∂ be the set of all the prefixes of η, including the empty history ε. For example, if $\eta_0 = \alpha_c\alpha_r\varphi[\alpha_c]$, then $(\eta_0^\flat)^\partial = ((\alpha_c\alpha_r[_\varphi\alpha_c]_\varphi)^\flat)^\partial = (\alpha_c\alpha_r\alpha_c)^\partial = \{\varepsilon, \alpha_c, \alpha_c\alpha_r, \alpha_c\alpha_r\alpha_c\}$.

Let η be a (well-formed) history. To have a short definition of S-set and L-set, it is convenient to balance all the safety framings of η, e.g. $[_\varphi\alpha$ becomes $[_\varphi\alpha]_\varphi = \varphi[\alpha]$. Then, the S-set $S(\eta)$, the L-set $L(\eta)$, and validity of histories and history expressions are defined as follows:

Safe/Live sets and validity

$$
\begin{array}{ll}
S(\varepsilon) = \emptyset & L(\varepsilon) = \emptyset \\
S(\eta\,\alpha) = S(\eta\,\langle_\psi) = S(\eta\,\rangle_\psi) = S(\eta) & L(\eta\,\alpha) = L(\eta\,[_\varphi) = L(\eta\,]_\varphi) = L(\eta) \\
S(\eta_0\,\varphi[\eta_1]) = S(\eta_0\,\eta_1) \cup \varphi[\eta_0^\flat\,(\eta_1^\flat)^\partial] & L(\eta_0\,\psi\langle\eta_1\rangle) = L(\eta_0\,\eta_1) \cup \psi\langle\eta_0^\flat\,(\eta_1^\flat)^\partial\rangle
\end{array}
$$

A history η is *valid* ($\models \eta$ in symbols) when:

$$
\begin{array}{rcl}
\varphi[\mathcal{H}] \in S(\eta) & \Longrightarrow & \forall \eta' \in \mathcal{H}.\, \eta' \models \varphi \\
\psi\langle\mathcal{H}\rangle \in L(\eta) & \Longrightarrow & \exists \eta' \in \mathcal{H}.\, \eta' \models \psi
\end{array}
$$

A history expression H is π-*valid* when $[\![H]\!]^\pi \neq \bot$ and $\eta \in [\![H]\!]^\pi \Longrightarrow \models \eta$

Example 4. Consider again the history $\eta_1 = \alpha\psi\langle\alpha\rangle\alpha_{sgn}$. We have that:

$$
\begin{array}{rl}
L(\eta_1) & = L(\alpha\psi\langle\alpha\rangle\alpha_{sgn}) \\
& = L(\alpha\psi\langle\alpha\rangle) = L(\alpha\alpha) \cup \psi\langle\alpha^\flat(\alpha^\flat)^\partial\rangle \\
& = \emptyset \cup \psi\langle\alpha\{\varepsilon, \alpha\}\rangle = \psi\langle\{\alpha, \alpha\alpha\}\rangle
\end{array}
$$

Since neither $\alpha \models \psi$ nor $\alpha\alpha \models \psi$, then η_1 is not valid. Consider now the history:

$$\eta = \langle_\psi[_\varphi\alpha_1]_\varphi\langle_\psi\alpha_2\rangle_\psi\alpha_3[_{\varphi'}\alpha_4$$

Then, after rewriting η as $\eta]_{\varphi'}$ to balance the safety framings, we have:

$$
\begin{array}{l}
S(\eta) = \{\ \varphi[\{\varepsilon, \alpha_1\}], \varphi'[\{\alpha_1\alpha_2\alpha_3, \alpha_1\alpha_2\alpha_3\alpha_4\}]\ \} \\
L(\eta) = \{\ \psi\langle\{\alpha_1, \alpha_1\alpha_2\}\rangle\ \}
\end{array}
$$

□

5 Type and effect system

We now introduce a type and effect system for our calculus, building upon [4, 42]. Types and type environments, ranged over by τ and Γ, are mostly standard and are defined in the following table. The history expression H in the functional type $\tau \xrightarrow{H} \tau'$ describes the latent effect associated with an abstraction, i.e. one of the histories represented by H is generated when a value is applied to an abstraction with that type. Note that we overload the symbol τ to range over both expression types and request types $\tau \xrightarrow{\varphi[\psi\langle\bullet\rangle]} \tau'$.

Types and Type Environments

$$\tau, \tau' ::= \mathit{unit} \mid \tau \xrightarrow{H} \tau' \qquad \Gamma ::= \emptyset \mid \Gamma; x : \tau \quad (x \notin \mathit{dom}(\Gamma))$$

A typing judgment $\Gamma, H \vdash_{\mathsf{Srv}} e : \tau$ means that, given a service repository Srv (omitted in the index, when immaterial), the expression e evaluates to a value of type τ, and produces a history belonging to the effect H. The relation $\Gamma, H \vdash_{\mathsf{Srv}} e : \tau$ is defined as the least relation closed under the rules below. Typing judgments are similar to those of the simply-typed λ-calculus. The effects in the rule for application are concatenated according to the evaluation order of the call-by-value semantics (function, argument, latent effect). The actual effect of an abstraction is the empty history expression, while the latent effect is equal to the actual effect of the function body. The rule for abstraction constraints the premise to equate the actual and latent effects, up to associativity, commutativity, idempotency and zero of $+$, associativity and zero of $\cdot$, α-conversion, unfolding of recursion, and elimination of vacuous μ-binders. The last rule allows for *weakening* of effects. A service invocation $\mathtt{req}_r\tau$ has an empty actual effect, and a functional type τ', whose latent effect is a planned selection that picks from Srv those services matching the constraints on the request type τ. A more detailed explanation will follow after Example 5.

Typing relation

$$\Gamma, \varepsilon \vdash * : \mathit{unit} \quad \text{(T-Unit)} \qquad \Gamma, \alpha \vdash \alpha : \mathit{unit} \quad \text{(T-Ev)}$$

$$\Gamma, \varepsilon \vdash x : \Gamma(x) \quad \text{(T-Var)} \qquad \frac{\Gamma; x : \tau; z : \tau \xrightarrow{H} \tau', H \vdash e : \tau'}{\Gamma, \varepsilon \vdash \lambda_z x.\, e : \tau \xrightarrow{H} \tau'} \quad \text{(T-Abs)}$$

$$\frac{\Gamma, H \vdash e : \tau \xrightarrow{H''} \tau' \quad \Gamma, H' \vdash e' : \tau}{\Gamma, H \cdot H' \cdot H'' \vdash e\, e' : \tau'} \quad \text{(T-App)}$$

$$\frac{\Gamma, H \vdash e : \tau}{\Gamma, \varphi[H] \vdash \varphi[e] : \tau} \quad \text{(T-SF)} \qquad \frac{\Gamma, H \vdash e : \tau}{\Gamma, \psi\langle H\rangle \vdash \psi\langle e\rangle : \tau} \quad \text{(T-LF)}$$

$$\frac{\tau' = \biguplus\{\, \tau \oplus_{r[\ell]} \tau_\ell \mid e_\ell : \tau_\ell \in \mathsf{Srv} \wedge \tau_\ell \approx \tau \,\}}{\Gamma, \varepsilon \vdash_{\mathsf{Srv}} \mathtt{req}_r\tau : \tau'} \quad \text{(T-Req)}$$

$$\frac{\Gamma, H \vdash e : \tau \quad \Gamma, H \vdash e' : \tau}{\Gamma, H \vdash \mathtt{if}\ b\ \mathtt{then}\ e\ \mathtt{else}\ e' : \tau} \quad \text{(T-If)} \qquad \frac{\Gamma, H \vdash e : \tau}{\Gamma, H + H' \vdash e : \tau} \quad \text{(T-Wkn)}$$

Example 5. Consider the following expression:

$$e \;=\; \mathtt{if}\ b\ \mathtt{then}\ \lambda_z x.\, \alpha\ \mathtt{else}\ \lambda_z x.\, \alpha'$$

Let $\tau = \mathit{unit}$, and $\Gamma = \{z : \tau \xrightarrow{\alpha+\alpha'} \tau; x : \tau\}$. Then, the following typing

derivation is possible:

$$\dfrac{\dfrac{\dfrac{\Gamma, \alpha \vdash \alpha : \tau}{\Gamma, \alpha + \alpha' \vdash \alpha : \tau}}{\emptyset, \varepsilon \vdash \lambda_z x.\, \alpha : \tau \xrightarrow{\alpha+\alpha'} \tau} \quad \dfrac{\dfrac{\Gamma, \alpha' \vdash \alpha' : \tau}{\Gamma, \alpha' + \alpha \vdash \alpha' : \tau}}{\emptyset, \varepsilon \vdash \lambda_z x.\, \alpha' : \tau \xrightarrow{\alpha'+\alpha} \tau}}{\emptyset, \varepsilon \vdash \mathtt{if}\ b\ \mathtt{then}\ \lambda_z x.\, \alpha\ \mathtt{else}\ \lambda_z x.\, \alpha' : \tau \xrightarrow{\alpha+\alpha'} \tau}$$

Note that we can equate the history expressions $\alpha + \alpha'$ and $\alpha' + \alpha$, because $+$ is commutative. The typing derivation above shows the use of the weakening rule to unify the latent effects on arrow types. Let now:

$$e' \;=\; \lambda_w x.\, \mathtt{if}\ b'\ \mathtt{then}\ *\ \mathtt{else}\ w(e\,x)$$

Let $\Gamma = \{w : \tau \xrightarrow{H} \tau, x : \tau\}$, where H is left undefined. Then, recalling that $\varepsilon \cdot H' = H' = H' \cdot \varepsilon$ for any history expression H', we have:

$$\dfrac{\Gamma, \varepsilon \vdash * : \tau \qquad \dfrac{\dfrac{\Gamma, \varepsilon \vdash w : \tau \xrightarrow{H} \tau \quad \dfrac{\Gamma, \varepsilon \vdash e : \tau \xrightarrow{\alpha+\alpha'} \tau \quad \Gamma, \varepsilon \vdash x : \tau}{\Gamma, \alpha + \alpha' \vdash e\,x : \tau}}{\Gamma, (\alpha + \alpha') \cdot H \vdash w(e\,x) : \tau}}{\Gamma, \varphi[(\alpha + \alpha') \cdot H] \vdash \varphi[w(e\,x)] : \tau}}{\Gamma, \varepsilon + \varphi[(\alpha + \alpha') \cdot H] \vdash \mathtt{if}\ b'\ \mathtt{then}\ *\ \mathtt{else}\ \varphi[w(e\,x)] : \tau}$$

To apply the typing rule for abstractions, the constraint $H = \varepsilon + \varphi[(\alpha + \alpha') \cdot H]$ must be solved. Let $H = \mu h.\, \varepsilon + \varphi[(\alpha + \alpha') \cdot h]$. It is easy to prove that:

$$[\![H]\!] \;=\; [\![\varepsilon + \varphi[(\alpha + \alpha') \cdot h]]\!]_{\{[\![H]\!]/h\}} \;=\; \{\varepsilon\} \cup \varphi[(\alpha + \alpha') \cdot [\![H]\!]]$$

We have then found a solution to the constraint above, so we can conclude that:

$$\emptyset, \varepsilon \vdash e' : \tau \xrightarrow{\mu h.\, \varepsilon + \varphi[(\alpha+\alpha') \cdot h]} \tau$$

Note in passing that a simple extension of the type inference algorithm of [42] suffices for solving constraints as the one above. The main difference concerns planned selections. To deal with that, our algorithm uses a straightforward implementation of the rule (T-Req) □

To give a type to requests, we need to define the auxiliary operators $\approx$, $\oplus$ and $\Cup$. We introduce them below, with the help of a running example.

We write $\tau \approx \tau'$, and say τ, τ' *compatible*, whenever, omitting the annotations on the arrows, τ and τ' are equal. Formally:

$$\mathit{unit} \approx \mathit{unit} \qquad (\tau_0 \xrightarrow{X} \tau_1) \approx (\tau_0' \xrightarrow{Y} \tau_1') \quad \text{iff } \tau_0 \approx \tau_0' \text{ and } \tau_1 \approx \tau_1'$$

Example 6. Let $\tau = \mathit{unit}$, let $e = \mathtt{req}_r \tau_r$, where $\tau_r = (\tau \to \tau) \xrightarrow{\varphi[\bullet]} (\tau \xrightarrow{\psi\langle\bullet\rangle} \tau)$, and let $\mathsf{Srv} = \{e_{\ell_1} : \tau_{\ell_1}, e_{\ell_2} : \tau_{\ell_2}\}$, where $\tau_{\ell_i} = (\tau \xrightarrow{h_i} \tau) \xrightarrow{\alpha_i \cdot h_i} (\tau \xrightarrow{\beta_i} \tau)$ for $i \in 1..2$. We have that $\tau_r \approx \tau_{\ell_1} \approx \tau_{\ell_2}$, i.e. both the services in Srv are compatible with the request in e. □

The operator $\oplus_{r[\ell]}$ combines a request type τ and a service type τ', when they are compatible. Given a request type $\hat{\tau} = \tau_0 \xrightarrow{\varphi[\psi\langle\bullet\rangle]} \tau_1$ and a service type $\hat{\tau}' = \tau'_0 \xrightarrow{H} \tau'_1$, the result of $\hat{\tau} \oplus_{r[\ell]} \hat{\tau}'$ is $\tau'_0 \xrightarrow{\{r[\ell]\rhd\varphi[\psi\langle H\rangle]\}} (\tau_1 \oplus'_{r[\ell]} \tau'_1)$, where:

$$\begin{array}{l} \mathit{unit} \oplus'_{r[\ell]} \mathit{unit} = \mathit{unit} \\ (\tau_0 \xrightarrow{\varphi[\psi\langle\bullet\rangle]} \tau_1) \oplus'_{r[\ell]} (\tau'_0 \xrightarrow{H} \tau'_1) = (\tau_0 \oplus'_{r[\ell]} \tau'_0) \xrightarrow{\{r[\ell]\rhd\varphi[\psi\langle H\rangle]\}} (\tau_1 \oplus'_{r[\ell]} \tau'_1) \end{array}$$

Note that combining functional types does not affect the type of the argument. This reflects the intuition that the type of the argument to be passed to the selected service cannot be constrained by the request.

Example 6 (cont.). The request type τ_r is composed with the service types in Srv as follows:

$$\begin{array}{l} \tau_r \oplus_{r[\ell_1]} \tau_{\ell_1} = (\tau \xrightarrow{h_1} \tau) \xrightarrow{\{r[\ell_1]\rhd\varphi[\alpha_1\cdot h_1]\}} (\tau \xrightarrow{\{r[\ell_1]\rhd\psi\langle\beta_1\rangle\}} \tau) \\ \tau_r \oplus_{r[\ell_2]} \tau_{\ell_2} = (\tau \xrightarrow{h_2} \tau) \xrightarrow{\{r[\ell_2]\rhd\varphi[\alpha_2\cdot h_2]\}} (\tau \xrightarrow{\{r[\ell_2]\rhd\psi\langle\beta_2\rangle\}} \tau) \end{array} \qquad \square$$

Eventually, the operator $\Cup$ combines the types obtained by combining the request type with the service types. Given two compatible types $\hat{\tau} = \tau_0 \xrightarrow{H} \tau_1$ and $\hat{\tau}' = \tau'_0 \xrightarrow{H'} \tau'_1$, the result of $\hat{\tau} \Cup \hat{\tau}'$ is $\tau''_0 \xrightarrow{H\cup H'} (\tau_1\sigma \Cup' \tau'_1\sigma)$, where σ unifies τ_0 and τ'_0 (i.e. $\tau_0\sigma = \tau'_0\sigma = \tau''_0$), and:

$$\begin{array}{l} \mathit{unit} \Cup' \mathit{unit} = \mathit{unit} \\ (\tau_0 \xrightarrow{H} \tau_1) \Cup' (\tau'_0 \xrightarrow{H'} \tau'_1) = (\tau_0 \Cup' \tau'_0) \xrightarrow{H\cup H'} (\tau_1 \Cup' \tau'_1) \end{array}$$

Example 6 (cont.). Now, we can unify the combination of the request type τ_r with the service types, obtaining:

$$\tau' \;=\; (\tau \xrightarrow{h} \tau) \xrightarrow{\{r[\ell_1]\rhd\varphi[\alpha_1\cdot h],\, r[\ell_2]\rhd\varphi[\alpha_2\cdot h]\}} (\tau \xrightarrow{\{r[\ell_1]\rhd\psi\langle\beta_1\rangle,\, r[\ell_2]\rhd\psi\langle\beta_2\rangle\}} \tau)$$

where $\sigma = \{h_1/h, h_2/h\}$ is the selected unifier between $\tau \xrightarrow{h_1} \tau$ and $\tau \xrightarrow{h_2} \tau$. $\square$

The following example further illustrates how requests and services are typed.

Example 7. Let e and Srv as in Example 6, and consider the expression $(e(\lambda.\gamma))*$. Note that applying (any service resulting from) the request to the function $\lambda.\gamma$ yields a new function, which we eventually apply to the value $*$. We have the following typing derivation, where we omit the component $\Gamma = \emptyset$:

$$\dfrac{\dfrac{\varepsilon \vdash e : \tau' \qquad \varepsilon \vdash (\lambda.\gamma) : \tau \xrightarrow{\gamma} \tau}{\{r[\ell_1] \rhd \varphi[\alpha_1\cdot\gamma],\ r[\ell_2] \rhd \varphi[\alpha_2\cdot\gamma]\} \vdash e(\lambda.\gamma) : \tau \xrightarrow{\{r[\ell_1]\rhd\psi\langle\beta_1\rangle,\, r[\ell_2]\rhd\psi\langle\beta_2\rangle\}} \tau}}{\{r[\ell_1] \rhd \varphi[\alpha_1\cdot\gamma],\ r[\ell_2] \rhd \varphi[\alpha_2\cdot\gamma]\} \cdot \{r[\ell_1] \rhd \psi\langle\beta_1\rangle,\ r[\ell_2] \rhd \psi\langle\beta_2\rangle\} \vdash (e(\lambda.\gamma))* : \tau}$$

Evaluating the resulting history expression H with the plan $\pi = r[\ell_1]$ yields:

$$\begin{array}{l} [\![H]\!]^\pi = [\![\{r[\ell_1] \rhd \varphi[\alpha_1\cdot\gamma],\ r[\ell_2] \rhd \varphi[\alpha_2\cdot\gamma]\}]\!]^\pi \cdot [\![\{r[\ell_1] \rhd \psi\langle\beta_1\rangle,\ r[\ell_2] \rhd \psi\langle\beta_2\rangle\}]\!]^\pi \\ \qquad = (\{\varphi[\alpha_1\gamma]\} \cup \bot) \cdot (\{\psi\langle\beta_1\rangle\} \cup \bot) \;=\; \{\varphi[\alpha_1\gamma]\ \psi\langle\beta_1\rangle\} \end{array} \qquad \square$$

A plan is *well-typed* if, when it associates a service to a request $\mathtt{req}_r\tau$, then the type of the service is compatible with the type of the request:

$$\pi = r[\ell] \mid \pi' \ \wedge \ e_\ell : \tau_\ell \in \mathsf{Srv} \implies \tau_\ell \approx \tau$$

The next theorem states that our type and effect system over-approximates the actual run-time histories, for a given expression and a given service repository Srv. If H is the effect of e, then the histories η generated when evaluating e under the plan π are the prefixes of the histories in $[\![H]\!]^\pi$, stripped of all the framing events, i.e. $\eta \in ([\![H]\!]^\pi)^{\flat\partial}$. As usual, precision is lost when reducing the conditional construct to non-determinism, and when dealing with recursive functions (see Example 5). Additionally, we over-approximate the set of services satisfying the calling requirements.

Theorem 1. Given a service repository Srv, if $\Gamma, H \vdash_{\mathsf{Srv}} e : \tau$ and $\varepsilon, e \to_\pi^* \eta, e'$ (with π well-typed), then $\eta \in ([\![H]\!]^\pi)^{\flat\partial}$.

Example 8. Let $e = \alpha_0; \varphi[\mathtt{if}\ b\ \mathtt{then}\ \alpha_1\ \mathtt{else}\ \alpha_2]$, with φ requiring "never α_2". Assume that the guard b always evaluates to true. Then, for any plan π, we have the following computation:

$$\varepsilon, e \to_\pi \alpha_0, \varphi[\mathtt{if}\ b\ \mathtt{then}\ \alpha_1\ \mathtt{else}\ \alpha_2] \to_\pi \alpha_0, \varphi[\alpha_1] \to_\pi \alpha_0\alpha_1, \varphi[*] \to_\pi \alpha_0\alpha_1, *$$

The history expression extracted from e is $H = \alpha_0 \cdot \varphi[\alpha_1 + \alpha_2]$. Then, for all plans π, $[\![H]\!]^\pi = \{\alpha_0[_\varphi\alpha_1]_\varphi, \alpha_0[_\varphi\alpha_2]_\varphi\}$, and:

$$(([\![H]\!]^\pi)^\flat)^\partial = \{\alpha_0\alpha_1, \alpha_0\alpha_2\}^\partial = \{\varepsilon, \alpha_0, \alpha_0\alpha_1, \alpha_0\alpha_2\}$$

which (strictly) contains all the run-time histories in the computation above. □

We can now state the type safety property. A plan π is *viable* for e when the reduction of e with plan π does not go wrong, i.e. $\varepsilon, e \to_\pi^* \eta', e'$, and either e' is a value, or there exists a transition $\eta', e' \to_\pi \eta'', e''$ for some η'', e''. For example, a computation goes wrong when attempting to execute an event forbidden by a currently active policy, or when the plan π offers no choices for a request.

Theorem 2 (Type Safety). Given a service repository Srv, let $\Gamma, H \vdash_{\mathsf{Srv}} e : \tau$, with e closed. If H is π-valid for some well-typed plan π, then π is viable for e.

Example 9. Let e and Srv as in Example 6, and let $e_{\ell_i} = \lambda x.\,(\alpha_i; (x*); (\lambda.\beta_i))$ for $i \in 1..2$. Assume the constraints φ, ψ on the service request in e are such that φ is true, and ψ requires "eventually β_1". Consider now the expression $e' = \varphi_0[(e(\lambda.\gamma))*]$, where φ_0 requires "never α_2". Let $\pi = r[\ell_1]$. The history expression $\varphi_0[H]$ of e' (where H has been inferred for (e $(\lambda.\gamma))*$ in Example 7) is π-valid, because $[\![\varphi_0[H]]\!]^\pi = \{\varphi_0[\varphi[\alpha_1\gamma]\psi\langle\beta_1\rangle]\}$ contains exactly one valid history. We have exactly one computation of e' with plan π, and, as predicted by Theorem 2, it does not go wrong:

$$\begin{aligned}
\varepsilon, \varphi_0[(e(\lambda.\gamma))*] &\to_\pi \varepsilon, \varphi_0[(e_{\ell_1}(\lambda.\gamma))*] \ \to_\pi \ \varepsilon, \varphi_0[(\alpha_1; (\lambda.\gamma)*; (\lambda.\beta_1))*] \\
&\to_\pi \alpha_1, \varphi_0[(\lambda.\gamma)*; (\lambda.\beta_1))*] \ \to_\pi \ \alpha_1, \varphi_0[(\gamma; (\lambda.\beta_1))*] \\
&\to_\pi \alpha_1\gamma, \varphi_0[(\lambda.\beta_1)*] \ \to_\pi \ \alpha_1\gamma, \varphi_0[\beta_1] \ \to_\pi \ \alpha_1\gamma\beta_1, \varphi_0[*] \\
&\to_\pi \alpha_1\gamma\beta_1, *
\end{aligned}$$

Consider now the plan $\pi' = r[\ell_2]$. Then H is not π'-valid, because e.g. the event α_2 violates φ_0. In this case the computation of e':

$$\varepsilon, \varphi_0[(e(\lambda.\gamma))*] \to_{\pi'} \varepsilon, \varphi_0[(e_{\ell_2}(\lambda.\gamma))*] \to_{\pi'} \varepsilon, \varphi_0[(\alpha_2; (\lambda.\gamma)*; (\lambda.\beta_1))*]$$

is correctly aborted, because $\alpha_2 \not\models \varphi_0$. So, π' is not viable for e'. □

Validity of history expressions also guarantees an additional correctness property about the "liveness" of a service. Roughly, it says that if $\psi\langle e\rangle$ has a valid effect, then e will eventually escape the liveness framing, i.e. the policy ψ will be eventually obeyed. We need the notion of evaluation context $\mathcal{C}(\bullet)$, defined by the following grammar: $\bullet \mid v\,\mathcal{C}(\bullet) \mid \mathcal{C}(\bullet)\,e \mid \varphi[\mathcal{C}(\bullet)] \mid \psi\langle\mathcal{C}(\bullet)\rangle$. Additionally, a λ-abstraction $\lambda_z x.\, e$ is *guarded* if $e = \alpha; e'$ for some event α and expression e'.

Theorem 3. Let $\Gamma, H \vdash_{\mathsf{Srv}} e : \tau$, with e closed, and H valid for some well-typed plan π. Let $\varepsilon, e \to_\pi^* \eta, \mathcal{C}(\psi\langle e'\rangle)$, with e' having guarded λ-abstractions only. Then, there exists k such that, for each computation:

$$\eta, \mathcal{C}(\psi\langle e'\rangle) \to_\pi \eta_1, e_1 \to_\pi \cdots \to_\pi \eta_k, e_k$$

there exists $n \leq k$ such that $e_n = \mathcal{C}(\psi\langle e''\rangle)$ and $\eta_n \models \psi$.

Intuitively, if a computation is long enough (i.e. at least k steps), it will always escape all liveness framings in a finite number of steps $n \leq k$ — because, by rule (E-LF2), $\eta_n, \mathcal{C}(\psi\langle e''\rangle) \to_\pi \eta_n, \mathcal{C}(e'')$.

Example 10. Consider the expressions $e_1 = \psi\langle(\lambda_z x.\, \mathtt{if}\ b\ \mathtt{then}\ \beta\ \mathtt{else}\ \alpha; \beta; zx)*\rangle$ and $e_2 = \psi\langle(\lambda_z x.\, \mathtt{if}\ b\ \mathtt{then}\ \beta\ \mathtt{else}\ \alpha; zx; \beta)*\rangle$, where ψ ask "eventually β". Let $H_1 = \psi\langle\mu h.\, \alpha \cdot \beta \cdot h + \beta\rangle$ and $H_2 = \psi\langle\mu h.\, \alpha \cdot h \cdot \beta + \beta\rangle$ be the history expressions associated with e_1 and e_2, respectively. Since H_1 is valid, Theorem 3 guarantees that *all* the computations of e_1 will satisfy the policy ψ. Instead, since H_2 is *not* valid, the expression e_2 might never obey ψ, as shown by the following computation (where $b = \mathit{ff}$ and $Z = \lambda_z x.\, \mathtt{if}\ b\ \mathtt{then}\ \beta\ \mathtt{else}\ \alpha; zx; \beta$):

$$\begin{aligned}
\varepsilon, e_2 &\to_\pi \varepsilon, \psi\langle \mathtt{if}\ b\ \mathtt{then}\ \beta\ \mathtt{else}\ \alpha; Z*; \beta\rangle \\
&\to_\pi \varepsilon, \psi\langle \alpha; Z*; \beta\rangle \to_\pi \alpha, \psi\langle Z*; \beta\rangle \\
&\to_\pi \alpha, \psi\langle \mathtt{if}\ b\ \mathtt{then}\ \beta\ \mathtt{else}\ \alpha; Zx; \beta; \beta\rangle \\
&\to_\pi \alpha, \psi\langle \alpha; Z*; \beta; \beta\rangle \to_\pi \alpha\alpha, \psi\langle Z*; \beta; \beta\rangle \to_\pi \cdots
\end{aligned}$$

□

Note however that, since our notion of validity is extensional rather than intensional, we still do not have a decision procedure to tell us for which plans π (if any) a history expression is π-valid. This problem is addressed by the planning and verification methods presented in Sections 6 and 7.

6 Planning service composition

Once extracted a history expression H from an expression e, we have to analyse H to find if there is any viable plan for the execution of e. This issue is not trivial, because the effect of selecting a given service for a request is not confined to the execution of that service. For instance, the history generated while running a

service may later on violate a policy that will become active after the service has returned, as shown in Example 11 below. Since each service selection affects the *whole* execution of a program, we cannot simply devise a viable plan by selecting services that satisfy the constraints imposed by the requests, only.

Example 11. Let $e = \varphi[(\lambda x.\,(\mathtt{req}_{r_2}\,\tau_2)x)\,((\mathtt{req}_{r_1}\,\tau_1)*)]$, where φ requires "never γ after α", $\tau = \mathit{unit}$, $\tau_1 = \tau \to (\tau \to \tau)$ and $\tau_2 = (\tau \to \tau) \to \tau$. Intuitively, the service selected upon the request r_1 returns a function, which is then passed as an argument to the service selected upon r_2. Assume the repository Srv comprises exactly the following four services:

$$e_{\ell_1} : \tau \xrightarrow{\alpha} (\tau \xrightarrow{\beta} \tau) \qquad e_{\ell_2} : (\tau \xrightarrow{h} \tau) \xrightarrow{h\cdot\gamma} \tau$$
$$e_{\ell'_1} : \tau \xrightarrow{\alpha'} (\tau \xrightarrow{\beta'} \tau) \qquad e_{\ell'_2} : (\tau \xrightarrow{h} \tau) \xrightarrow{\varphi'[h]} \tau$$

where φ' requires "never β'". Since the request type τ_1 imposes no constraints and matches the types of e_{ℓ_1} and $e_{\ell'_1}$, both these services can be selected for the request r_1. Similarly, both e_{ℓ_2} and $e_{\ell'_2}$ can be chosen for r_2. Therefore, we have to consider four possible plans when evaluating the history expression H of e:

$$H = \varphi[\{r_1[\ell_1] \rhd \alpha, r_1[\ell'_1] \rhd \alpha'\}\cdot \\ \{r_2[\ell_2] \rhd \{r_1[\ell_1] \rhd \beta, r_1[\ell'_1] \rhd \beta'\}\cdot\gamma,\ r_2[\ell'_2] \rhd \varphi'[\{r_1[\ell_1] \rhd \beta, r_1[\ell'_1] \rhd \beta'\}]\}]$$

Consider first the plan $\pi_1 = r_1[\ell_1] \mid r_2[\ell_2]$. Then, $[\![H]\!]^{\pi_1} = \varphi[\alpha\beta\gamma]$ is *not* valid and π_1 is not viable for e, because the policy φ is violated. Consider now $\pi_2 = r_1[\ell'_1] \mid r_2[\ell'_2]$. Then, $[\![H]\!]^{\pi_2} = \varphi[\alpha'\varphi'[\beta']]$ is *not* valid and π_2 is not viable for e, because the policy φ' is violated. Instead, the remaining two plans, $r_1[\ell_1] \mid r_2[\ell'_2]$ and $r_1[\ell'_1] \mid r_2[\ell_2]$ are viable for e. □

As shown above, the tree-shaped structure of planned selections makes it difficult to determine the plans π under which a history expression is valid. Things become easier if we "linearize" such a tree structure into a set of history expressions, forming an equivalent planned selection $\{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}$, where no H_i has further selections. E.g., the linearization of H in Example 11 is:

$$\{r_1[\ell_1] \mid r_2[\ell_2] \rhd \varphi[\alpha\cdot\beta\cdot\gamma],\quad r_1[\ell_1] \mid r_2[\ell'_2] \rhd \varphi[\alpha\cdot\varphi'[\beta]], \\ r_1[\ell'_1] \mid r_2[\ell_2] \rhd \varphi[\alpha'\cdot\beta'\cdot\gamma],\ r_1[\ell'_1] \mid r_2[\ell'_2] \rhd \varphi[\alpha'\cdot\varphi'[\beta']]\}$$

Formally, we say that H is *equivalent* to H' ($H \equiv H'$ in symbols) when $[\![H]\!]^\pi_\rho = [\![H']\!]^\pi_\rho$, for each ρ and plan π. The following properties of $\equiv$ hold.

Theorem 4. The relation $\equiv$ is a congruence, and it satisfies the equations between planned selections displayed in the following table.

Equational properties of planned selections

$$H \equiv \{0 \rhd H\} \tag{1}$$
$$\{\pi_i \rhd H_i\}_{i\in I} \cdot \{\pi'_j \rhd H'_j\}_{j\in J} \equiv \{\pi_i \mid \pi'_j \rhd H_i \cdot H'_j\}_{i\in I, j\in J} \tag{2}$$
$$\{\pi_i \rhd H_i\}_{i\in I} + \{\pi'_j \rhd H'_j\}_{j\in J} \equiv \{\pi_i \mid \pi'_j \rhd H_i + H'_j\}_{i\in I, j\in J} \tag{3}$$
$$\varphi[\{\pi_i \rhd H_i\}_{i\in I}] \equiv \{\pi_i \rhd \varphi[H_i]\}_{i\in I} \tag{4}$$
$$\psi\langle\{\pi_i \rhd H_i\}_{i\in I}\rangle \equiv \{\pi_i \rhd \psi\langle H_i\rangle\}_{i\in I} \tag{5}$$
$$\mu h.\, \{\pi_i \rhd H_i\} \equiv \{\pi_i \rhd \mu h.\, H_i\}_{i\in I} \tag{6}$$
$$\{\pi_i \rhd \{\pi'_{i,j} \rhd H_{i,j}\}_{j\in J}\}_{i\in I} \equiv \{\pi_i \mid \pi'_{i,j} \rhd H_{i,j}\}_{i\in I, j\in J} \tag{7}$$

Example 12. Consider again the history expression computed in Example 7:

$$H = \{r[\ell_1] \rhd \varphi[\alpha_1 \cdot \gamma],\ r[\ell_2] \rhd \varphi[\alpha_2 \cdot \gamma]\} \cdot \{r[\ell_1] \rhd \psi\langle\beta_1\rangle,\ r[\ell_2] \rhd \psi\langle\beta_2\rangle\}$$

Applying the equation (2) of Theorem 4, we obtain:

$$\begin{aligned} H \equiv\ & \{r[\ell_1] \mid r[\ell_1] \rhd \varphi[\alpha_1 \cdot \gamma] \cdot \psi\langle\beta_1\rangle,\ r[\ell_1] \mid r[\ell_2] \rhd \varphi[\alpha_1 \cdot \gamma] \cdot \psi\langle\beta_2\rangle, \\ & r[\ell_2] \mid r[\ell_1] \rhd \varphi[\alpha_2 \cdot \gamma] \cdot \psi\langle\beta_1\rangle,\ r[\ell_2] \mid r[\ell_2] \rhd \varphi[\alpha_2 \cdot \gamma] \cdot \psi\langle\beta_2\rangle\} \\ =\ & \{r[\ell_1] \rhd \varphi[\alpha_1 \cdot \gamma] \cdot \psi\langle\beta_1\rangle,\ r[\ell_2] \rhd \varphi[\alpha_2 \cdot \gamma] \cdot \psi\langle\beta_2\rangle\} \end{aligned}$$

In the last step, we have used idempotency of $|$, and we have purged the resulting selection from the plan $r[\ell_1] \mid r[\ell_2]$, which is not injective. □

Example 13. Let $H = \mu h.\, \{r[\ell_1] \rhd \alpha_1,\ r[\ell_2] \rhd \alpha_2\} \cdot h$. Then, using equations (1), (2) and (6) of Theorem 4, and the identity of the plan 0, we obtain:

$$\begin{aligned} H &\equiv \mu h.\, \{r[\ell_1] \rhd \alpha_1,\ r[\ell_2] \rhd \alpha_2\} \cdot \{0 \rhd h\} \\ &\equiv \mu h.\, \{r[\ell_1] \mid 0 \rhd \alpha_1 \cdot h,\ r[\ell_2] \mid 0 \rhd \alpha_2 \cdot h\} \\ &= \mu h.\, \{r[\ell_1] \rhd \alpha_1 \cdot h,\ r[\ell_2] \rhd \alpha_2 \cdot h\} \\ &\equiv \{r[\ell_1] \rhd \mu h.\, \alpha_1 \cdot h,\ r[\ell_2] \rhd \mu h.\, \alpha_2 \cdot h\} \end{aligned}$$

Note that the original H can choose a service among ℓ_1 and ℓ_2 at *each* iteration of the loop. Instead, in the linearization of H, the request r will be resolved into the *same* service at each iteration. □

Example 14. Consider again the history expression of Example 2. Applying equations (1), (2) and (7) of Theorem 4, we obtain:

$$\begin{aligned} H &= \{r[\ell_1] \rhd \alpha_1 \cdot \{r'[\ell'_1] \rhd \beta_1, r'[\ell'_2] \rhd \beta_2\},\ r[\ell_2] \rhd \alpha_2\} \\ &\equiv \{r[\ell_1] \rhd \{0 \rhd \alpha_1\} \cdot \{r'[\ell'_1] \rhd \beta_1, r'[\ell'_2] \rhd \beta_2\},\ r[\ell_2] \rhd \alpha_2\} \\ &\equiv \{r[\ell_1] \rhd \{0 \mid r'[\ell'_1] \rhd \alpha_1 \cdot \beta_1, 0 \mid r'[\ell'_2] \rhd \alpha_1 \cdot \beta_2\},\ r[\ell_2] \rhd \alpha_2\} \\ &= \{r[\ell_1] \rhd \{r'[\ell'_1] \rhd \alpha_1 \cdot \beta_1, r'[\ell'_2] \rhd \alpha_1 \cdot \beta_2\},\ r[\ell_2] \rhd \alpha_2\} \\ &\equiv \{r[\ell_1] \mid r'[\ell'_1] \rhd \alpha_1 \cdot \beta_1,\ r[\ell_1] \mid r'[\ell'_2] \rhd \alpha_1 \cdot \beta_2,\ r[\ell_2] \rhd \alpha_2\} \end{aligned}$$ □

We say that a history expression H is *linear* when $H = \{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}$, the plans are pairwise *independent* (i.e. $\pi_i \neq \pi_j|\pi$ for all $i \neq j$ and π) and no H_i has planned selections.

Given a history expression H, we obtain its linearization in three steps. First, we apply the first equation of Theorem 4 to each event, variable and ε in H. Then, we orient the equations of Theorem 4 from left to right, obtaining a rewriting system that is easily proved finitely terminating and confluent – up to the equational laws of the algebra of plans. The resulting planned selection $H' = \{\pi_1 \triangleright H_1 \cdots \pi_k \triangleright H_k\}$ has no further selections in H_i, but there may be non-independent plans $\pi_i \sqsubseteq \pi_j$ (recall that we discard $\pi_i \triangleright H_i$ when π_i is not injective). In the third linearization step, for each such pairs, we update H' by inserting $\pi_i \triangleright H_i + H_j$, and removing $\pi_j \triangleright H_j$.

The following result enables us to detect the viable plans for service composition: executions driven by any of them will never violate the security constraints on demand.

Theorem 5. If $H = \{\pi_1 \triangleright H_1 \cdots \pi_k \triangleright H_k\}$ is linear, and H_i is valid for some $i \in 1..k$, then H is π_i-valid.

Summing up, we extract from an expression e a history expression H, we linearize it into $\{\pi_1 \triangleright H_1 \cdots \pi_k \triangleright H_k\}$, and if some H_i is valid, then we can deduce that H is π_i-valid. By Theorem 2, the plan π_i safely drives the execution of e, without resorting to any run-time monitor. It remains then to verify the validity of history expressions that, like the H_i above, have no planned selections.

7 Verifying validity

The last step in our technical development mechanically verifies the validity of history expressions with no planned selections. Our technique is based on model checking Basic Process Algebras (BPAs) with Finite State Automata (FSA). The standard decision procedure for verifying that a BPA process p satisfies a ω-regular property φ amounts to constructing the pushdown automaton for p and the Büchi automaton for the negation of φ. Then, the property holds if it is empty the (context-free) language accepted by the conjunction of the above, which is still a pushdown automaton. This problem is known to be decidable, and several algorithms and tools show this approach feasible [22]. Since our histories are always finite, it turns out that we can use Finite State Automata instead of Büchi Automata.

Recall however that, as it is, our notion of validity is non-regular, because of the arbitrary nesting of framings. As an example, consider again the history expression $H = \mu h.\, \alpha + h \cdot h + \varphi[h]$. The language $[\![H]\!]^\pi$ is context-free and non-regular, because it contains unbounded pairs of balanced $[_\varphi$ and $]_\varphi$ (for all plans π, as H contains no planned selections, due to Theorem 5). Since context-free languages are not closed under intersection, the emptiness problem is undecidable. To apply the procedure sketched above, we will first manipulate history expressions in order to make validity a regular property.

7.1 Redundant framings

History expressions can generate histories with *redundant framings*, i.e. nesting of the same framing. For example, the history $\eta = \alpha\varphi[\alpha'\varphi'[\varphi[\alpha'']]]$ has an inner redundant safety framing φ around α''. Since α'' is already under the scope of the outermost φ-framing, it happens that η is valid if and only if $\alpha\varphi[\alpha'\varphi'[\alpha'']]$

is valid. Formally, the S-sets of η comprise $\varphi[\{\alpha, \alpha\alpha', \alpha\alpha'\alpha''\}]$ for the outer framing, and $\varphi[\{\alpha\alpha', \alpha\alpha'\alpha''\}]$ for the inner one. Validity requires that all the histories in $\{\alpha, \alpha\alpha', \alpha\alpha'\alpha''\}$ and $\{\alpha\alpha', \alpha\alpha'\alpha''\}$ obey φ. Since the second set is strictly included in the first one, then the *inner* safety framing is redundant.

Similarly, consider the history $\eta' = \alpha\psi\langle\alpha'\psi\langle\alpha''\rangle\rangle$. The L-sets of η' are $\psi[\{\alpha, \alpha\alpha', \alpha\alpha'\alpha''\}]$ for the outer framing and $\psi[\{\alpha\alpha', \alpha\alpha'\alpha''\}]$ for the inner one. Since the first set includes the second one, and validity requires that there exists a history in the L-set satisfying ψ, then the *outer* framing is redundant.

Removing redundant framings from a history preserves its validity. But one needs the expressive power of a pushdown automaton, because framings openings and closings are to be matched in pairs. For example, consider the following history:

$$\eta \;=\; \alpha \overbrace{[_\varphi \cdots [_\varphi}^{n} \overbrace{]_\varphi \cdots]_\varphi}^{m} [_\varphi$$

The last $[_\varphi$ is redundant if $n > m$, and is *not* if $n = m$.

As a matter of fact, it turns out that it is possible to regularize the safety side of validity, by removing the redundant safety framings. For the liveness side, this approach seems not feasible, but, surprisingly enough, a tailored construction of finite state automata suffices.

7.2 Regularization of safety framings

Below, we define a transformation that, given a history expression H, yields a H' that does not generate redundant safety framings, and H' is valid if and only if H is such. Recall that there is no need for regularizing planned selections, because, by Theorem 5, we will always verify the validity of history expressions with no selections.

Let $\tilde{H}$ be a history expression with a hole $\bullet$, and let $H = \tilde{H}\{H'/\bullet\}$ be a history expression, for some H'. We say that H' *is guarded by* φ *in* H when $\varphi \in \mathit{guard}(\tilde{H})$, defined as the smallest set satisfying the following equations.

Guards

$$\begin{array}{rcll}
\mathit{guard}(H_0 \cdot H_1) & = & \mathit{guard}(H_i) & \text{if } \bullet \in H_i \\
\mathit{guard}(H_0 + H_1) & = & \mathit{guard}(H_i) & \text{if } \bullet \in H_i \\
\mathit{guard}(\varphi[H]) & = & \{\varphi\} \cup \mathit{guard}(H) & \\
\mathit{guard}(\psi\langle H\rangle) & = & \mathit{guard}(H) & \\
\mathit{guard}(\mu h.\ H) & = & \mathit{guard}(H) &
\end{array}$$

Example 15. Let $H = \varphi[\alpha\cdot h_0\cdot\varphi'[\alpha'+h_1]]\cdot h_2$, and let $\tilde{H} = \varphi[\alpha\cdot h_0\cdot\varphi'[\alpha'+\bullet]]\cdot h_2$. Then, $H = \tilde{H}\{h_1/\bullet\}$, and so h_1 is guarded by $\mathit{guard}(\tilde{H}) = \{\varphi, \varphi'\}$. Similarly, h_0 is guarded by $\{\varphi\}$, and h_2 is unguarded (i.e. guarded by $\emptyset$). □

Let H be a (possibly non-closed) history expression. Without loss of generality, assume that all the variables in H have distinct names. We define below $H \downarrow_{\Phi,\Omega}$, the history expression produced by the *regularization* of H against a set of policies Φ and a mapping Ω from variables to history expressions.

Regularization of safety framings

$$\varepsilon\downarrow_{\Phi,\Omega} = \varepsilon \qquad h\downarrow_{\Phi,\Omega} = h \qquad \alpha\downarrow_{\Phi,\Omega} = \alpha$$

$$(H \cdot H')\downarrow_{\Phi,\Omega} = H\downarrow_{\Phi,\Omega} \cdot H'\downarrow_{\Phi,\Omega} \qquad (H + H')\downarrow_{\Phi,\Omega} = H\downarrow_{\Phi,\Omega} + H'\downarrow_{\Phi,\Omega}$$

$$\varphi[H]\downarrow_{\Phi,\Omega} = \begin{cases} H\downarrow_{\Phi,\Omega} & \text{if } \varphi \in \Phi \\ \varphi[H\downarrow_{\Phi\cup\{\varphi\},\Omega}] & \text{otherwise} \end{cases}$$

$$\psi\langle H\rangle\downarrow_{\Phi,\Omega} = \begin{cases} H\downarrow_{\Phi,\Omega} & \text{if } \psi \in \Phi \\ \psi\langle H\downarrow_{\Phi,\Omega}\rangle & \text{otherwise} \end{cases}$$

$$(\mu h.\, H)\downarrow_{\Phi,\Omega} = \mu h.\, (H'\sigma'\downarrow_{\Phi,\Omega\{(\mu h.H)\Omega/h\}} \sigma)$$

where $H = H'\{h/h_i\}_i$, h_i fresh, $h \notin fv(H')$, and

$$\sigma(h_i) \;=\; (\mu h.H)\Omega\downarrow_{\Phi\cup guard(H'\{\bullet/h_i\}),\Omega}$$

$$\sigma'(h_i) \;=\; \begin{cases} h & \text{if } guard(H'\{\bullet/h_i\}) \subseteq \Phi \\ h_i & \text{otherwise} \end{cases}$$

Intuitively, $H\downarrow_{\Phi,\Omega}$ results from H by eliminating all the redundant safety framings, and all the framings in Φ. The environment Ω is needed to deal with free variables in the case of nested μ-expressions. We feel free to omit the component Ω when unneeded, and, when H is closed, we abbreviate $H\downarrow_{\emptyset,\emptyset}$ with $H\downarrow$.

The last three regularization rules would benefit from some explanation. Consider first a history expression of the form $\varphi[H]$ to be regularized against a set of policies Φ. To eliminate the redundant safety framings, we must ensure that H has neither φ-framings, nor redundant safety framings itself. This is accomplished by regularizing H against $\Phi \cup \{\varphi\}$.

A history expression $\psi\langle H\rangle$ is dealt with by removing the liveness framing if $\psi \in \Phi$ (because there is an outer safety framing enforcing ψ), otherwise the framing remains. Note that we could end up with redundant liveness framings, but they will not prevent us from verifying validity of history expressions.

Consider a history expression of the form $\mu h.H$. Its regularization against Φ and Ω proceeds as follows. Each free occurrence of h in H guarded by some $\Phi' \not\subseteq \Phi$ is unfolded and regularized against $\Phi \cup \Phi'$. The substitution Ω is used to bind the free variables to closed history expressions. Technically, the i-th free occurrence of h in H is picked up by the substitution $\{h/h_i\}$, for h_i fresh. Note also that $\sigma(h_i)$ is computed only if $\sigma'(h_i) = h_i$. As a matter of fact, regularization is a total function, and its definition above can be easily turned into a finitely terminating rewriting system.

Example 16. Consider the history expression $H_0 = \mu h.\, H$, where $H = \alpha + h \cdot h + \varphi[h]$. Then, H can be written as $H'\{h/h_i\}_{i\in 0..2}$, where $H' = \alpha + h_0 \cdot h_1 + \varphi[h_2]$.

Since $guard(H'\{\bullet/h_2\}) = guard(\alpha + h_0 \cdot h_1 + \varphi[\bullet]) = \{\varphi\} \not\subseteq \emptyset$, then:

$$\begin{aligned} H_0\downarrow_\emptyset &= \mu h.\, H'\{h/h_0, h/h_1\}\downarrow_\emptyset \{H_0\downarrow_\varphi / h_2\} \\ &= \mu h.\, \alpha + h \cdot h + \varphi[H_0\downarrow_\varphi] \end{aligned}$$

To compute $H_0\downarrow_\varphi$, note that no occurrence of h is guarded by $\Phi \not\subseteq \{\varphi\}$. Then:

$$H_0\downarrow_\varphi \;=\; \mu h.\, (\alpha + h \cdot h + \varphi[h])\downarrow_\varphi \;=\; \mu h.\, \alpha + h \cdot h + h$$

Since $[\![H_0\downarrow_\varphi]\!] = \{\alpha\}^*$ has no φ-framings, we have that $[\![H_0\downarrow]\!] = \big(\{\alpha\}^*\varphi[\{\alpha\}^*]\big)^*$ has no redundant framings. □

Example 17. Let $H_0 = \mu h.\, H_1$, where $H_1 = \mu h'.\, H_2$, and $H_2 = \alpha + h \cdot \varphi[h']$. Since $guard(H_1\{\bullet/h\}) = \emptyset$, we have that:

$$H_0\downarrow_{\emptyset,\emptyset} \;=\; \mu h.\, (H_1\downarrow_{\emptyset,\{H_0/h\}})$$

Note that H_2 can be written as $H_2'\{h/h_0\}$, where $H_2' = \alpha + h \cdot \varphi[h_0]$. Since $guard(H_2'\{\bullet/h_0\}) = \{\varphi\} \not\subseteq \emptyset$, it follows that:

$$\begin{aligned} H_1\downarrow_{\emptyset,\{H_0/h\}} &= \mu h'.\, H_2'\downarrow_{\emptyset,\{H_0/h, H_1\{H_0/h\}/h'\}} \{H_1\{H_0/h\}\downarrow_{\varphi,\{H_0/h\}} / h_0\} \\ &= \mu h'.\, \alpha + h \cdot \varphi[h_0]\ \{(\mu h'.\, \alpha + H_0 \cdot \varphi[h'])\downarrow_{\varphi,\{H_0/h\}} / h_0\} \\ &= \mu h'.\, \alpha + h \cdot \varphi[H_3\downarrow_{\varphi,\{H/h\}}] \\ &= \alpha + h \cdot \varphi[H_3\downarrow_{\varphi,\{H/h\}}] \end{aligned}$$

where $H_3 = \mu h'.\, \alpha + H_0 \cdot \varphi[h']$, and the last step is possible because the outermost μ binds no variable. Since $guard(\alpha + H_0 \cdot \varphi[\bullet]) = \{\varphi\} \subseteq \{\varphi\}$:

$$H_3\downarrow_\varphi \;=\; \mu h'.\, (\alpha + H_0 \cdot \varphi[h'])\downarrow_\varphi \;=\; \mu h'.\, \alpha + H_0\downarrow_\varphi \cdot h'$$

Since $\{\varphi\}$ contains both $guard(H_1\{\bullet/h\}) = \emptyset$, and $guard(H_2\{\bullet/h'\}) = \{\varphi\}$:

$$\begin{aligned} H_0\downarrow_\varphi &= \mu h.\, (\mu h'.\, \alpha + h \cdot \varphi[h'])\downarrow_\varphi = \ \mu h.\, \mu h'.\, (\alpha + h \cdot \varphi[h'])\downarrow_\varphi \\ &= \mu h.\, \mu h'.\, \alpha + h \cdot h' \end{aligned}$$

Summing up, we have that:

$$\begin{aligned} H_0\downarrow_\emptyset &= \mu h.\, \alpha + h \cdot \varphi[H_3\downarrow_\varphi] \\ H_3\downarrow_\varphi &= \mu h'.\, \alpha + \big(\mu h.\, \mu h'.\, \alpha + h \cdot h'\big) \cdot h' \end{aligned}$$ □

We now establish the following basic properties of regularization, stating its correctness.

Theorem 6. For any history expression H:

- $H\downarrow$ has no redundant safety framings.
- $H\downarrow$ is valid if and only if H is valid.

7.3 From history expressions to Basic Process Algebras

Basic Process Algebras [5] (BPAs) provide a natural characterization of histories. A *BPA process* is given by the following abstract syntax, where ε denotes the terminated process, $\alpha \in \mathsf{Ev} \cup \mathsf{Frm}$, $\cdot$ denotes sequential composition, $+$ represents (nondeterministic) choice, and X is a variable.

Syntax of BPA processes

$$p, p' \quad ::= \quad \varepsilon \mid \alpha \mid p \cdot p' \mid p + p' \mid X$$

A BPA process p is *guarded* if each variable occurrence in p is within a subexpression $\alpha \cdot q$ of p. We assume a finite set $\Delta = \{X \triangleq p\}$ of definitions: for each variable X, there exists a single, guarded p such that $\{X \triangleq p\} \in \Delta$. As usual, we consider the process $\varepsilon \cdot p$ to be equivalent to p.

The operational semantics of BPAs is given by the following labelled transition system, in the SOS style. We denote with $[\![p_0, \Delta]\!]$ the set of the strings that label finite computations, i.e. $\{(a_i)_i \,|\, p_0 \xrightarrow{a_1} \cdots \xrightarrow{a_i} p_i\}$. Note that we only consider finite computations, since our histories are always such.

Operational Semantics of BPA processes

$$\frac{}{\alpha \xrightarrow{\alpha} \varepsilon} \qquad \frac{p \xrightarrow{\alpha} p'}{p + q \xrightarrow{\alpha} p'} \qquad \frac{q \xrightarrow{\alpha} q'}{p + q \xrightarrow{\alpha} q'} \qquad \frac{p \xrightarrow{\alpha} p'}{p \cdot q \xrightarrow{\alpha} p' \cdot q} \qquad \frac{p \xrightarrow{\alpha} p' \quad X \triangleq p \in \Delta}{X \xrightarrow{\alpha} p'}$$

We now introduce a mapping from history expressions to BPAs, in the line of [4, 42]. Again, note that there is no need for transforming planned selections into BPAs, because we are only interested in the validity of history expressions with no selections.

The mapping takes as input a history expression H and an injective function Θ from history variables h to BPA variables X, and it outputs a BPA process p and a finite set of definitions Δ. Without loss of generality, we assume that all the variables in H have distinct names.

The rules that transform history expressions into BPAs are rather natural. Events, variables, concatenation and choice are mapped into the corresponding BPA counterparts. A history expression $\varphi[H]$ is mapped to the BPA for H, surrounded by the opening and closing of the φ-framing; similarly for $\psi\langle H \rangle$. A history expression $\mu h.H$ is mapped to a fresh BPA variable X, bound to the translation of H in the set of definitions Δ. To avoid the problem of unguarded BPA processes, we assume a standard preprocessing step, that inserts a dummy event before each unguarded occurrence of a variable in a history expression. Dummy events are eventually discarded before the verification phase.

Mapping history expressions to BPAs

$$
\begin{array}{rcl}
BPA(\varepsilon,\Theta) & = & (\varepsilon,\emptyset) \\
BPA(\alpha,\Theta) & = & (\alpha,\emptyset) \\
BPA(h,\Theta) & = & (\Theta(h),\emptyset) \\
BPA(H_0 \cdot H_1,\Theta) & = & (p_0 \cdot p_1, \Delta_0 \cup \Delta_1), \text{where } BPA(H_i,\Theta) = (p_i,\Delta_i) \\
BPA(H_0 + H_1,\Theta) & = & (p_0 + p_1, \Delta_0 \cup \Delta_1), \text{where } BPA(H_i,\Theta) = (p_i,\Delta_i) \\
BPA(\varphi[H],\Theta) & = & ([_\varphi \cdot p \cdot]_\varphi, \Delta), \text{where } BPA(H,\Theta) = (p,\Delta) \\
BPA(\psi\langle H\rangle,\Theta) & = & (\langle_\psi \cdot p \cdot \rangle_\psi, \Delta), \text{where } BPA(H,\Theta) = (p,\Delta) \\
BPA(\mu h.H,\Theta) & = & (X, \Delta \cup \{X \triangleq p\}), \text{where } BPA(H,\Theta\{X/h\}) = (p,\Delta)
\end{array}
$$

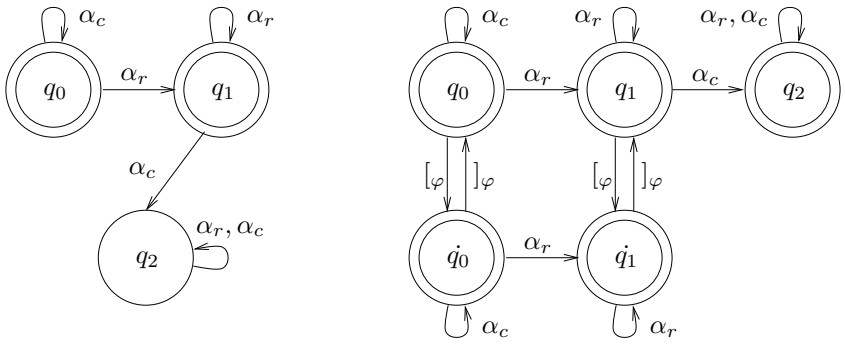
We now state the correspondence between history expressions and BPAs. The prefixes of the histories generated by a history expression H (i.e. $[\![H]\!]^\partial$, where we omit the plan π because immaterial) are all and only the strings that label the finite computations of $BPA(H)$.

Lemma 7. $[\![H]\!]^\partial = [\![BPA(H)]\!]$.

7.4 Finite State Automata for validity

Given a policy φ, we are interested in defining a formula $\varphi_{[\,]}$ and a formula $\varphi_{\langle\rangle}$ to be used in verifying the validity of a history η with respect to security policies within their framings.

As mentioned above, since our histories are always finite and our properties are regular, FSA suffice for defining the safety and liveness properties we are using. As an example, let φ be the policy saying that no event α_c can occur after an α_r (see Section 2). The finite state automata A_φ and $A_{\varphi_{[\,]}}$ are shown below, which define φ and $\varphi_{[\,]}$, respectively. It is immediate checking that the history $[_\varphi \alpha_r]_\varphi \alpha_c$ is accepted by $A_{\varphi_{[\,]}}$, while $\alpha_r [_\varphi \alpha_c]_\varphi$ is not.



Intuitively, the automaton $A_{\varphi_{[\,]}}$ is partitioned into two layers. The first layer is a copy of A_φ, where all the states are final. This models the fact that we are outside the scope of φ, i.e. the history leading to any state in this layer has balanced safety framings of φ (or none). The second layer is reachable from

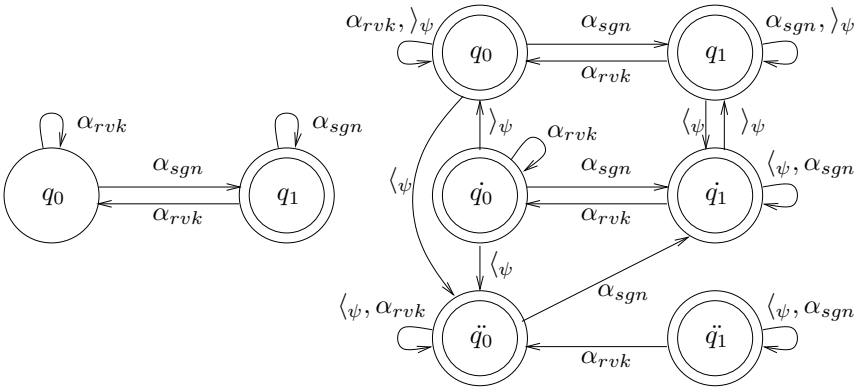
the first one when opening a safety framing for φ, while closing the framing gets back. The transitions in the second layer are a copy of those connecting final states in A_φ. Consequently, the states in the second layer are exactly the final states in A_φ. Since $A_{\varphi_{[\,]}}$ is only concerned with the verification of φ, the transitions for opening and closing safety framings $\varphi' \neq \varphi$, as well as those for liveness framings ψ, are rendered as self-loops.

We require that the history η to be verified against $\varphi_{[\,]}$ has no redundant safety framings, i.e. η has been regularized. Hereafter, let the formula φ be defined by the finite state automaton $A_\varphi = (\Sigma, Q, q_0, \rho, F)$, which we assume to have a distinguished non-final sink state. Then, the property $\varphi_{[\,]}$ is defined through the finite state automaton $A_{\varphi_{[\,]}}$ defined below.

Finite state automaton for $\varphi_{[\,]}$

$$
\begin{aligned}
A_{\varphi_{[\,]}} &= (\Sigma', Q', q_0, \rho', F') \\
\Sigma' &= \Sigma \cup \{\, [_{\varphi'},]_{\varphi'}, \langle_\psi, \rangle_\psi \mid \varphi', \psi \in \mathsf{Pol} \,\} \\
Q' &= F' = Q \cup \{\, \dot{q} \mid q \in F \,\} \\
\rho' &= \rho \cup \{\, (q, [_\varphi, \dot{q}) \mid q \in F \,\} \cup \{\, (\dot{q},]_\varphi, q) \mid q \in Q \,\} \\
&\quad \cup \{\, (\dot{q}_i, \alpha, \dot{q}_j) \mid (q_i, \alpha, q_j) \in \rho \wedge q_j \in F \,\} \\
&\quad \cup \{\, (q, [_{\varphi'}, q) \cup (q,]_{\varphi'}, q) \mid q \in Q' \wedge \varphi' \neq \varphi \,\} \\
&\quad \cup \{\, (q, \langle_\psi, q) \cup (q, \rangle_\psi, q) \mid q \in Q' \,\}
\end{aligned}
$$

Likewise, we introduce in a while the finite state automaton $A_{\psi_{\langle\rangle}}$ that defines the property $\psi_{\langle\rangle}$. However, in this case we allow histories to have redundant framings. As an example, let ψ be the policy saying that α_{sgn} eventually occurs with no subsequent α_{rvk} (see Section 2). The finite state automata A_ψ and $A_{\psi_{\langle\rangle}}$ for ψ and $\psi_{\langle\rangle}$, respectively, are shown below. The history $\alpha_{rvk}\langle_\psi \alpha_{sgn}\rangle_\psi$ is accepted by $A_{\psi_{\langle\rangle}}$, while $\alpha_{sgn}\alpha_{rvk}\langle_\psi \alpha_{rvk}\rangle_\psi$ is not. Also, note that $\langle_\psi \alpha_{sgn}\alpha_{rvk}\rangle_\psi$ is accepted, and indeed it represents a computation that leaves the liveness framing as soon as α_{sgn} has occurred.



The automaton $A_{\psi_{\langle\rangle}}$ consists of three layers. The first layer is a copy of A_ψ, and it represents being outside of the liveness framing $\psi\langle\cdots\rangle$. The second and

the third layer model being inside the framing. If you are in the second layer, then you have already found a history that satisfies the property ψ, while if you are in the third layer, you are still looking for. Suppose now that a new framing $\psi\langle\cdots\rangle$ is opened when you are in the second layer, so this is a redundant liveness framing. If you were in a state that was final in A_ψ, then you remain in the second layer; otherwise, you go to the corresponding state in the third layer. If the redundant framing is opened when you are in the third layer, then you stay there. If a framing is closed when you are in the second layer, then you can go back to the first layer, but if you are in the third layer, then you get stuck. The automaton $A_{\psi_{\langle\,\rangle}}$ is defined in the following table.

Finite state automaton for $\psi_{\langle\,\rangle}$

$$
\begin{aligned}
A_{\psi_{\langle\,\rangle}} &= (\Sigma', Q', q_0, \rho', F') \\
\Sigma' &= \Sigma \cup \{\, [_\varphi,]_\varphi, \langle_{\psi'}, \rangle_{\psi'} \mid \varphi, \psi' \in \mathsf{Pol} \,\} \\
Q' &= F' = Q \cup \{\, \dot{q}, \ddot{q} \mid q \in Q \,\} \\
\rho' &= \rho \cup \{\, (q, \langle_\psi, \dot{q}) \mid q \in F \,\} \cup \{\, (q, \langle_\psi, \ddot{q}) \mid q \notin F \,\} \\
&\quad \cup \{\, (\dot{q}, \langle_\psi, \dot{q}) \mid q \in F \,\} \cup \{\, (\dot{q}, \langle_\psi, \ddot{q}) \mid q \notin F \,\} \cup \{\, (\ddot{q}, \langle_\psi, \ddot{q}) \mid q \in Q \,\} \\
&\quad \cup \{\, (\dot{q_i}, \alpha, \dot{q_j}) \mid (q_i, \alpha, q_j) \in \rho \,\} \\
&\quad \cup \{\, (\ddot{q_i}, \alpha, \dot{q_j}) \mid (q_i, \alpha, q_j) \in \rho \wedge q_j \in F \,\} \\
&\quad \cup \{\, (\ddot{q_i}, \alpha, \ddot{q_j}) \mid (q_i, \alpha, q_j) \in \rho \wedge q_j \notin F \,\} \\
&\quad \cup \{\, (\dot{q}, \rangle_\psi, q), (q, \rangle_\psi, q) \mid q \in Q \,\} \\
&\quad \cup \{\, (q, \langle_{\psi'}, q),\ (q, \rangle_{\psi'}, q) \mid q \in Q' \wedge \psi' \neq \psi \,\} \\
&\quad \cup \{\, (q, [_\varphi, q),\ (q,]_\varphi, q) \mid q \in Q' \,\}
\end{aligned}
$$

Although the policies enforced by the security framings can always inspect the whole past history, we can easily limit the scope from the side of the past. It suffices to mark in the history the point in time β_φ from which checking a policy φ has to start. The corresponding automaton ignores all the events before β_φ, and then behaves like the standard automaton enforcing φ.

We now relate validity of histories with the formulae $\varphi_{[\,]}$ and $\psi_{\langle\rangle}$ for the policies φ, ψ spanning over η.

Lemma 8. Let η be a history with no redundant safety framings. Then, η is valid if and only if $\eta \models \varphi_{[\,]}$ and $\eta \models \psi_{\langle\rangle}$ for all φ, ψ such that $[_\varphi, \langle_\psi \in \eta$.

Since finite state automata are closed under intersection, a valid history η is accepted by the intersection of the automata $A_{\varphi_{[\,]}}$ and $A_{\psi_{\langle\rangle}}$, for all φ, ψ in η.

Validity of a closed history expression H with no planned selections can be decided by showing that the BPA generated by the regularization of H satisfies the given regular formula. Together with Theorem 2, the execution of an expression in our calculus never violates security if its effect is verified valid. Thus we are dispensed from using an execution monitor to enforce the safety properties, and we additionally guarantee the liveness properties on demand.

Theorem 9. A history expression H with no planned selections is valid iff:

$$[\![BPA(H\downarrow)]\!] \models \bigwedge_{\varphi\in H} \varphi_{[\,]} \wedge \bigwedge_{\psi\in H} \psi_{\langle\,\rangle}$$

8 Discussion

The essence of the SOC approach resides in a programming paradigm where loosely coupled, reusable components can be invoked and composed by clients. A key aspect is the use of the "WS-*" standards, e.g. WSDL for service description, WS-BPEL for service orchestration, WS-CDL for choreography, WS-Security for security policies, etc. All these standards feature a call-by-name mechanism for service invocation. Recent initiatives aim at relaxing this *a priori* agreement on the syntax of service interactions, and instead rely on an informal semantics, based on ontologies. The semantic-web initiative is an example of this approach [6].

Our type and effects for services extend the WSDL notion of published interfaces. Besides the standard WSDL attributes, our effects add semantic information about a service behaviour, in the spirit of WSLA proposal [31]. This additional attribute, namely a history expression, formally is a context-free grammar that over-approximates the run-time behaviour of services. Also, this extension impacts on service discovery and selection. In the current Web service technology, these operations use UDDI registries of services or matchmaking algorithms based on semantic ontologies. Our notion of call-by-contract suggests instead to select a service with a matchmaking algorithm based on formal semantic abstractions.

Call-by-contract makes the picture complex, because several components need to cooperate in a trusted manner. First, registries are assumed to be trusted, in that they certify the agreement between a published type and effect and the actual behaviour of the service considered. Formally, a registry statically analyses service code to infer its interface, by the type and effect system of Section 5. Also, the infrastructure running services must only execute the published ones, and code updates require certification before use. The actual mechanism to enforce these aspects of trust are outside the scope of the present paper, which instead concentrates on certification and planning. In our abstract model we assume the interconnecting network is reliable, and we do not address here the classical security problem of confidentiality, identity and availibility.

Our proposal for a semantically-based orchestration outlines the design of an infrastructure for secure service composition. The call-by-contract invocation mechanism adds a further layer to the standard remote procedure call. Before starting the execution of a service, the orchestrator collects the relevant components, by inquiring the (possibly distributed) registries. The plans provided by the orchestrator resolve all the requests in the initiator service, as well as those in the invoked services. This mechanism differs from the standard one, e.g. WS-BPEL, with the advantage of offering a way to enforce all the security policies imposed. The trustfulness of the orchestrator (Theorem 10) follows from our formal approach, in particular from the soundness of the type and effect system (Theorem 2), and the correctness of planning (Theorem 4) and of verification (Theorem 9).

9 Related work

Process calculi techniques have been used to study the foundation of services. The main goal of some of these proposals, e.g. [23, 15, 28, 33] is to formalise various aspects of standards for the description, execution and orchestration of services (WSDL, SOAP and WS-BPEL). The Global Calculus [18] addresses the problem of relating orchestration and choreography. As a matter of fact, our λ^{req} builds over the standard service infrastructure the above calculi formalise. Indeed, our call-by-contract supersedes standard invocation mechanisms and allows for verified planning.

The secure composition of components has been the main concern underlying the design of Sewell and Vitek's box-π [41], an extension of the π-calculus that allows for expressing safety policies in the form of *security wrappers*. These are programs that encapsulate a component to control the interactions with other (possibly untrusted) components. The calculus is equipped with a type system that statically captures the allowed causal information flows between components. Our safety framings are closely related to wrappers, but in [41] there is no analog of our liveness framings.

Gorla, Hennessy and Sassone [27] consider a calculus for mobile agents which may migrate between sites in a controlled manner. Each site has a *membrane*, representing both a security policy and a classification of external sites with respect to their levels of trust. A membrane guards the incoming agents before allowing them to execute. Three classes of membranes are studied, the most complex being the class of policies enforceable by finite state automata. When an agent comes from an untrusted site, *all* its code must be checked. Instead, an agent coming from a trusted site must only provide the destination site with a *digest* of its behaviour, so allowing for more efficient checks.

A different approach is Cook and Misra's Orc [36], a programming model for structured orchestration of services. The basic computational entities orchestrated by Orc expressions are sites. A site computation can start other orchestrations, locally store the effects of a computation, and make them visible to clients. Orc provides three basic composition operators, that can be used to model some common workflow patterns, identified by Van der Aalst et al. [21].

Another solution to planning service composition has been proposed in [34], where the problem of achieving a given composition goal is expressed as a constraint satisfaction problem.

From a technical point of view, the work of Skalka and Smith [42] is the closest to this paper. We share with them the use of a type and effect system and that of model checking validity of effects. In [42], a static approach to history-based access control is proposed. The λ-calculus is enriched with access events and local checks on the past event history. Local checks make validity a regular property, so regularization is unneeded. The programming model and the type system of [42] allow for access events parametrized by constants, and for let-polymorphism. We have omitted these features for simplicity, but they can be easily recovered by using similar techniques.

A related line of research addresses the issue of modelling and analysing resource usage. Igarashi and Kobayashi [30] introduce a type systems to check whether a program accesses resources according to a user-defined usage policy. Our model is less general than the framework of [30], but we provide a

static verification technique, while [30] does not. Colcombet and Fradet [19] and Marriot, Stuckey and Sulzmann [35] mix static and dynamic techniques to transform programs in order to make them obey a given safety property. Besson, de Grenier de Latour and Jensen [7] tackle the problem of characterizing when a program can call a stack-inspecting method while respecting a global security policy. Compared to [19, 35, 7], our programming model allows for local policies, while the other only considers global ones.

Recently, increasing attention has been devoted to express service contracts as behavioural (or session) types. These synthetise the essential aspects of the interaction behaviour of services, while allowing for efficient static verification of properties of composed systems. Session types [29] have been exploited to formalize compatibility of components [45] and to describe adaptation of web services [17]. Security issues have been recently considered in terms of session types, e.g. in [13], which proves the decidability of type-checking in an extension of the π-calculus with session types and correspondence assertions [50]. Our λ^{req} has no explicit primitive for sessions. However, they can be suitably encoded, via higher-order functions.

Other papers have proposed type-based methodologies to check security properties of distributed systems. For instance, Gordon and Jeffrey [26] use a type and effect system to prove authenticity properties of security protocols. Web service authentication has been recently modelled and analysed in [8, 9] through a process calculus enriched with cryptographic primitives. In particular, [10] builds security libraries using the WS-Security policies of [3]. These libraries are then mechanically analysed with ProVerif [11].

10 Conclusions

We have enriched the λ-calculus with primitives to express service composition under security constraints, as a first step towards the design of programming constructs to publish, select and compose services on top of a semantic theory. The security requirements are safety and liveness properties over (an abstraction of) execution histories. These properties have a local scope, possibly nested.

We have used a type and effect system to extract from a given program a history expression, i.e. a safe approximation of its run-time behaviour that also includes plans, i.e. which services can be selected to serve requests. Indeed, plans drive service composition, so to achieve the task assigned while respecting the security constraints. A history expression is valid (under a plan π) when it represents execution histories (driven by π) that never violate the security policies within their scope.

To verify the validity of history expressions, we have exploited model checking over Basic Process Algebras and Finite State Automata. However, nesting of scopes makes validity non-regular, and has required us to transform history expressions (technically, to linearize and regularize them) so that model checking is feasible. When a history expression of a program e is verified valid under a plan π, then e will never go wrong. Therefore, at run-time it suffices to follow the plan π to enforce the required safety and liveness properties without resorting to any run-time monitoring.

The main result of this paper is resumed by the following theorem:

Theorem 10. Let $\Gamma, H \vdash e : \tau$, for e closed, and let $H' = \{\pi_1 \triangleright H_1 \cdots \pi_k \triangleright H_k\}$ be the linearization of H. If H_i is verified valid for some $i \in 1..k$, then the execution of e with plan π_i will never go wrong.

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A Proofs

B Type safety

To prove that our type and effect system for λ^{req} is correct, it is convenient to define a variant of λ^{req} where explicit framing events replace safety and liveness framings. Indeed, λ^{req} only generates plain event histories, while the semantic interpretation of history expressions comprises histories with framing events.

This new language is called $\lambda^\sharp$. Its syntax is much the same as λ^{req}, the only differences being that there are no policy framings, and events β may belong to $\mathsf{Ev} \cup \mathsf{Frm} \cup \{\, \rangle\!\rangle_\psi \mid \psi \in \mathsf{Pol} \,\}$. The special framing event $\rangle\!\rangle_\psi$ delimits an expression that has already fulfilled a liveness requirement ψ, but it has not exited from the framing $\psi\langle \cdots \rangle$ yet. The small-step operational semantics of $\lambda^\sharp$ is given below. The rules are similar to those of λ^{req}, with the difference that in $\lambda^\sharp$ a transition is possible only if the new history is valid. Since execution starts from the empty history (which is valid), then validity is preserved by evaluation. Note that, for simplicity, we allow $\lambda^\sharp$ to generate also non well-formed histories (i.e. histories that are not prefixes of balanced ones). However, we define below the encoding $\sharp$ of λ^{req} in $\lambda^\sharp$, which ensures that, for all $e \in \lambda^{req}$, the histories generated during the evaluation of $e^\sharp$ are always well-formed. The set Cls below denotes all the events that close the scope of a framing, i.e. $\mathsf{Cls} = \{\,]_\varphi, \rangle_\varphi, \rangle\!\rangle_\varphi \mid \varphi \in \mathsf{Pol} \,\}$.

Operational semantics of $\lambda^\sharp$

$$\frac{\eta, e_1 \twoheadrightarrow_\pi \eta', e_1' \quad \models \eta' \quad (e_2 \not\rangle_\psi \text{ or } \not\models \eta\rangle_\psi)}{\eta, e_1 e_2 \twoheadrightarrow_\pi \eta', e_1' e_2} \quad (\text{E}\sharp\text{-App1})$$

$$\frac{\eta, e_2 \twoheadrightarrow_\pi \eta', e_2' \quad \models \eta' \quad e_2 \notin \mathsf{Cls}}{\eta, v\, e_2 \twoheadrightarrow_\pi \eta', v\, e_2'} \quad (\text{E}\sharp\text{-App2})$$

$$\eta, (\lambda_z x.e) v \twoheadrightarrow_\pi \eta, e\{v/x, \lambda_z x.e/z\} \quad (\text{E}\sharp\text{-AbsApp})$$

$$\frac{\models \eta\beta}{\eta, \beta \twoheadrightarrow_\pi \eta\beta, *} \quad (\text{E}\sharp\text{-Ev}) \qquad \eta, \mathtt{if}\ b\ \mathtt{then}\ e_{tt}\ \mathtt{else}\ e_{ff} \twoheadrightarrow_\pi \eta, e_{\mathcal{B}(b)} \quad (\text{E}\sharp\text{-If})$$

$$\frac{\models \eta\,]_\varphi}{\eta, v]_\varphi \twoheadrightarrow_\pi \eta\,]_\varphi, v} \quad (\text{E}\sharp\text{-SF}) \qquad \frac{e_\ell : \tau_\ell \in \mathsf{Srv}^\sharp \quad \tau_\ell \approx \tau \quad \pi = r[\ell] \mid \pi'}{\eta, (\mathtt{req}_r \tau) v \twoheadrightarrow_\pi \eta, e_\ell v} \quad (\text{E}\sharp\text{-Req})$$

$$\frac{\models \eta\rangle_\psi}{\eta, e\rangle_\psi \twoheadrightarrow_\pi \eta, e\rangle\!\rangle_\psi} \quad (\text{E}\sharp\text{-LF1}) \qquad \eta, v\rangle\!\rangle_\psi \twoheadrightarrow_\pi \eta\,\rangle_\psi, v \quad (\text{E}\sharp\text{-LF2})$$

We now define an encoding $\sharp$ of λ^{req} in $\lambda^\sharp$. It will be used in Lemma 12 to prove that the behaviour of a λ^{req} expression e is equivalent (in a sense that will be formalized later) to the behaviour of its encoding $e^\sharp$. Note that the repository $\mathsf{Srv}^\sharp$ in the rule (E$\sharp$-LF2) comprises all the services $e_\ell^\sharp : \tau_\ell'$ such that $e_\ell : \tau_\ell \in \mathsf{Srv}$ (types are computed by the type systems for $\lambda^\sharp$, given below). The mapping $\sharp$

is defined inductively as follows:

$$x^\sharp = x \qquad \alpha^\sharp = \alpha \qquad (\mathtt{if}\ b\ \mathtt{then}\ e_0\ \mathtt{else}\ e_1)^\sharp = \mathtt{if}\ b\ \mathtt{then}\ e_0^\sharp\ \mathtt{else}\ e_1^\sharp$$

$$(\lambda_z x.\, e)^\sharp = \lambda_z x.\, e^\sharp \qquad (e_0\, e_1)^\sharp = e_0^\sharp\, e_1^\sharp$$

$$\varphi[e]^\sharp = [_\varphi; e^\sharp]_\varphi \qquad \psi\langle e\rangle^\sharp = \langle_\varphi; e^\sharp\rangle_\varphi \qquad (\mathtt{req}_r\tau)^\sharp = \mathtt{req}_r\tau$$

Note that a safety framing $\varphi[e]$ is mapped to the expression $[_\varphi; e^\sharp]_\varphi$. This is justified by looking at the following computation of $\varphi[e]^\sharp$:

$$\eta,\, [_\varphi; e^\sharp]_\varphi \;\twoheadrightarrow\; \eta[_\varphi,\, e^\sharp]_\varphi \;\twoheadrightarrow^*\; \eta[_\varphi\eta',\, v]_\varphi \;\twoheadrightarrow\; \eta[_\varphi\eta']_\varphi,\, v$$

Note that translating $\varphi[e]$ to $[_\varphi; e;]_\varphi$ is incorrect, because the computation of $\varphi[e]^\sharp$ would have the following form:

$$\eta,\, [_\varphi; e^\sharp;]_\varphi \;\twoheadrightarrow\; \eta[_\varphi,\, e^\sharp;]_\varphi \;\twoheadrightarrow^*\; \eta[_\varphi\eta',\, v;]_\varphi \;\twoheadrightarrow\; \eta[_\varphi\eta',\,]_\varphi$$

and thus the value v would be incorrectly discarded.

It is convenient to characterize the $\lambda^\sharp$ expressions that are obtained by encoding λ^{req} expressions. This will be used later, together with Lemma 11 below, when establishing our type safety result. The following well-formedness criterion, telling that a closing framing event may only occur as the second term of an application, will suffice for our purposes.

Definition 1. We say that a $\lambda^\sharp$ expression e is *well-formed* when:

- $e = *$, $e = x$, $e = \alpha$, $e = \mathtt{req}_r\tau$, $e \in \mathsf{Frm} \setminus \mathsf{Cls}$, or
- $e = \lambda_z x.e'$, and e' is well-formed, or
- $e = \mathtt{if}\ b\ \mathtt{then}\ e_0\ \mathtt{else}\ e_1$, and both e_0 and e_1 are well-formed, or
- $e = e_0 e_1$, e_0 is well-formed, and either e_1 is well-formed or $e_1 \in \mathsf{Cls}$.

Note that not all well-formed $\lambda^\sharp$ expressions (e.g. $\alpha]_\varphi$) are encodings of λ^{req} expressions. However, the following lemma states that the converse is true: the encoding of a λ^{req} expression is well-formed. Moreover, well-formedness is preserved by the semantics of $\lambda^\sharp$.

Lemma 11. For each $e_0 \in \lambda^{req}$, well-formed $e \in \lambda^\sharp$, and history η:

(11a) $e_0^\sharp$ is well-formed.

(11b) if $\eta, e \twoheadrightarrow_\pi \eta', e'$, then e' is well-formed.

Proof. For (11a), we proceed by induction on the structure of e_0. The only interesting cases are $e_0 = \varphi[e_1]$ and $e_0 = \psi\langle e_1\rangle$. In the fist case, we have that $e_0^\sharp = [_\varphi; e_1^\sharp\,]_\varphi = (\lambda.e_1^\sharp\,]_\varphi)\, [_\varphi$ is well-formed, because both $[_\varphi$ and $e_1^\sharp$ are such. The second case is similar.

For (11b), we proceed by induction on the depth of the proof of $\eta, e \twoheadrightarrow_\pi \eta', e'$. The base cases are trivial. For the inductive case, we have to consider the last rule applied in the derivation.

- case (E♯-App1). Let $e = e_0 e_1$, and $e' = e'_0 e_1$. Then:

$$\frac{\eta, e_0 \twoheadrightarrow_\pi \eta', e'_0 \quad \models \eta' \quad (e_1 \neq\rangle_\psi \text{ or } \not\models \eta\rangle_\psi)}{\eta, e_0 e_1 \twoheadrightarrow_\pi \eta', e'_0 e_1}$$

  By definition, either e_1 is well-formed, or $e_1 \in \mathsf{Cls}$. By the induction hypothesis, e'_0 is well-formed. Then, $e' = e'_0 e_1$ is well-formed, too.

- case (E♯-App2). Let $e = v e_1$, and $e' = v e'_1$. Then:

$$\frac{\eta, e_1 \twoheadrightarrow_\pi \eta', e'_1 \quad e_1 \notin \mathsf{Cls} \quad \models \eta'}{\eta, v\, e_1 \twoheadrightarrow_\pi \eta', v\, e'_1}$$

  Since $e = v e_1$ is well-formed and $e_1 \notin \mathsf{Cls}$, then both v and e_1 are well-formed. By the induction hypothesis, e'_1 is well-formed. Then, $e' = v e'_1$ is well-formed, too.

- case (E♯-SF).

$$\frac{\models \eta]_\varphi}{\eta, v]_\varphi \twoheadrightarrow_\varphi \eta]_\varphi, v}$$

  Since $e = v]_\varphi$ is well-formed, then $e' = v$ is well-formed, too.

- case (E♯-LF1).

$$\frac{\models \eta\rangle_\psi}{\eta, e_0\rangle_\psi \twoheadrightarrow_\pi \eta, e_0\rangle\!\rangle_\psi}$$

  Since $e = e_0\rangle_\psi$ is well-formed, then e_0 is well-formed, and so also $e' = e_0\rangle\!\rangle_\psi$.

- case (E♯-LF2).

$$\eta, v\rangle\!\rangle_\psi \twoheadrightarrow_\pi \eta\,\rangle_\psi, v$$

  Since $e = v\rangle\!\rangle_\psi$ is well-formed, then $e' = v$ is well-formed, too.

- case (E♯-AbsApp). Let $e = (\lambda_z x.e_0)v$, and $e' = e_0\{v/x, \lambda_z x.e_0/z\}$.

$$\eta, (\lambda_z x.e_0)v \twoheadrightarrow_\pi \eta, e_0\{v/x, \lambda_z x.e_0/z\}$$

  Since e is well-formed, then both e_0 and v are such (note in passing that it cannot be $v =]_\varphi$ or $v =\rangle_\varphi$, because $]_\varphi, \rangle_\varphi$ are not values). It is easy to prove the following statement, by induction on the structure of e:

$$\forall x, e, e' \in \lambda^\sharp.\ e, e' \text{ well-formed} \implies e\{e'/x\} \text{ well-formed}$$

  Therefore, $e' = e_0\{v/x, \lambda_z x.e_0/z\}$ is well-formed.

- case (E♯-If). Let $e = \texttt{if}\, b\, \texttt{then}\, e_{tt}\, \texttt{else}\, e_{ff}$, and $e' = e_{\mathcal{B}(b)}$.

$$\eta, \texttt{if}\, b\, \texttt{then}\, e_{tt}\, \texttt{else}\, e_{ff} \twoheadrightarrow_\pi \eta, e_{\mathcal{B}(b)}$$

  Since e is well-formed, both e_{tt} and e_{ff} are well-formed, too.

- case (E♯-Req). Let $e = \mathtt{req}_r\tau$ and $e' = e_\ell$. Then:

$$\frac{e_\ell : \tau_\ell \in \mathsf{Srv}^\sharp \quad \tau_\ell \approx \tau \quad \pi = r[\ell] \mid \pi'}{\eta, (\mathtt{req}_r\tau)v \twoheadrightarrow_\pi \eta, e_\ell^\sharp v}$$

  Since $e_\ell^\sharp$ is well-formed by (11a), so is $e_\ell^\sharp\, v$. □

To establish the correspondence between λ^{req} and $\lambda^\sharp$, we introduce an alternative semantics for λ^{req}, called *policy-tracking* semantics. This new semantics is used as a bridge between the two languages, and it has transitions $\eta, e, \Phi, \Psi \to_\pi \eta', e', \Phi', \Psi'$ that explicitly record the sequences Φ/Ψ of the active safety/liveness framings. To record entering a framing, we use the events $[_\varphi$ and $\langle_\psi$, i.e. $\varphi[e]$ is rewritten as $[_\varphi; \varphi[e]$. We record the fact that an expression e has succesfully exited the scope of ψ with the special framing $\psi\langle\!\langle e \rangle\!\rangle$, i.e. $\eta, \psi\langle e\rangle \to_\pi \eta, e$ is rendered as $\eta, \psi\langle e\rangle, \Phi, \Psi \to_\pi \eta, \psi\langle\!\langle e\rangle\!\rangle, \Phi, \Psi$. Values v can always exit from a framing $\psi\langle\!\langle v\rangle\!\rangle$. For well-typed expressions, the "policy-tracking" semantics is equivalent to the "policy-framing" one, modulo the events that tell when framings are entered, and the special framings $\psi\langle\!\langle e\rangle\!\rangle$.

We now define a backward translation $\flat$ from $\lambda^\sharp$ to the policy-tracking λ^{req} :

$$x^\flat = x \qquad \alpha^\flat = \alpha \qquad (\mathtt{if}\ b\ \mathtt{then}\ e_0\ \mathtt{else}\ e_1)^\flat = \mathtt{if}\ b\ \mathtt{then}\ e_0^\flat\ \mathtt{else}\ e_1^\flat$$

$$[_\varphi{}^\flat = [_\varphi \qquad \langle_\psi{}^\flat = \langle_\psi \qquad (\lambda_z x.\, e)^\flat = \lambda_z x.\, e^\flat \qquad (e_0\, e_1)^\flat = e_0^\flat\, e_1^\flat \quad (e_1 \notin \mathsf{Cls})$$

$$(e\,]_\varphi)^\flat = \varphi[e^\flat] \qquad (e\,\rangle_\psi)^\flat = \psi\langle e^\flat\rangle \qquad (e\,\rangle\!\rangle_\psi)^\flat = \psi\langle\!\langle e^\flat\rangle\!\rangle \qquad (\mathtt{req}_r\tau)^\flat = \mathtt{req}_r\tau$$

Note that $\flat$ is not the direct inverse of $\sharp$, but it becomes such when we consider the policy-tracking λ^{req} . For example, we have that $\varphi[e]^\sharp = [_\varphi; e^\sharp]_\varphi$, while $([_\varphi; e^\sharp]_\varphi)^\flat = [_\varphi^\flat; (e^\sharp]_\varphi)^\flat = [_\varphi; \varphi[(e^\sharp)^\flat]$. Recall that $[_\varphi; \varphi[e]$ is just the way a standard framing $\varphi[e]$ is encoded in the policy-tracking λ^{req} .

The following lemma establishes the equivalence between λ^{req} and $\lambda^\sharp$: each transition in $\lambda^\sharp$ may be mimicked by a (policy-tracking) transition in λ^{req} , and *viceversa*. The sequences $\Phi(\eta)/\Psi(\eta)$ contain the active safety/liveness policies of η, and are defined as follows:

$$\begin{array}{lll} \Phi(\varepsilon) = \varepsilon & \Phi(\eta\alpha) = \Phi(\eta) & \Phi(\eta[_\varphi) = \Phi(\eta)\varphi \\ \Phi(\eta\langle_\psi) = \Phi(\eta) & \Phi(\eta\rangle_\psi) = \Phi(\eta) & \Phi(\eta\varphi[\eta']) = \Phi(\eta) \end{array}$$

$$\begin{array}{lll} \Psi(\varepsilon) = \varepsilon & \Psi(\eta\alpha) = \Psi(\eta) & \Psi(\eta\langle_\psi) = \Psi(\eta)\psi \\ \Psi(\eta[_\varphi) = \Psi(\eta) & \Psi(\eta]_\varphi) = \Psi(\eta) & \Psi(\eta\psi\langle\eta'\rangle) = \Psi(\eta) \end{array}$$

Lemma 12. For all $\lambda^\sharp$ expressions e_0, e_1, histories η and plans π:

$$\varepsilon, e_0 \twoheadrightarrow_\pi^* \eta, e_1 \implies \varepsilon, e_0^\flat, \varepsilon, \varepsilon \to_\pi^* \eta^\flat, e_1^\flat, \Phi(\eta), \Psi(\eta) \tag{12a}$$

Moreover, for all λ^{req} expressions e_0, e_1, histories η and plans π:

$$\varepsilon, e_0, \varepsilon, \varepsilon \to_\pi^* \eta, e_1, \Phi, \Psi \implies \exists \bar\eta, \bar e_1 : \varepsilon, e_0^\sharp \twoheadrightarrow_\pi^* \bar\eta, \bar e_1 \tag{12b}$$

where $\bar\eta^\flat = \eta$, $\bar e_1{}^\flat = e_1$, $\Phi = \Phi(\bar\eta)$, and $\Psi = \Psi(\bar\eta)$.

Operational semantics of λ^{req} with active framings

$$\frac{\eta, e_1, \Phi, \Psi \to_\pi \eta', e_1', \Phi', \Psi'}{\eta, e_1 e_2, \Phi, \Psi \to_\pi \eta', e_1' e_2, \Phi', \Psi'} \quad \text{(E2-App1)}$$

$$\frac{\eta, e_2, \Phi, \Psi \to_\pi \eta', e_2', \Phi', \Psi'}{\eta, v e_2, \Phi, \Psi \to_\pi \eta', v e_2', \Phi', \Psi'} \quad \text{(E2-App2)}$$

$$\eta, (\lambda_z x.e)v, \Phi, \Psi \to_\pi \eta, e\{v/x, \lambda_z x.e/z\}, \Phi, \Psi \quad \text{(E2-AbsApp)}$$

$$\eta, \alpha, \Phi, \Psi \to_\pi \eta\alpha, *, \Phi, \Psi \quad \text{(E2-Ev)}$$

$$\eta, \texttt{if}\ b\ \texttt{then}\ e_{tt}\ \texttt{else}\ e_{ff}, \Phi, \Psi \to_\pi \eta, e_{\mathcal{B}(b)}, \Phi, \Psi \quad \text{(E2-If)}$$

$$\frac{\eta \models \varphi}{\eta, [_\varphi, \Phi, \Psi \to_\pi \eta, *, \Phi\varphi, \Psi} \quad \text{(E2-SF1)}$$

$$\frac{\eta, e, \Phi, \Psi \to_\pi \eta', e', \Phi', \Psi' \quad \eta' \models \varphi}{\eta, \varphi[e], \Phi, \Psi \to_\pi \eta', \varphi[e'], \Phi', \Psi'} \quad \text{(E2-SF2)}$$

$$\frac{\eta \models \varphi}{\eta, \varphi[v], \Phi\varphi, \Psi \to_\pi \eta, v, \Phi, \Psi} \quad \text{(E2-SF3)}$$

$$\eta, \langle_\psi, \Phi, \Psi \to_\pi \eta, *, \Phi, \Psi\psi \quad \text{(E2-LF1)}$$

$$\frac{\eta, e, \Phi, \Psi \to_\pi \eta', e', \Phi', \Psi' \quad \eta' \not\models \psi}{\eta, \psi\langle e\rangle, \Phi, \Psi \to_\pi \eta', \psi\langle e'\rangle, \Phi', \Psi'} \quad \text{(E2-LF2)}$$

$$\frac{\eta \models \psi}{\eta, \psi\langle e\rangle, \Phi, \Psi \to_\pi \eta, \psi\langle\langle e\rangle\rangle, \Phi, \Psi} \quad \text{(E2-LF3)}$$

$$\frac{\eta, e, \Phi, \Psi \to_\pi \eta', e', \Phi', \Psi'}{\eta, \psi\langle\langle e\rangle\rangle, \Phi, \Psi \to_\pi \eta', \psi\langle\langle e'\rangle\rangle, \Phi', \Psi'} \quad \text{(E2-LF4)}$$

$$\eta, \psi\langle\langle v\rangle\rangle, \Phi, \Psi\psi \to_\pi \eta, v, \Phi, \Psi \quad \text{(E2-LF5)}$$

$$\frac{e_\ell : \tau_\ell \in \mathsf{Srv} \quad \tau_\ell \approx \tau \quad \pi = r[\ell] \mid \pi'}{\eta, (\texttt{req}_r \tau)v \to_\pi \eta, e_\ell\, v} \quad \text{(E2-Req)}$$

Proof. For (12a), we proceed by induction on the number of steps. Assume that $\varepsilon, e_0 \twoheadrightarrow_\pi^* \eta_1, e_1$. We prove that:

$$\eta_1, e_1 \twoheadrightarrow_\pi \eta_2, e_2 \implies \eta_1^\flat, e_1^\flat, \Phi(\eta_1), \Psi(\eta_1) \to_\pi^* \eta_2^\flat, e_2^\flat, \Phi(\eta_2), \Psi(\eta_2)$$

We further induce on the derivation of $\eta_1, e_1 \twoheadrightarrow_\pi \eta_2, e_2$. There are the following exhaustive cases:

- case (E♯-Ev). Let $e_1 = \beta$, $e_2 = *$ and $\eta_2 = \eta_1\beta$. Then:

$$\frac{\models \eta_1\beta}{\eta_1, \beta \twoheadrightarrow_\pi \eta_1\beta, *}$$

There are two cases. If $\beta \in \mathsf{Ev}$, then $\eta_2^\flat = (\eta_1\beta)^\flat = \eta_1^\flat\beta$, $\Phi(\eta_2) = \Phi(\eta_1)$, $\Psi(\eta_2) = \Psi(\eta_1)$, $e_1^\flat = \beta^\flat = \beta$, and $e_2^\flat = *^\flat = *$. Thus, by (E2-Ev):

$$\eta_1^\flat, \beta, \Phi(\eta_1), \Psi(\eta_1) \rightarrow_\pi \eta_1^\flat\beta, *, \Phi(\eta_2), \Psi(\eta_2)$$

Otherwise $\beta \in \mathsf{Frm} \setminus \mathsf{Cls}$, because β is well-formed by Lemma 11. If $\beta = [_\varphi$, then $\eta_2^\flat = (\eta_1[_\varphi)^\flat = \eta_1^\flat$, $\Phi(\eta_2) = \Phi(\eta_1)\varphi$, $\Psi(\eta_2) = \Psi(\eta_1)$, $e_1^\flat = [_\varphi^\flat = [_\varphi$, $e_2^\flat = *^\flat = *$. Then, by (E2-SF1):

$$\eta_1^\flat, [_\varphi, \Phi(\eta_1), \Psi(\eta_1) \rightarrow_\pi^1 \eta_1^\flat, *, \Phi(\eta_1)\varphi, \Psi(\eta_1)$$

The case $\beta = \langle_\psi$ is similar, and it uses the rule (E2-LF1).

- case (E♯-AbsApp): Let $e_1 = (\lambda_z x.e)v$, $e_2 = e\{v/x, \lambda_z x.e/z\}$, $\eta_2 = \eta_1$.

$$\eta_1, (\lambda_z x.e)v \twoheadrightarrow_\pi \eta_1, e\{v/x, \lambda_z x.e/z\}$$

Then, $e_1^\flat = (\lambda_z x.e^\flat)v^\flat$. By (E2-AbsApp), we have that:

$$\eta^\flat, (\lambda_z x.e^\flat)v^\flat, \Phi(\eta_1), \Psi(\eta_1) \rightarrow_\pi \eta^\flat, e^\flat\{v^\flat/x, \lambda_z x.e^\flat/z\}, \Phi(\eta_1), \Psi(\eta_1)$$

The following statement can be easily proved by structural induction:

$$\forall e_0, e_1 \in \lambda^\sharp.\ (e_0\{e_1/x\})^\flat = e_0^\flat\{e_1^\flat/x\} \tag{8}$$

By (8), $e_2^\flat = (e\{v/x, \lambda_z x.e/z\})^\flat = (e^\flat\{v^\flat/x, \lambda_z x.e^\flat/z\})$.

- case (E♯-SF): Let $e_1 = v\,]_\varphi$ and $e_2 = v$.

$$\frac{\models \eta_1]_\varphi}{\eta_1, v]_\varphi \twoheadrightarrow_\pi \eta_1]_\varphi, v}$$

Here $e_1^\flat = (v]_\varphi)^\flat = \varphi[v^\flat]$, $e_2^\flat = v^\flat$, $\Phi(\eta_1]_\varphi) = \Phi$ for some Φ such that $\Phi(\eta_1) = \Phi\varphi$, and $\Psi(\eta_1) = \Psi(\eta_1]_\varphi)$. Since $\eta_1]_\varphi$ is valid, then $\eta_1^\flat \models \varphi$. Then, by (E2-SF3):

$$\frac{\eta_1^\flat \models \varphi}{\eta_1^\flat, \varphi[v^\flat], \Phi\varphi, \Psi \rightarrow_\pi \eta_1^\flat, v^\flat, \Phi, \Psi}$$

- case (E♯-LF1): Let $e_1 = e\rangle_\psi$ and $e_2 = e\rangle\!\rangle_\psi$.

$$\frac{\models \eta_1\rangle_\psi}{\eta_1, e\rangle_\psi \twoheadrightarrow_\pi \eta_1, e\rangle\!\rangle_\psi}$$

Here $e_1^\flat = (e\rangle_\psi)^\flat = \psi\langle e^\flat\rangle$, and $e_2^\flat = (e\rangle\!\rangle_\psi)^\flat = \psi\langle\langle e\rangle\rangle$. Since $\eta_1\rangle_\psi$ is valid, then $\eta_1^\flat \models \psi$ (by construction, none of the past histories inside the framing satisfies ψ). Then, by (E2-LF3):

$$\frac{\eta_1^\flat \models \varphi}{\eta_1^\flat, \varphi[v^\flat], \Phi, \Psi \rightarrow_\pi \eta_1^\flat, v^\flat, \Phi, \Psi}$$

- case (E♯-LF2): Let $e_1 = v\rangle\!\rangle_\psi$ and $e_2 = v$.

$$\eta_1, v\rangle\!\rangle_\psi \twoheadrightarrow_\pi \eta_1\rangle_\psi, v$$

  Here $e_1^\flat = (v\rangle\!\rangle_\psi)^\flat = \psi\langle\!\langle v^\flat\rangle\!\rangle$, $e_2^\flat = v^\flat$, $\Psi(\eta_1\rangle_\psi) = \Psi$ for some Ψ such that $\Psi(\eta_1) = \Psi\psi$, and $\Phi(\eta_1) = \Phi(\eta_1\rangle_\varphi)$. Then, by (E2-LF5):

$$\eta_1^\flat, \psi\langle\!\langle v^\flat\rangle\!\rangle, \Phi, \Psi\psi \to_\pi \eta_1^\flat, v^\flat, \Phi, \Psi$$

- the cases (E♯-App1), (E♯-App2), (E♯-If), (E♯-Req) are straightforward.

For (12b), we prove that, for all λ^{req} expressions e_0, e_1 and histories η:

$$\varepsilon, e_0, \varepsilon, \varepsilon \to_\pi^n \eta, e_1, \Phi, \Psi \implies \exists \bar\eta, \bar{e_1} : \bar{e_1}^\flat = e_1, \bar\eta^\flat = \eta, \Phi(\bar\eta) = \Phi, \Psi(\bar\eta) = \Psi \text{ and } \varepsilon, e_0^\sharp \twoheadrightarrow^n \bar\eta, \bar{e_1}$$

We proceed by induction on the number of steps in $\varepsilon, e_0, \varepsilon, \varepsilon \to_\pi^* \eta, e_1, \Phi$. For the base case, we have that $\varepsilon, e_0, \varepsilon, \varepsilon \to_\pi^0 \varepsilon, e_0, \varepsilon, \varepsilon$, and $\varepsilon, e_0^\sharp \twoheadrightarrow_\pi^0 \varepsilon, e_0^\sharp$. Then we are done, because $(e_0^\sharp)^\flat = e_0$. For the inductive case, assume that $\varepsilon, e_0, \varepsilon, \varepsilon \to_\pi^* \eta_1, e_1, \Phi_1, \Psi_1$. We prove the following inductive step:

$$\eta_1, e_1, \Phi_1, \Psi_1 \to_\pi \eta_2, e_2, \Phi_2, \Psi_2 \implies \exists \bar{\eta_2}, \bar{e_2} : \bar{e_2}^\flat = e_2, \bar{\eta_2}^\flat = \eta_2, \Phi(\bar{\eta_2}) = \Phi_2, \Psi(\bar{\eta_2}) = \Psi_2, \text{ and } \bar{\eta_1}, \bar{e_1} \twoheadrightarrow_\pi \bar{\eta_2}, \bar{e_2}$$

We proceed by a further induction on the derivation of $\eta_1, e_1, \Phi_1, \Psi_1 \to \eta_2, e_2, \Phi_2, \Psi_2$. There are the following cases (the omitted ones are straightforward):

- case (E2-Ev). Let $e_1 = \alpha$, and $e_2 = *$.

$$\eta_1, \alpha, \Phi_1, \Psi_1 \to \eta_1\alpha, *, \Phi_1, \Psi_1$$

  By the induction hypothesis (on the number of steps), $\bar{e_1} = \alpha$, $\bar{\eta_1}^\flat = \eta_1$, $\Phi(\bar{\eta_1}) = \Phi_1$, $\Psi(\bar{\eta_1}) = \Psi_1$ and $\bar{\eta_1}$ is valid. Choose $\bar{e_2} = *$ and $\bar{\eta_2} = \bar{\eta_1}\alpha$. Then, $\bar{e_2}^\flat = * = e_2$, $\bar{\eta_2}^\flat = \bar{\eta_1}^\flat\alpha = \eta_1\alpha = \eta_2$. Since $\bar{\eta_1}$ is valid, by definition of $\Phi()$, we have $\eta_1\alpha \models \varphi$ for all $\varphi \in \Phi_1$. Thus, $\bar{\eta_1}\alpha$ is valid, and, by (E♯-Ev):

$$\frac{\models \bar{\eta_1}\alpha}{\bar{\eta_1}, \alpha \twoheadrightarrow_\pi \bar{\eta_1}\alpha, *}$$

- case (E2-AbsApp). Let $e_1 = (\lambda_z x.e)v$, and $e_2 = e\{v/x, \lambda_z x.e/z\}$. Then:

$$\eta_1, (\lambda_z x.e)v, \Phi_1 \to_\pi \eta_1, e\{v/x, \lambda_z x.e/z\}, \Phi_1$$

  By the induction hypothesis (on the number of steps), we have $\bar{\eta_1}$ and $\bar{e_1}$ such that $\bar{\eta_1}^\flat = \eta_1$, $\bar{e_1}^\flat = e_1$, $\Phi(\bar{\eta_1}) = \Phi_1$, and $\bar{\eta_1}$ is valid. By definition of $\flat$, it follows that $\bar{e_1}$ is of the form $(\lambda_z x.\bar{e})\bar{v}$, where $\bar{e}^\flat = e$ and $\bar{v}^\flat = v$. Then, by (E♯-AbsApp),

$$\bar{\eta_1}, (\lambda_z x.\bar{e})\bar{v}, \twoheadrightarrow_\pi \bar{\eta_1}, \bar{e}\{\bar{v}/x, \lambda_z x.\bar{e}/z\}$$

  By (8), $(\bar{e}\{\bar{v}/x, \lambda_z x.\bar{e}/z\})^\flat = \bar{e}^\flat\{\bar{v}^\flat/x, (\lambda_z x.\bar{e})^\flat/z\} = e\{v/x, \lambda_z x.\bar{e}^\flat/z\} = e\{v/x, \lambda_z x.e/z\} = e_2$.

- case (E2-SF1). Let $e_1 = [_\varphi$, $e_2 = *$, then:

$$\frac{\eta_1 \models \varphi}{\eta_1, [_\varphi, \Phi_1, \Psi_1 \rightarrow_\pi \eta_1, *, \Phi_1\varphi, \Psi_1}$$

  By the induction hypothesis (on the number of steps), $\bar{e_1} = [_\varphi$, $\bar{\eta_1}^\flat = \eta_1$, $\Phi(\bar{\eta_1}) = \Phi_1$, $\Psi(\bar{\eta_1}) = \Psi_1$ and $\bar{\eta_1}$ is valid. Choose $\bar{e_2} = *$ and $\bar{\eta_2} = \bar{\eta_1}[_\varphi$. Then $\bar{e_1}^\flat = [_\varphi$, $\bar{e_2}^\flat = *$, $\Phi(\bar{\eta_2}) = \Phi(\bar{\eta_1}[_\varphi) = \Phi(\bar{\eta_1})\varphi = \Phi_1\varphi$, and $\Psi(\bar{\eta_2}) = \Psi(\bar{\eta_1}[_\varphi) = \Psi_1$. Since $\bar{\eta_1}^\flat = \eta_1 \models \varphi$, then $\bar{\eta_1}[_\varphi$ is valid, and, by (E♯-Ev):

$$\frac{\models \bar{\eta_1}[_\varphi}{\bar{\eta_1}, [_\varphi \twoheadrightarrow_\pi \bar{\eta_1}[_\varphi, *}$$

- case (E2-SF2). Let $e_1 = \varphi[e_3]$ and $e_2 = \varphi[e_4]$.

$$\frac{\eta_1, e_3, \Phi_1, \Psi_1 \rightarrow_\pi \eta_2, e_4, \Phi_2, \Psi_2 \quad \eta_2 \models \varphi}{\eta_1, \varphi[e_3], \Phi_1, \Psi_1 \rightarrow_\pi \eta_2, \varphi[e_4], \Phi_2, \Psi_2}$$

  By the induction hypothesis (on the derivation), there are $\bar{\eta_1}, \bar{\eta_2}, \bar{e_3}, \bar{e_4}$ such that $\bar{\eta_1}^\flat = \eta_1$, $\bar{\eta_2}^\flat = \eta_2$, $\bar{e_3}^\flat = e_3$, $\bar{e_4}^\flat = e_4$, $\Phi(\bar{\eta_2}) = \Phi_2$, $\Psi(\bar{\eta_2}) = \Psi_2$ and $\bar{\eta_1}, \bar{e_3} \twoheadrightarrow_\pi \bar{\eta_2}, \bar{e_4}$, with $\bar{\eta_2}$ valid. Let $\bar{e_1} = \bar{e_3}]_\varphi$, and $\bar{e_2} = \bar{e_4}]_\varphi$. Then, $\bar{e_1}^\flat = \varphi[\bar{e_3}^\flat] = \varphi[e_3]$, and $\bar{e_2}^\flat = \varphi[\bar{e_4}^\flat] = \varphi[e_4]$. Thus, by (E♯-App1):

$$\frac{\bar{\eta_1}, \bar{e_3} \twoheadrightarrow \bar{\eta_2}, \bar{e_4} \quad \models \bar{\eta_2} \quad]_\varphi \neq\rangle_\psi}{\bar{\eta_1}, \bar{e_3}]_\varphi \twoheadrightarrow \bar{\eta_2}, \bar{e_4}]_\varphi}$$

- case (E2-SF3). Let $e_1 = \varphi[v]$, and $e_2 = v$.

$$\eta_1, \varphi[v], \Phi\varphi, \Psi \rightarrow_\pi \eta_1, v, \Phi, \Psi$$

  By the induction hypothesis (on the number of steps), $\bar{e_1} = \bar{v}]_\varphi$ with $\bar{v}$ such that $\bar{v}^\flat = v$, $\bar{\eta_1}^\flat = \eta_1$, $\Phi(\bar{\eta_1}) = \Phi\varphi$, $\Psi(\bar{\eta_1}) = \Psi$, $\eta_1 \models \varphi$ and $\bar{\eta_1}$ is valid. Choose $\bar{e_2} = \bar{v}$. Then, $\bar{e_1}^\flat = \varphi[\bar{v}^\flat] = \varphi[v]$, $\bar{e_2}^\flat = \bar{v}^\flat = v$, $\Phi(\bar{\eta_1}]_\varphi) = \Phi$, $\Psi(\bar{\eta_1}]_\varphi) = \Psi$. Choose $\bar{\eta_2} = \bar{\eta_1}]_\varphi$. So, $\bar{\eta_2}$ is valid and $\bar{\eta_2}^\flat = \eta_1$. By (E♯-SF):

$$\frac{\models \bar{\eta_1}]_\varphi}{\bar{\eta_1}, \bar{v}]_\varphi \twoheadrightarrow_\pi \bar{\eta_1}]_\varphi, \bar{v}}$$

- case (E2-LF1). Let $e_1 = \langle_\psi$, $e_2 = *$, then:

$$\eta_1, \langle_\psi, \Phi_1, \Psi_1 \rightarrow_\pi \eta_1, *, \Phi_1, \Psi_1\psi$$

  By the induction hypothesis (on the number of steps), $\bar{e_1} = \langle_\varphi$, $\bar{\eta_1}^\flat = \eta_1$, $\Phi(\bar{\eta_1}) = \Phi_1$, $\Psi(\bar{\eta_1}) = \Psi_1$ and $\bar{\eta_1}$ is valid. Choose $\bar{e_2} = *$ and $\bar{\eta_2} = \bar{\eta_1}\langle_\psi$. Then $\bar{e_1}^\flat = \langle_\varphi$, $\bar{e_2}^\flat = *$, $\Phi(\bar{\eta_2}) = \Phi(\bar{\eta_1}\langle_\psi) = \Phi_1$, and $\Psi(\bar{\eta_2}) = \Psi(\bar{\eta_1}\langle_\psi) = \Psi(\bar{\eta_1})\psi = \Psi_1\psi$. Since $\bar{\eta_1}^\flat$ is valid, then $\bar{\eta_1}\langle_\psi$ is valid, and, by (E♯-Ev):

$$\frac{\models \bar{\eta_1}\langle_\psi}{\bar{\eta_1}, \langle_\varphi \twoheadrightarrow_\pi \bar{\eta_1}\langle_\varphi, *}$$

- case (E2-LF2). Let $e_1 = \psi\langle e_3\rangle$ and $e_2 = \psi\langle e_4\rangle$.

$$\frac{\eta_1, e_3, \Phi_1, \Psi_1 \rightarrow_\pi \eta_2, e_4, \Phi_2, \Psi_2 \quad \eta_2 \not\models \varphi}{\eta_1, \varphi\langle e_3\rangle, \Phi_1, \Psi_1 \rightarrow_\pi \eta_2, \psi\langle e_4\rangle, \Phi_2, \Psi_2}$$

By the induction hypothesis (on the derivation), there are $\bar{\eta_1}, \bar{\eta_2}, \bar{e_3}, \bar{e_4}$ such that $\bar{\eta_1}^\flat = \eta_1$, $\bar{\eta_2}^\flat = \eta_2$, $\bar{e_3}^\flat = e_3$, $\bar{e_4}^\flat = e_4$, $\Phi(\bar{\eta_2}) = \Phi_2$, $\Psi(\bar{\eta_2}) = \Psi_2$ and $\bar{\eta_1}, \bar{e_3} \twoheadrightarrow_\pi \bar{\eta_2}, \bar{e_4}$, with $\bar{\eta_2}$ valid. Let $\bar{e_1} = \bar{e_3}\rangle_\psi$, and $\bar{e_2} = \bar{e_4}\rangle_\psi$. Then, $\bar{e_1}^\flat = \psi\langle\bar{e_3}^\flat\rangle = \psi\langle e_3\rangle$, and $\bar{e_2}^\flat = \psi\langle\bar{e_4}^\flat\rangle = \psi\langle e_4\rangle$. Since $\eta_2 \not\models \psi$, then $\bar{\eta_2}\rangle_\psi$ is not valid. Thus, by (E♯-App1):

$$\frac{\bar{\eta_1}, \bar{e_3} \twoheadrightarrow \bar{\eta_2}, \bar{e_4} \quad \models \bar{\eta_2} \quad \not\models \bar{\eta_2}\rangle_\psi}{\bar{\eta_1}, \bar{e_3}\langle\psi\rangle \twoheadrightarrow \bar{\eta_2}, \bar{e_4}\langle\psi\rangle}$$

- case (E2-LF3). Let $e_1 = \psi\langle e\rangle$ and $e_2 = \psi\langle\!\langle e\rangle\!\rangle$.

$$\frac{\eta_1 \models \psi}{\eta_1, \psi\langle e\rangle, \Phi_1, \Psi_1 \rightarrow_\pi \eta_1, \psi\langle\!\langle e\rangle\!\rangle, \Phi_1, \Psi_1}$$

By the induction hypothesis (on the number of steps), we have $\bar{e_1}$ and $\bar{\eta_1}$ such that $\bar{e_1}^\flat = e_1$ and $\bar{\eta_1}^\flat = \eta_1$, with $\bar{\eta_1}$ valid. Then, $\bar{e_1} = \bar{e}\rangle_\psi$ for some $\bar{e}$ such that $\bar{e}^\flat = e$. Choose $\bar{e_2} = \bar{e}\rangle\!\rangle_\psi$. Then, $\bar{e_2}^\flat = (\bar{e}\rangle\!\rangle_\psi)^\flat = \psi\langle\!\langle\bar{e}^\flat\rangle\!\rangle = \psi\langle\!\langle e\rangle\!\rangle$. Since $\eta_1 \models \psi$ and $\bar{\eta_1}$ is valid, then also $\bar{\eta_1}\rangle_\psi$ is valid. Then, by (E♯-LF1):

$$\frac{\models \bar{\eta_1}\rangle_\psi}{\bar{\eta_1}, \bar{e}\rangle_\psi \twoheadrightarrow_\pi \bar{\eta_1}, \bar{e}\rangle\!\rangle_\psi}$$

- case (E2-LF4). Let $e_1 = \psi\langle\!\langle e_3\rangle\!\rangle$ and $e_2 = \psi\langle\!\langle e_4\rangle\!\rangle$.

$$\frac{\eta_1, e_3, \Phi_1, \Psi_1 \rightarrow_\pi \eta_2, e_4, \Phi_2, \Psi_2}{\eta_1, \varphi\langle\!\langle e_3\rangle\!\rangle, \Phi_1, \Psi_1 \rightarrow_\pi \eta_2, \psi\langle\!\langle e_4\rangle\!\rangle, \Phi_2, \Psi_2}$$

Similarly to the case (E2-LF2), let $\bar{e_1} = \bar{e_3}\rangle_\psi$, and $\bar{e_2} = \bar{e_4}\rangle_\psi$. Then, $\bar{e_1}^\flat = \psi\langle\!\langle\bar{e_3}^\flat\rangle\!\rangle = \psi\langle\!\langle e_3\rangle\!\rangle$, and $\bar{e_2}^\flat = \psi\langle\!\langle\bar{e_4}^\flat\rangle\!\rangle = \psi\langle\!\langle e_4\rangle\!\rangle$. Thus, by (E♯-App1):

$$\frac{\bar{\eta_1}, \bar{e_3} \twoheadrightarrow \bar{\eta_2}, \bar{e_4} \quad \models \bar{\eta_2}}{\bar{\eta_1}, \bar{e_3}\rangle\!\rangle_\psi \twoheadrightarrow \bar{\eta_2}, \bar{e_4}\rangle\!\rangle_\psi}$$

- case (E2-LF5). Let $e_1 = \psi\langle\!\langle v\rangle\!\rangle$, and $e_2 = v$.

$$\eta_1, \psi\langle\!\langle v\rangle\!\rangle, \Phi, \Psi\psi \rightarrow_\pi \eta_1, v, \Phi, \Psi$$

By the induction hypothesis (on the number of steps), $\bar{e_1}$ is of the form $\bar{v}\rangle\!\rangle_\psi$ with $\bar{v}$ such that $\bar{v}^\flat = v$, $\bar{\eta_1}^\flat = \eta_1$, $\Phi(\bar{\eta_1}) = \Phi$, $\Psi(\bar{\eta_1}) = \Psi\psi$ and $\bar{\eta_1}$ is valid. Choose $\bar{e_2} = \bar{v}$. Then, $\bar{e_1}^\flat = \psi\langle\!\langle\bar{v}^\flat\rangle\!\rangle = \psi\langle\!\langle v\rangle\!\rangle$, $\bar{e_2}^\flat = \bar{v}^\flat = v$, $\Phi(\bar{\eta_1}\rangle_\varphi) = \Phi$ and $\Psi(\bar{\eta_1}\rangle_\varphi) = \Psi$. Choose $\bar{\eta_2} = \bar{\eta_1}\rangle_\psi$. Then, $\bar{\eta_2}^\flat = \bar{\eta_1}^\flat = \eta_1$, and, by (E♯-LF2):

$$\bar{\eta_1}, \bar{v}\rangle\!\rangle_\psi \twoheadrightarrow \bar{\eta_1}\rangle_\psi, \bar{v}$$

□

Type and effect system for $\lambda^\sharp$

$$\Gamma, \varepsilon \vdash_\sharp * : \mathit{unit} \quad (\text{T}\sharp\text{-Unit}) \qquad \Gamma, \beta \vdash_\sharp \beta : \mathit{unit} \quad (\text{T}\sharp\text{-Ev})$$

$$\Gamma, \varepsilon \vdash_\sharp x : \Gamma(x) \quad (\text{T}\sharp\text{-Var}) \qquad \frac{\Gamma; x : \tau; z : \tau \xrightarrow{H} \tau', H \vdash_\sharp e : \tau'}{\Gamma, \varepsilon \vdash_\sharp \lambda_z x.e : \tau \xrightarrow{H} \tau'} \quad (\text{T}\sharp\text{-Abs})$$

$$\frac{\Gamma, H \vdash_\sharp e : \tau \xrightarrow{H''} \tau' \quad \Gamma, H' \vdash_\sharp e' : \tau \quad e' \notin \mathsf{Cls}}{\Gamma, H \cdot H' \cdot H'' \vdash_\sharp e e' : \tau'} \quad (\text{T}\sharp\text{-App})$$

$$\frac{\Gamma, H \vdash_\sharp e : \tau}{\Gamma, H\cdot]_\varphi \vdash_\sharp e]_\varphi : \tau} \quad (\text{T}\sharp\text{-SF}) \qquad \frac{\Gamma, H \vdash_\sharp e : \tau \quad \beta \in \{\rangle_\psi, \rangle\!\rangle_\psi\}}{\Gamma, H\cdot\rangle_\psi \vdash_\sharp e\beta : \tau} \quad (\text{T}\sharp\text{-LF})$$

$$\frac{\Gamma, H \vdash_\sharp e : \tau \quad \Gamma, H \vdash_\sharp e' : \tau}{\Gamma, H \vdash_\sharp \mathtt{if}\, b\, \mathtt{then}\, e\, \mathtt{else}\, e' : \tau} \quad (\text{T}\sharp\text{-If}) \qquad \frac{\Gamma, H \vdash_\sharp e : \tau}{\Gamma, H + H' \vdash_\sharp e : \tau} \quad (\text{T}\sharp\text{-Wkn})$$

$$\frac{\tau' = \biguplus\{\, \tau \oplus^\sharp_{r[\ell]} \tau_\ell \mid e_\ell : \tau_\ell \in \mathsf{Srv}^\sharp \wedge \tau_\ell \approx \tau \,\}}{\Gamma, \varepsilon \vdash \mathtt{req}_r \tau : \tau'} \quad (\text{T}\sharp\text{-Req})$$

We now introduce a type and effect system for $\lambda^\sharp$, that resembles that for λ^{req}, with the exception that it also deals with framing events. By (T♯-Ev), an opening framing event $[_\varphi$ is typed in the same way as standard access events. A closing event $]_\varphi$ cannot be typed alone: the only rule that deals such an event is (T♯-SF), which requires $]_\varphi$ to be applied to an expression e. Roughly, this is because our encoding of λ^{req} in $\lambda_\sharp$ maps a policy framing $\varphi[e]$ to the expression $[_\varphi; e^\sharp]_\varphi$ (see the proof of the following lemma for details). The operator $\oplus^\sharp$ acts as $\oplus$, but it discards the constraints on the request type, e.g.:

$$(\mathit{unit} \xrightarrow{\varphi[\bullet]} \mathit{unit}) \oplus^\sharp_{r[\ell]} (\mathit{unit} \xrightarrow{\alpha} \mathit{unit}) = \mathit{unit} \xrightarrow{\{r[\ell] \rhd \alpha\}} \mathit{unit}$$

We now prove that typing a λ^{req} expression and its encoding in $\lambda^\sharp$ yields "equivalent" history expressions. The only difference is in that constraints on request types are maintained in λ^{req} and discarded in $\lambda^\sharp$. For example, consider $e = (\mathtt{req}_r \mathit{unit} \xrightarrow{\varphi[\bullet]} \mathit{unit})*$ and $e_{\ell_1} = \lambda.\alpha$, with $\mathsf{Srv}^\sharp = \mathsf{Srv} = \{e_{\ell_1} : \mathit{unit} \xrightarrow{\alpha} \mathit{unit}\}$. Then, $\{r[\ell_1] \rhd \varphi[\alpha]\} \vdash e : \mathit{unit}$, while $\{r[\ell_1] \rhd \alpha\} \vdash_\sharp e : \mathit{unit}$.

Lemma 13. If $\Gamma, H \vdash e : \tau$, then there exist H', τ' such that (i) $\Gamma, H' \vdash_\sharp e^\sharp : \tau'$, (ii) $(\llbracket H \rrbracket^\pi)^\flat = (\llbracket H' \rrbracket^\pi)^\flat$, and (iii) if H is π-valid, then H' is π-valid.

Proof. By induction on the depth of the proof of $\Gamma, H \vdash e : \tau$. The only interesting case is when the rules (T-SF) or (T-LF) have been applied. The proofs are similar, so we only consider the first case. We have:

$$\frac{\Gamma, H \vdash e : \tau}{\Gamma, \varphi[H] \vdash \varphi[e] : \tau}$$

By definition of $\sharp$, $\varphi[e]^\sharp = [_\varphi; e^\sharp\,]_\varphi = (\lambda_z.e^\sharp]_\varphi)\,[_\varphi$ for z fresh. By the induction hypothesis, $\Gamma, H' \vdash_\sharp e^\sharp : \tau'$. It is easy to prove, by induction on the typing derivation, the following weakening result:

$$\Gamma, H' \vdash_\sharp e : \tau \implies \Gamma; x : \tau', H' \vdash_\sharp e : \tau \quad \text{if } x \notin fv(e) \tag{9}$$

Now let $z : unit \xrightarrow{H'\cdot]_\varphi} \tau'$. Then, $\Gamma; z : unit \xrightarrow{H'\cdot]_\varphi}, H' \vdash_\sharp e^\sharp : \tau'$, and we have the following typing derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma; z : unit \xrightarrow{H'\cdot]_\varphi} \tau', H' \vdash_\sharp e^\sharp : \tau'}{\Gamma; z : unit \xrightarrow{H'\cdot]_\varphi} \tau', H'\cdot]_\varphi \vdash_\sharp e^\sharp\,]_\varphi : \tau'}}{\Gamma, \varepsilon \vdash_\sharp \lambda_z.e^\sharp]_\varphi : unit \xrightarrow{H'\cdot]_\varphi} \tau'} \qquad \dfrac{}{\Gamma, [_\varphi \vdash_\sharp [_\varphi : unit}}{\Gamma, [_\varphi \cdot H' \cdot]_\varphi \vdash_\sharp (\lambda.e^\sharp]_\varphi)\,[_\varphi : \tau'}$$

The obtained history expression fulfils the lemma, because, by the induction hypothesis, $([\![[_\varphi \cdot H' \cdot]_\varphi]\!]^\pi)^\flat = ([\![H']\!]^\pi)^\flat = ([\![H]\!]^\pi)^\flat = ([\![\varphi[H]]\!]^\pi)^\flat$.

Note in passing that the case (T$\sharp$-App) is trivial, because if $e = e_0 e_1$, then $e^\sharp = e_0^\sharp e_1^\sharp$ never takes the form $e']_\varphi$, by definition of the encoding $\sharp$. □

Subject reduction holds for $\lambda^\sharp$. It states that, if a $\lambda^\sharp$ expression e_0 is typable with effect H_0, and there is a transition $\eta_0, e_0 \twoheadrightarrow_\pi \eta_1, e_1$, then e_1 is also typable, with an effect H_1 such that $\eta_1[\![H_1]\!]^\pi$ is more precise (i.e. it contains fewer histories) than $\eta_0[\![H_0]\!]^\pi$. Note that we only admit well-formed expressions. As stated in Lemma 11 above, this is not a restriction, because the transformation of a λ^{req} expression e under $\sharp$ is always well-formed.

Lemma 14 (Subject Reduction). If $\Gamma, H_0 \vdash_\sharp e_0 : \tau$ and $\eta_0, e_0 \twoheadrightarrow_\pi \eta_1, e_1$, for e_0 closed and well-formed, then there exists H_1 such that $\Gamma, H_1 \vdash_\sharp e_1 : \tau$ and $\eta_1[\![H_1]\!]^\pi \subseteq \eta_0[\![H_0]\!]^\pi$.

Proof. By induction on the depth of the proof of $\Gamma, H_0 \vdash_\sharp e_0 : \tau$. For the base case, note that neither (T$\sharp$-Var) nor (T$\sharp$-Unit) can be used, because the former rule types variables x, which are not closed, while the latter rule types $*$, which does not admit further transitions. Therefore, the base case involves the rule (T$\sharp$-Ev). Let $e_0 = \beta$, and $H_0 = \beta$. Then, by (T$\sharp$-Ev):

$$\Gamma, \beta \vdash_\sharp \beta : unit$$

Let $e_1 = *$, and $\eta_1 = \eta_0\beta$. By (E$\sharp$-Ev), we have that:

$$\dfrac{\models \eta_0\beta}{\eta_0, \beta \twoheadrightarrow_\pi \eta_0\beta, *}$$

By (T$\sharp$-Unit), $\Gamma, \varepsilon \vdash_\sharp * : unit$. Let $H_1 = \varepsilon$. Then, $\eta_1[\![H_1]\!] = \eta_0\beta\varepsilon = \eta_0[\![H_0]\!]$.

For the inductive case, consider the last step of the typing derivation. We only have the following cases:

- case (T$\sharp$-App). Let $e_0 = ee'$, with $e' \notin \mathsf{Cls}$, and $H_0 = H \cdot H' \cdot H''$. Then:

$$\dfrac{\Gamma, H \vdash_\sharp e : \tau \xrightarrow{H''} \tau' \quad \Gamma, H' \vdash_\sharp e' : \tau \quad e' \notin \mathsf{Cls}}{\Gamma, H \cdot H' \cdot H'' \vdash_\sharp ee' : \tau'}$$

We have the following three further subcases, according to the rule used to deduce $\eta_0, e_0 \twoheadrightarrow_\pi \eta_1, e_1$.

1. If (T♯-App1) has been used, then:

$$\frac{\eta_0, e \twoheadrightarrow_\pi \eta_1, e'' \quad \models \eta_1}{\eta_0, ee' \twoheadrightarrow_\pi \eta_1, e''e'}$$

Let $e_1 = e''e'$. By the induction hypothesis, there exists H_1' such that $\Gamma, H_1' \vdash e'' : \tau \xrightarrow{H''} \tau'$ and $\eta_1[\![H_1']\!] \subseteq \eta_0[\![H]\!]$. Then, by (T♯-App):

$$\frac{\Gamma, H_1' \vdash_\sharp e'' : \tau \xrightarrow{H''} \tau' \quad \Gamma, H' \vdash_\sharp e' : \tau \quad e' \notin \mathsf{Cls}}{\Gamma, H_1' \cdot H' \cdot H'' \vdash_\sharp e''e' : \tau'}$$

Let $H_1 = H_1' \cdot H' \cdot H''$. Then, $\eta_1[\![H_1]\!] = \eta_1[\![H_1']\!][\![H' \cdot H'']\!] \subseteq \eta_0[\![H]\!][\![H' \cdot H'']\!] = \eta_0[\![H \cdot H' \cdot H'']\!] = \eta_0[\![H_0]\!]$.

2. If (E♯-App2) has been used, then $e = v$, and:

$$\frac{\eta_0, e' \twoheadrightarrow_\pi \eta_1, e'' \quad \models \eta_1 \quad e' \notin \mathsf{Cls}}{\eta_0, ve' \twoheadrightarrow_\pi \eta_1, ve''}$$

Let $e_1 = ve''$. By the induction hypothesis, there exists H_1' such that $\Gamma, H_1' \vdash e'' : \tau$ and $\eta_1[\![H_1']\!] \subseteq \eta_0[\![H']\!]$. Since e is a closed value, then $H = \varepsilon$. By Lemma 11, e'' is well-formed, so $e'' \notin \mathsf{Cls}$. Then, by (T♯-App):

$$\frac{\Gamma, \varepsilon \vdash_\sharp e : \tau \xrightarrow{H''} \tau' \quad \Gamma, H_1' \vdash_\sharp e'' : \tau \quad e'' \neq]_\varphi}{\Gamma, \varepsilon \cdot H_1' \cdot H'' \vdash_\sharp e''e' : \tau'}$$

Let $H_1 = \varepsilon \cdot H_1' \cdot H''$. Then, $\eta_1[\![H_1]\!] = \eta_1[\![H_1']\!][\![H'']\!] \subseteq \eta_0[\![H']\!][\![H'']\!] = \eta_0[\![H' \cdot H'']\!] = \eta_0 H_0$.

3. If (E♯-Req) has been used, then $e = \mathtt{req}_r\tau_r$, $e' = v$, and:

$$\frac{e_\ell : \tau_\ell \in \mathsf{Srv} \quad \tau_\ell \approx \tau_r \quad \pi = r[\ell] \mid \pi'}{\eta_0, (\mathtt{req}_r\tau_r)v \twoheadrightarrow_\pi \eta_0, e_\ell^\sharp\, v}$$

By (T♯-Req), it follows that:

$$\tau \xrightarrow{H''} \tau' \;=\; \uplus\{\, \tau_r \oplus^\sharp_{r[\ell']} \tau_{\ell'} \mid e_{\ell'} : \tau_{\ell'} \in \mathsf{Srv} \wedge \tau_r \approx \tau_{\ell'} \,\}$$

Let $e_{\ell_1} \ldots e_{\ell_n}$ be the services such that $\tau_r \approx \tau_{\ell_i}$, and let $\tau_{\ell_i} = \tau_i \xrightarrow{H_i} \tau_i'$ for $i \in 1..n$. Then, $H = H' = \varepsilon$, and $H'' = \{r[\ell_1] \rhd H_1 \cdots r[\ell_n] \rhd H_n\}$. Since $\tau_\ell \approx \tau_r$, there exists $k \in 1..n$ such that $\ell = \ell_k$. By construction of Srv, we have that $e_{\ell_k} : \tau_{\ell_k} \in \mathsf{Srv}$ implies $H_k \vdash e_{\ell_k} : \tau_{\ell_k}$. Let $H_1 = H_k$. Then:

$$\eta_0[\![H_0]\!]^\pi = \eta_0[\![\varepsilon \cdot \varepsilon \cdot \{r[\ell_1] \rhd H_1 \cdots r[\ell_n] \rhd H_n\}]\!]^{r[\ell_k]|\pi'} = \eta_0[\![H_k]\!]^\pi = \eta_0[\![H_1]\!]^\pi$$

- case (T♯-SF). Let $H_0 = H]_\varphi$ and $e_0 = e]_\varphi$. Then:

$$\frac{\Gamma, H \vdash_\sharp e : \tau}{\Gamma, H\cdot]_\varphi \vdash_\sharp e]_\varphi : \tau}$$

We have two further subcases, according to the rule used to deduce $\eta_0, e_0 \twoheadrightarrow_\pi \eta_1, e_1$ (note that the rule (E♯-App2) cannot be used, because it requires $e_1 \notin \mathsf{Cls}$). If (E♯-App1) has been used, then:

$$\frac{\eta_0, e \twoheadrightarrow_\pi \eta_1, e' \quad \models \eta_1}{\eta_0, e]_\varphi \twoheadrightarrow_\pi \eta_1, e']_\varphi}$$

Let $e_1 = e']_\varphi$. By the induction hypothesis, there exists H' such that $\Gamma, H' \vdash_\sharp e' : \tau$, and $\eta_1[\![H']\!] \subseteq \eta_0[\![H]\!]$. Then, by (T♯-SF):

$$\frac{\Gamma, H' \vdash_\sharp e' : \tau}{\Gamma, H'\cdot]_\varphi \vdash_\sharp e']_\varphi : \tau}$$

Let $H_1 = H'\cdot]_\varphi$. Then, $\eta_1[\![H_1]\!] = \eta_1[\![H']\!]\,]_\varphi \subseteq \eta_0[\![H]\!]\,]_\varphi = \eta_0[\![H_0]\!]$.

Otherwise, if (E♯-SF) has been used, then:

$$\frac{\models \eta_0]_\varphi}{\eta_0, v]_\varphi \twoheadrightarrow_\pi \eta_0]_\varphi, v}$$

Let $e_0 = v]_\varphi$, $e_1 = v$, and $\eta_1 = \eta_0]_\varphi$. As a premise of (T♯-SF), we have that $\Gamma, H \vdash_\sharp v : \tau$. Since v is a closed value, then $H = \varepsilon$, and hence $H = \varepsilon]_\varphi =]_\varphi$. Let $H_1 = \varepsilon$. Then, $\eta_1[\![H_1]\!] = \eta_0]_\varphi\varepsilon = \eta_0[\![H_0]\!]$.

- case (T♯-LF). Let $H_0 = H\rangle_\psi$ and $e_0 = e\beta$, with $\beta \in \{\rangle_\psi, \rangle\!\rangle_\psi\}$. Then:

$$\frac{\Gamma, H \vdash_\sharp e : \tau}{\Gamma, H\cdot\rangle_\psi \vdash_\sharp e\beta : \tau}$$

We have the following three subcases, according to the event β and to the rule used to deduce $\eta_0, e_0 \twoheadrightarrow_\pi \eta_1, e_1$ (note that the rule (E♯-App2) cannot be used, because it requires $e_1 \notin \mathsf{Cls}$). If (E♯-App1) has been used, then:

$$\frac{\eta_0, e \twoheadrightarrow_\pi \eta_1, e' \quad \models \eta_1 \quad (\beta \neq\rangle_\psi \text{ or } \not\models \eta_0\rangle_\psi)}{\eta_0, e\beta \twoheadrightarrow_\pi \eta_1, e'\beta}$$

Let $\beta =\rangle_\psi$ and $e_1 = e'\rangle_\varphi$. Then, $\eta' \not\models \psi$, and by the induction hypothesis, there exists H' such that $\Gamma, H' \vdash_\sharp e' : \tau$, and $\eta_1[\![H']\!] \subseteq \eta_0[\![H]\!]$. By (T♯-LF):

$$\frac{\Gamma, H' \vdash_\sharp e' : \tau}{\Gamma, H'\cdot\rangle_\psi \vdash_\sharp e'\rangle_\psi : \tau}$$

Let $H_1 = H'\cdot\rangle_\varphi$. Then, $\eta_1[\![H_1]\!] = \eta_1[\![H']\!]\,\rangle_\psi \subseteq \eta_0[\![H]\!]\,\rangle_\varphi = \eta_0[\![H_0]\!]$.

Otherwise, if $\beta =\rangle_\psi$ and (E♯-LF1) has been used, then:

$$\frac{\models \eta_0\rangle_\psi}{\eta_0, e\rangle_\psi \twoheadrightarrow_\pi \eta_0, e\rangle\!\rangle_\psi}$$

Let $e_0 = e\rangle_\psi$, $e_1 = e\rangle\!\rangle_\psi$, and $\eta_1 = \eta_0$. By (T♯-LF):

$$\frac{\Gamma, H \vdash_\sharp e : \tau}{\Gamma, H\cdot\rangle_\psi \vdash_\sharp e\rangle\!\rangle_\psi : \tau}$$

Let $H_1 = H_0$. Then, $\eta_1[\![H_1]\!] = \eta_0[\![H_0]\!]$.

The last subcase is when $\beta = \rangle\!\rangle_\psi$ and (E♯-LF2) has been applied:

$$\eta_0, v\rangle\!\rangle_\psi \twoheadrightarrow_\pi \eta_0\rangle_\psi, v$$

Let $e_0 = v\rangle\!\rangle_\psi$, $e_1 = v$, and $\eta_1 = \eta_0\rangle_\psi$. As a premise of (T♯-LF), we have that $\Gamma, H \vdash_\sharp e : \tau$. Since v is a closed value, then $H = \varepsilon$. Thereby, $H = \varepsilon\rangle_\psi = \rangle_\psi$. Let $H_1 = \varepsilon$. Then, $\eta_1[\![H_1]\!] = \eta_0\rangle_\psi\varepsilon = \eta_0[\![H_0]\!]$.

- case (T♯-If). Let $e_0 = \texttt{if}\, b\, \texttt{then}\, e_{tt}\, \texttt{else}\, e_{ff}$, then:

$$\frac{\Gamma, H_0 \vdash_\sharp e_{tt} : \tau \quad \Gamma, H_0 \vdash_\sharp e_{ff} : \tau}{\Gamma, H_0 \vdash_\sharp \texttt{if}\, b\, \texttt{then}\, e_{tt}\, \texttt{else}\, e_{ff} : \tau}$$

Then, by (E♯-If):

$$\eta_0, \texttt{if}\, b\, \texttt{then}\, e_{tt}\, \texttt{else}\, e_{ff} \twoheadrightarrow_\pi \eta_0, e_{\mathcal{B}(b)}$$

Let $\eta_1 = \eta_0$ and $H_1 = H_0$. Then, $\eta_1[\![H_1]\!] = \eta_0[\![H_0]\!]$.

- case (T♯-Wkn). Let $H_0 = H + H'$. Then:

$$\frac{\Gamma, H \vdash_\sharp e_0 : \tau}{\Gamma, H + H' \vdash_\sharp e_0 : \tau}$$

By the induction hypothesis, there exists H_1 such that $\Gamma, H_1 \vdash_\sharp e_1 : \tau$ and $\eta_1[\![H_1]\!] \subseteq \eta_0[\![H]\!]$. Then, $\eta_1[\![H_1]\!] \subseteq \eta_0[\![H]\!] \subseteq \eta_0[\![H]\!] \cup \eta_0[\![H']\!] = \eta_0[\![H + H']\!] = \eta_0[\![H_0]\!]$. □

Theorem 15. If $\Gamma, H \vdash e : \tau$ and $\varepsilon, e \to_\pi^* \eta, e'$, then $\eta \in ([\![H]\!]^\pi)^{\flat\partial}$.

Proof. By Lemma 13, $\Gamma, H^\sharp \vdash_\sharp e^\sharp : \tau$, with $([\![H^\sharp]\!]^\pi)^\flat = ([\![H]\!]^\pi)^\flat$. By Lemma 12, $\varepsilon, e^\sharp \twoheadrightarrow_\pi^* \bar{\eta}, \bar{e}$, with $\bar{\eta}^\flat = \eta$ and $\bar{e}^\flat = e'$. We now prove the following statement:

$$\Gamma, H^\sharp \vdash_\sharp e : \tau \ \wedge\ \varepsilon, e \twoheadrightarrow_\pi^* \eta, e' \implies \eta \in ([\![H^\sharp]\!]^\pi)^\partial \tag{10}$$

This follows by Lemma 14, using a straightforward inductive argument. In particular, Lemma 14 ensures that $\Gamma, H' \vdash_\sharp e' : \tau$ and $\varepsilon[\![H^\sharp]\!]^\pi \supseteq \eta[\![H']\!]^\pi$, for some H'. Thus, $\bar{\eta} \in (\bar{\eta}[\![H']\!]^\pi)^\partial \subseteq ([\![H^\sharp]\!]^\pi)^\partial$, and:

$$\eta = \bar{\eta}^\flat \in \left(([\![H^\sharp]\!]^\pi)^\partial\right)^\flat = \left(([\![H^\sharp]\!]^\pi)^\flat\right)^\partial = ([\![H]\!]^\pi)^{\flat\partial}$$ □

The following lemma states that a well-typed $\lambda_\sharp$ expression with a valid effect is never stuck, i.e. either it is a value or it can take a transition. This propery if usually referred to as *progress*, and it will be used together with subject reduction to prove that our type and effect system enjoys type safety.

Lemma 16 (Progress). Let $\Gamma, H \vdash_\sharp e : \tau$, for e closed, and let ηH be π-valid for some η, π. Then, either e is a value, or there exists a transition $\eta, e \twoheadrightarrow_\pi \eta', e'$.

Proof. By induction on the depth of the proof of the typing derivation. We only have to consider the following cases:

- case (T♯-Ev). Let $e = \beta$, and $H = \beta$. Then:

$$\Gamma, \beta \vdash_\sharp \beta : \mathit{unit}$$

Since $\eta H = \eta\beta$ is valid, then, by (E♯-Ev):

$$\frac{\models \eta\beta}{\eta, \beta \twoheadrightarrow_\pi \eta\beta, *}$$

- case (T♯-App). Let $e = e_0 e_1$, and $H = H_0 \cdot H_1 \cdot H_2$. Then:

$$\frac{\Gamma, H_0 \vdash_\sharp e_0 : \tau \xrightarrow{H_2} \tau' \quad \Gamma, H_1 \vdash_\sharp e_1 : \tau \quad e_1 \notin \mathsf{Cls}}{\Gamma, H_0 \cdot H_1 \cdot H_2 \vdash_\sharp e_0 e_1 : \tau'}$$

There are four further subcases.

1. If e_0 is not a value, then, by induction hypothesis, $\eta, e_0 \twoheadrightarrow_\pi \eta', e_0'$ for some η' and e_0'. By Lemma 14, $\Gamma, H' \vdash e_0' : \tau$ for some H' such that $\eta'[\![H']\!] \subseteq \eta[\![H_0]\!]$. Since $\eta[\![H_0 \cdot H_1 \cdot H_2]\!]$ is valid, and validity is prefix-closed, then $\eta[\![H_0]\!]$ is also valid. Then, $\eta'[\![H']\!]$ is valid, as well as its prefix η'. By (E♯-App1):

$$\frac{\eta, e_0 \twoheadrightarrow_\pi \eta', e_0' \quad \models \eta'}{\eta, e_0 e_1 \twoheadrightarrow_\pi \eta', e_0' e_1}$$

2. If e_0 is a (closed) value and e_1 is not a value, then $H_0 = \varepsilon$. By the induction hypothesis, $\eta, e_1 \twoheadrightarrow_\pi \eta', e_1'$ for some η' and e_1'. By Lemma 14, $\Gamma, H' \vdash e_1' : \tau$ for some H' such that $\eta'[\![H']\!] \subseteq \eta[\![H_1]\!]$. Since $\eta[\![\varepsilon \cdot H_1 \cdot H_2]\!]$ is valid, then $\eta[\![H_1]\!]$ is also valid, as well as its subset $\eta'[\![H']\!]$ and its prefix η'. Then, by (E♯-App2):

$$\frac{\eta, e_1 \twoheadrightarrow_\pi \eta', e_1' \quad \models \eta' \quad e_1 \notin \mathsf{Cls}}{\eta, e_0 e_1 \twoheadrightarrow_\pi \eta', e_0 e_1'}$$

3. If $e_0 = \lambda_z x.\, e''$ and e_1 is a closed value, then $H_0 = H_1 = \varepsilon$. Since $\eta[\![H_0 \cdot H_1 \cdot H_2]\!] = \eta[\![\varepsilon \cdot \varepsilon \cdot H_2]\!]$ is valid, then η is valid. By (E♯-AbsApp):

$$\eta, (\lambda_z x.e'')e_1 \twoheadrightarrow_\pi \eta, e''\{e_1/x, \lambda_z x.e''/z\}$$

4. If $e_0 = \mathtt{req}_r \tau_r$ and e_1 is a closed value, then $H_0 = H_1 = \varepsilon$, and:

$$\tau \xrightarrow{H_2} \tau' \;=\; \Cup\{\, \tau_r \oplus_{r[\ell]} \tau_\ell \mid e_\ell : \tau_\ell \in \mathsf{Srv}^\sharp \wedge \tau_r \approx \tau_\ell \,\}$$

Let $e_{\ell_1} \ldots e_{\ell_n}$ be the services such that $\tau_r \approx \tau_{\ell_i}$, and let $\tau_{\ell_i} = \tau_i \xrightarrow{K_i} \tau_i'$ for $i \in 1..n$. Then, $H_2 = \{r[\ell_1] \rhd K_1 \cdots r[\ell_1] \rhd K_n\}$. By contradiction, if π is not of the form $r[\ell_i] \mid \pi'$ for any π' and $i \in 1..k$, then $[\![H_2]\!]^\pi = \bot$. Thereby, $H_0 \cdot H_1 \cdot H_2$ would not be π-valid. Then, $\pi = r[\ell_i] \mid \pi'$ for some i and π'. By (E♯-Req):

$$\frac{e_{\ell_i} : \tau_{\ell_i} \in \mathsf{Srv}^\sharp \quad \tau_{\ell_i} \approx \tau_r \quad \pi = r[\ell_i] \mid \pi'}{\eta, (\mathtt{req}_r \tau_r)e_1 \twoheadrightarrow_\pi \eta, e^\sharp_{\ell_i} e_1}$$

- case (T♯-SF). Let $e = e']_\varphi$, and $H = H'\cdot]_\varphi$. Then:

$$\frac{\Gamma, H' \vdash_\sharp e' : \tau}{\Gamma, H'\cdot]_\varphi \vdash_\sharp e']_\varphi : \tau}$$

  Since $\eta[\![H'\cdot]_\varphi]\!]$ is valid, then $\eta[\![H']\!]$ is valid, too. There are two subcases.

  If e' is not a value, then, by the induction hypothesis, $\eta, e' \twoheadrightarrow \eta', e''$ for some η' and e''. By Lemma 14, $\Gamma, H'' \vdash_\sharp e'' : \tau$ for some H'' such that $\eta''[\![H'']\!] \subseteq \eta[\![H']\!]$. Since $\eta[\![H']\!]$ contains only valid histories, then all the histories in $\eta'[\![H'']\!]$ are valid, and η' is valid, too, because validity is prefix-closed. Thus, by (E♯-App1):

$$\frac{\eta, e' \twoheadrightarrow_\pi \eta', e'' \quad \eta' \text{ valid}}{\eta, e']_\varphi \twoheadrightarrow_\pi \eta', e'']_\varphi}$$

  Otherwise, if e' is a (closed) value v, then $H' = \varepsilon$, and $\eta[\![H'\cdot]_\varphi]\!] = \eta\,]_\varphi$ is valid. Thus, by (E♯-SF):

$$\frac{\eta]_\varphi \text{ valid}}{\eta, v]_\varphi \twoheadrightarrow_\pi \eta]_\varphi, v}$$

- case (T♯-LF). Let $e = e'\beta$, $\beta \in \{\rangle_\psi, \rangle\!\rangle_\psi\}$ and $H = H'\cdot\rangle_\psi$.

$$\frac{\Gamma, H' \vdash_\sharp e' : \tau \quad \beta \in \{\rangle_\psi, \rangle\!\rangle_\psi\}}{\Gamma, H'\cdot\rangle_\psi \vdash_\sharp e'\beta : \tau}$$

  Since $\eta[\![H'\cdot\rangle_\psi]\!]$ is valid, then $\eta[\![H']\!]$ is valid. By the induction hypothesis, if e' is not a value, then $\eta, e' \twoheadrightarrow_\pi \eta', e''$ with η' valid. We have the following four subcases.

  1. If e' is not a value, $\beta = \rangle_\psi$ and $\eta\rangle_\psi$ is not valid, then, by (E♯-App1):

$$\frac{\eta, e' \twoheadrightarrow_\pi \eta', e'' \quad \models \eta' \quad \not\models \eta\rangle_\psi}{\eta, e'\rangle_\psi \twoheadrightarrow_\pi \eta', e''\rangle_\psi}$$

  2. If $\beta = \rangle_\psi$ and $\eta\rangle_\psi$ is valid, then, by (E♯-LF1):

$$\frac{\models \eta\rangle_\psi}{\eta, e'\rangle_\psi \twoheadrightarrow_\pi \eta, e'\rangle\!\rangle_\psi}$$

  3. If e' is not a value and $\beta = \rangle\!\rangle_\psi$, then, by (E♯-App1):

$$\frac{\eta, e' \twoheadrightarrow_\pi \eta', e'' \quad \models \eta'}{\eta, e'\rangle\!\rangle_\psi \twoheadrightarrow_\pi \eta', e''\rangle\!\rangle_\psi}$$

  4. If e' is not a value v and $\beta = \rangle\!\rangle_\psi$, then, by (E♯-LF2):

$$\eta, v\rangle\!\rangle_\psi \twoheadrightarrow_\pi \eta\,\rangle_\psi, v$$

- case (T♯-If). Let $e = \texttt{if}\, b\, \texttt{then}\, e_{tt}\, \texttt{else}\, e_{ff}$. Then:

$$\frac{\Gamma, H \vdash_\sharp e_{tt} : \tau \quad \Gamma, H \vdash_\sharp e_{ff} : \tau}{\Gamma, H \vdash_\sharp \texttt{if}\, b\, \texttt{then}\, e_{tt}\, \texttt{else}\, e_{ff} : \tau}$$

  Then, by the rule (E♯-If):

$$\eta, \texttt{if}\, b\, \texttt{then}\, e_{tt}\, \texttt{else}\, e_{ff} \twoheadrightarrow_\pi \eta, e_{\mathcal{B}}(b)$$

- case (T♯-Wkn). Let $H = H_0 + H_1$. Then:

$$\frac{\Gamma, H_0 \vdash_\sharp e : \tau}{\Gamma, H_0 + H_1 \vdash_\sharp e : \tau}$$

  Since $\eta H = \eta(H_0 + H_1)$ is valid, and $\eta[\![H_0]\!] \subseteq \eta[\![H]\!]$, then ηH_0 is valid. Then, by the induction hypothesis, $\eta, e \twoheadrightarrow_\pi \eta', e'$ for some η' and e'. □

Theorem 17 (Type Safety). If $\Gamma, H \vdash e : \tau$, with e closed. If H is π-valid, then the plan π is viable for e.

Proof. By contradiction, let $\varepsilon, e \to_\pi^* \eta, e_0$, and let η, e_0 be a stuck configuration, i.e. e_0 is not a value, and there is no outgoing transition from η, e_0. Since e is well-typed, there exists an equivalent policy-tracking computation $\varepsilon, e, \varepsilon, \varepsilon \to_\pi^* \eta, e_0', \Phi, \Psi$. By Lemma 12b, there exists $\bar\eta$ and $\bar{e_0}$ such that $\varepsilon, e^\sharp \twoheadrightarrow \bar\eta, \bar{e_0}$, with $\bar\eta^\flat = \eta$ and $\bar{e_0}^\flat = e_0'$. By Lemma 13, $H' \vdash_\sharp \bar e : \tau'$ for some π-valid H'. By Lemma 14, $\bar H \vdash_\sharp \bar{e_0} : \tau'$, for some $\bar H$ such that $\bar\eta[\![\bar H]\!] \subseteq [\![H']\!]$. Since H' is π-valid, then $\bar\eta[\![\bar H]\!]^\pi$ is valid, and $\bar\eta$ is valid, too, because validity is prefix-closed. By Lemma 12a, since η, e_0 is stuck, then also $\bar\eta, \bar{e_0}$ is stuck. Then, by Lemma 16, $\bar{e_0}$ must be a value. By definition of $\flat$, since $\bar{e_0}^\flat = e_0'$, then e_0' would be a value, and so also e_0: this contradicts the hypothesis that η, e_0 is stuck. □

Theorem 3. Let $\Gamma, H \vdash_{\mathsf{Srv}} e : \tau$, with e closed, and H valid for some well-typed plan π. Let $\varepsilon, e \to_\pi^* \eta, \mathcal{C}(\psi\langle e'\rangle)$, with e' having guarded λ-abstractions only. Then, there exists k such that, for each computation:

$$\eta, \mathcal{C}(\psi\langle e'\rangle) \to_\pi \eta_1, e_1 \to_\pi \cdots \to_\pi \eta_k, e_k$$

there exists $n \leq k$ such that $e_n = \mathcal{C}(\psi\langle e''\rangle)$ and $\eta_n \models \psi$.

Proof. Let H' be the effect of $\mathcal{C}(\psi\langle e'\rangle)$. Since $\mathcal{C}$ is an evaluation context, H' has the form $\psi\langle H_0\rangle \cdot H_1$, for some H_0 and H_1. By Lemma 14, $\eta[\![H']\!] \subseteq [\![H]\!]$, and so the validity of H implies that of $\eta \cdot H'$. Therefore, for each $\mathcal{H} \in \eta[\![H']\!] = \eta[\![\psi\langle H_0\rangle]\!][\![H_1]\!]$, there exists a history η' in $\eta[\![H_0]\!]$ obeying ψ. Now consider a computation:

$$\eta, \mathcal{C}(\psi\langle e'\rangle) \to_\pi \eta_1, e_1 \to_\pi \cdots$$

By Theorem 15, all $\eta\eta_i \in [\![H']\!]$. Thus, there exists n such that $\eta\eta_n \models \psi$. □

Note that the condition on guarded abstractions is needed to ensure that the effect of any recursive functions has the form $\mu h.\, H$. If an abstraction (with no events) were not guarded, it would be possible to assign it a non-recursive,

arbitrary effect — possibly valid w.r.t. any liveness policy ψ. This would clearly invalidate the statement of Theorem 3.

However, since all the λ-abstractions in e' are guarded (say by an event α), the effect H of $\psi\langle e'\rangle$ has the form $H = \psi\langle H_0 \cdot (\mu h.\, \alpha \cdot H') \cdot H_1\rangle$, where H_0 has no recursion. Indeed, by rule (T-Abs), the typing of a guarded recursive function $\lambda_z x.\, \alpha; e$ requires to equate the actual effect $\alpha \cdot H''$ of the function body with the latent effect of the function — and this can only be obtained by folding $\alpha \cdot H''$ into $\mu h.\, \alpha \cdot H'$, where $H'' = H'\{\mu h.\ \alpha \cdot H'/h\}$.

C Planning

Definition 2. A plan π is *less defined* than π' ($\pi \sqsubseteq \pi'$) when $\pi' = r[\ell] \mid \pi$. Two plans π, π' are *dependent* when $\pi \sqsubseteq \pi'$ or $\pi' \sqsubseteq \pi$.

Lemma 18. If $\pi' \sqsubseteq \pi$, then $[\![\{\pi' \rhd H\}]\!]^\pi_\rho = [\![H]\!]^\pi_\rho$.

Proof. Straightforward, by induction on the structure of π'. □

Lemma 19. Let $H = \{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}$. Then, for all plans π:

$$[\![H]\!]^\pi \;=\; \bigcup \{\, [\![H_i]\!]^\pi \mid \pi_i \sqsubseteq \pi \,\}$$

Proof. By Lemma 18:

$$\begin{aligned} [\![H]\!]^\pi = [\![\{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}]\!]^\pi \;&=\; \bigcup_{i \in 1..k} [\![\{\pi_i \rhd H_i\}]\!]^\pi \\ &= \bigcup \{\, [\![H_i]\!]^\pi \mid \pi_i \sqsubseteq \pi \,\} \end{aligned}$$

□

Lemma 20. The relation $\equiv$ is a congruence.

Proof. Let $\mathcal{C}(\_)$ be a history expression with a hole. We must prove that $\mathcal{C}(H) \equiv \mathcal{C}(H')$ whenever $H \equiv H'$. We proceed by induction on the structure of $\mathcal{C}$. The base cases $\mathcal{C} = \varepsilon$, $\mathcal{C} = \alpha$, $\mathcal{C} = h$ and $\mathcal{C} = \_$ are trivial. For the inductive case, we have the following subcases:

- if $\mathcal{C}(\_) = \mathcal{C}_1(\_) \cdot \mathcal{C}_2(\_)$ then, by the induction hypothesis:

$$\begin{aligned} [\![\mathcal{C}(H)]\!]^\pi_\rho = [\![\mathcal{C}_1(H) \cdot \mathcal{C}_2(H)]\!]^\pi_\rho \;&=\; [\![\mathcal{C}_1(H)]\!]^\pi_\rho \, [\![\mathcal{C}_2(H)]\!]^\pi_\rho \\ = [\![\mathcal{C}_1(H')]\!]^\pi_\rho \, [\![\mathcal{C}_2(H')]\!]^\pi_\rho \;&=\; [\![\mathcal{C}_1(H') \cdot \mathcal{C}_2(H')]\!]^\pi_\rho = [\![\mathcal{C}(H')]\!]^\pi_\rho \end{aligned}$$

- if $\mathcal{C}(\_) = \{\pi' \rhd \mathcal{C}_1(\_)\}$, then we have the following subcases, according to the structure of π':
  - if $\pi' = 0$, then, by the induction hypothesis:

$$[\![\{0 \rhd \mathcal{C}_1(H)\}]\!]^\pi_\rho = [\![\mathcal{C}_1(H)]\!]^\pi_\rho = [\![\mathcal{C}_1(H')]\!]^\pi_\rho = [\![\{0 \rhd \mathcal{C}_1(H')\}]\!]^\pi_\rho$$

– if $\pi' = \pi_0 \mid \pi_1$, we have two further subcases. If $\pi_0 \mid \pi_1 \sqsubseteq \pi$, then, by the induction hypothesis and by Lemma 18:

$$\begin{aligned}
[\![\{\pi_0 \mid \pi_1 \rhd \mathcal{C}_1(H)\}]\!]^\pi_\rho &= [\![\{\pi_0 \rhd \mathcal{C}_1(H)\}]\!]^\pi_\rho \ \cup_\perp \ [\![\{\pi_1 \rhd \mathcal{C}_1(H)\}]\!]^\pi_\rho \\
&= [\![\mathcal{C}_1(H)]\!]^\pi_\rho \ \cup_\perp \ [\![\mathcal{C}_1(H)]\!]^\pi_\rho \\
&= [\![\mathcal{C}_1(H')]\!]^\pi_\rho \ \cup_\perp \ [\![\mathcal{C}_1(H')]\!]^\pi_\rho \\
&= [\![\{\pi_0 \rhd \mathcal{C}_1(H')\}]\!]^\pi_\rho \ \cup_\perp \ [\![\{\pi_1 \rhd \mathcal{C}_1(H')\}]\!]^\pi_\rho \\
&= [\![\{\pi_0 \mid \pi_1 \rhd \mathcal{C}_1(H')\}]\!]^\pi_\rho
\end{aligned}$$

Otherwise, if π is not compatible with either π_0 or π_1:

$$\begin{aligned}
[\![\{\pi_0 \mid \pi_1 \rhd \mathcal{C}_1(H)\}]\!]^\pi_\rho &= [\![\{\pi_0 \rhd \mathcal{C}_1(H)\}]\!]^\pi_\rho \cup_\perp \ [\![\{\pi_1 \rhd \mathcal{C}_1(H)\}]\!]^\pi_\rho \\
&= \perp \ = \ [\![\{\pi_0 \rhd \mathcal{C}_1(H')\}]\!]^\pi_\rho \cup_\perp [\![\{\pi_1 \rhd \mathcal{C}_1(H')\}]\!]^\pi_\rho \\
&= [\![\{\pi_0 \mid \pi_1 \rhd \mathcal{C}_1(H')\}]\!]^\pi_\rho
\end{aligned}$$

– if $\pi' = r[\ell]$, then we have two further subcases. If $\pi = r[\ell] \mid \pi''$, then:

$$[\![\{r[\ell] \rhd \mathcal{C}_1(H)\}]\!]^\pi_\rho = [\![\mathcal{C}_1(H)]\!]^\pi_\rho = [\![\mathcal{C}_1(H')]\!]^\pi_\rho = [\![\{r[\ell] \rhd \mathcal{C}_1(H')\}]\!]^\pi_\rho$$

Otherwise, $[\![\mathcal{C}(H)]\!]^\pi_\rho = \perp = [\![\mathcal{C}(H')]\!]^\pi_\rho$.

- if $\mathcal{C}(\_) = \mu h.\, \mathcal{C}_1(\_)$, then recall that:

$$\begin{aligned}
[\![\mathcal{C}(H)]\!]^\pi_\rho &= \bigcup_{n\in\omega} f^n(\perp) & f(X) &= [\![\mathcal{C}_1(H)]\!]^\pi_{\rho\{X/h\}} \\
[\![\mathcal{C}(H')]\!]^\pi_\rho &= \bigcup_{n\in\omega} g^n(\perp) & g(X) &= [\![\mathcal{C}_1(H')]\!]^\pi_{\rho\{X/h\}}
\end{aligned}$$

By induction on n, we prove that $f^n(\perp) = g^n(\perp)$ for all n. The base case $n = 0$ is trivial: $f^0(\perp) = \perp = g^0(\perp)$. For the inductive case, we have:

$$\begin{aligned}
f^{n+1}(\perp) = f(f^n(\perp)) &= [\![\mathcal{C}_1(H)]\!]^\pi_{\rho\{f^n(\perp)/h\}} = [\![\mathcal{C}_1(H)]\!]^\pi_{\rho\{g^n(\perp)/h\}} \\
&= [\![\mathcal{C}_1(H')]\!]^\pi_{\rho\{g^n(\perp)/h\}} = g(g^n(\perp)) = g^{n+1}(\perp)
\end{aligned}$$

- the cases $\mathcal{C}(\_) = \mathcal{C}_1(\_) + \mathcal{C}_2(\_)$, $\mathcal{C}(\_) = \varphi[\mathcal{C}_1(\_)]$, and $\mathcal{C}(\_) = \psi\langle\mathcal{C}_1(\_)\rangle$ are similar to the first case. □

Lemma 21. $H \equiv \{0 \rhd H\}$

Proof. By definition, $[\![\{0 \rhd H\}]\!]^\pi_\rho = [\![H]\!]^\pi_\rho$. □

Lemma 22.

$$\begin{aligned}
&\{\pi_0 \rhd H_0\} + \{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\} \equiv \\
&\qquad \{\pi_0 \mid \pi_1 \rhd H_0 + H_1 \cdots \pi_0 \mid \pi_k \rhd H_0 + H_k\}
\end{aligned}$$

Proof. Let H and H' be the left term and the right term of the equation above, respectively. Let π be the evaluation plan, and let $I \subseteq 1..k$ be the set of indexes

i such that $\pi_i \sqsubseteq \pi$. Then, by Lemma 18:

$$\begin{aligned}
[\![H]\!]^\pi_\rho &= [\![\{\pi_0 \rhd H_0\}]\!]^\pi_\rho \cup_\perp [\![\{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}]\!]^\pi_\rho \\
&= [\![\{\pi_0 \rhd H_0\}]\!]^\pi_\rho \cup_\perp \bigcup_{i \in 1..k} [\![\{\pi_i \rhd H_i\}]\!]^\pi_\rho \\
&= [\![\{\pi_0 \rhd H_0\}]\!]^\pi_\rho \cup_\perp \bigcup_{i \in I} [\![H_i]\!]^\pi_\rho \\
[\![H']\!]^\pi_\rho &= \bigcup_{i \in 1..k} [\![\{\pi_0 \mid \pi_i \rhd H_0 + H_i\}]\!]^\pi_\rho \\
&= \bigcup_{i \in 1..k} \Big([\![\{\pi_0 \rhd H_0 + H_i\}]\!]^\pi_\rho \cup_\perp [\![\{\pi_i \rhd H_0 + H_i\}]\!]^\pi_\rho \Big) \\
&= \bigcup_{i \in I} \Big([\![\{\pi_0 \rhd H_0 + H_i\}]\!]^\pi_\rho \cup_\perp [\![H_0 + H_i]\!]^\pi_\rho \Big)
\end{aligned}$$

If $\pi_0 \sqsubseteq \pi$, then it follows that:

$$\begin{aligned}
[\![H]\!]^\pi_\rho = [\![H_0]\!]^\pi_\rho \cup_\perp \bigcup_{i \in I} [\![H_i]\!]^\pi_\rho \;&=\; \bigcup_{i \in I} [\![H_0]\!]^\pi_\rho \cup_\perp [\![H_i]\!]^\pi_\rho \;=\; \bigcup_{i \in I} [\![H_0 + H_i]\!]^\pi_\rho \\
=\; \bigcup_{i \in I} \Big([\![H_0 + H_i]\!]^\pi_\rho \cup_\perp [\![H_0 + H_i]\!]^\pi_\rho \Big) \;&=\; [\![H']\!]^\pi_\rho
\end{aligned}$$

Otherwise, we have that $[\![\{\pi_0 \rhd H_0\}]\!]^\pi_\rho = \perp = [\![\{\pi_0 \rhd H_0 \cdot H_i\}]\!]^\pi_\rho$, thus:

$$\begin{aligned}
[\![H]\!]^\pi_\rho = \perp \cup_\perp \bigcup_{i \in I} [\![H_i]\!]^\pi_\rho \;&=\; \perp \\
[\![H']\!]^\pi_\rho = \bigcup_{i \in I} \Big(\perp \cup_\perp [\![H_0 + H_i]\!]^\pi_\rho \Big) \;&=\; \bigcup_{i \in I} \perp \;=\; \perp
\end{aligned}$$

□

Lemma 23. $\{\pi_0 \rhd H_0\} \cdot \{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\} \equiv \{\pi_0 | \pi_1 \rhd H_0 \cdot H_1 \cdots \pi_0 | \pi_k \rhd H_0 \cdot H_k\}$

Proof. Let H and H' be the left term and the right term of the equation above, respectively. Let π be the evaluation plan, and let $I \subseteq 1..k$ be the set of indexes i such that $\pi_i \sqsubseteq \pi$. Then, by Lemma 18:

$$\begin{aligned}
[\![H]\!]^\pi_\rho &= [\![\{\pi_0 \rhd H_0\}]\!]^\pi_\rho [\![\{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}]\!]^\pi_\rho \\
&= [\![\{\pi_0 \rhd H_0\}]\!]^\pi_\rho \bigcup_{i \in 1..k} [\![\{\pi_i \rhd H_i\}]\!]^\pi_\rho \\
&= [\![\{\pi_0 \rhd H_0\}]\!]^\pi_\rho \bigcup_{i \in I} [\![H_i]\!]^\pi_\rho \\
[\![H']\!]^\pi_\rho &= \bigcup_{i \in 1..k} [\![\{\pi_0 \mid \pi_i \rhd H_0 \cdot H_i\}]\!]^\pi_\rho \\
&= \bigcup_{i \in 1..k} \Big([\![\{\pi_0 \rhd H_0 \cdot H_i\}]\!]^\pi_\rho \cup_\perp [\![\{\pi_i \rhd H_0 \cdot H_i\}]\!]^\pi_\rho \Big) \\
&= \bigcup_{i \in I} \Big([\![\{\pi_0 \rhd H_0 \cdot H_i\}]\!]^\pi_\rho \cup_\perp [\![H_0 \cdot H_i]\!]^\pi_\rho \Big)
\end{aligned}$$

If $\pi_0 \sqsubseteq \pi$, then it follows that:

$$\begin{aligned}[\![H]\!]^\pi_\rho = [\![H_0]\!]^\pi_\rho \bigcup_{i\in I} [\![H_i]\!]^\pi_\rho \;&=\; \bigcup_{i\in I} [\![H_0]\!]^\pi_\rho\, [\![H_i]\!]^\pi_\rho \;=\; \bigcup_{i\in I} [\![H_0 \cdot H_i]\!]^\pi_\rho \\ &=\; \bigcup_{i\in I} \Big([\![H_0 \cdot H_i]\!]^\pi_\rho \cup_\bot [\![H_0 \cdot H_i]\!]^\pi_\rho \Big) \;=\; [\![H']\!]^\pi_\rho\end{aligned}$$

Otherwise, we have that $[\![\{\pi_0 \rhd H_0\}]\!]^\pi_\rho = \bot = [\![\{\pi_0 \rhd H_0 \cdot H_i\}]\!]^\pi_\rho$, thus:

$$\begin{aligned}[\![H]\!]^\pi_\rho &= \bot \bigcup_{i\in I} [\![H_i]\!]^\pi_\rho \;=\; \bot \\ [\![H']\!]^\pi_\rho &= \bigcup_{i\in I} \Big(\bot \cup_\bot [\![H_0 \cdot H_i]\!]^\pi_\rho \Big) \;=\; \bigcup_{i\in I} \bot \;=\; \bot\end{aligned}$$ □

Lemma 24. $\{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\} \cdot \{\pi \rhd H\} \equiv \{\pi_1 | \pi \rhd H_1 \cdot H \cdots \pi_k | \pi \rhd H_k \cdot H\}$

Proof. Similar to the proof of Lemma 23.

Lemma 25. $\varphi[\{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}] \equiv \{\pi_1 \rhd \varphi[H_1] \cdots \pi_k \rhd \varphi[H_k]\}$

Proof. Let π be the evaluation plan, and let $I \subseteq 1..k$ be the set of indexes i such that $\pi_i \sqsubseteq \pi$. Then, by Lemma 18:

$$\begin{aligned}[\![\varphi[\{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}]]\!]^\pi_\rho &= \varphi[[\![\{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}]\!]^\pi_\rho] \\ &= \varphi[\bigcup_{i\in 1..k} [\![\{\pi_i \rhd H_i\}]\!]^\pi_\rho] \;=\; \varphi[\bigcup_{i\in I} [\![H_i]\!]^\pi_\rho] \\ &= \bigcup_{i\in I} \varphi[[\![H_i]\!]^\pi_\rho] \;=\; \bigcup_{i\in I} [\![\varphi[H_i]]\!]^\pi_\rho \;=\; \bigcup_{i\in 1..k} [\![\varphi[H_i]]\!]^\pi_\rho \\ &= [\![\{\pi_1 \rhd \varphi[H_1] \cdots \pi_k \rhd \varphi[H_k]\}]\!]^\pi_\rho\end{aligned}$$ □

Lemma 26. $\psi\langle\{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}\rangle \equiv \{\pi_1 \rhd \psi\langle H_1\rangle \cdots \pi_k \rhd \psi\langle H_k\rangle\}$

Proof. Similar to the proof of Lemma 25.

Lemma 27. $\mu h.\, \{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\} \equiv \{\pi_1 \rhd \mu h.\, H_1 \cdots \pi_k \rhd \mu h.\, H_k\}$

Proof. By definition, we have that:

$$\begin{aligned}[\![\mu h.\, \{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}]\!]^\pi_\rho &= \bigcup_{n\in\omega} f^n(\bot) \\ [\![\{\pi_1 \rhd \mu h.\, H_1 \cdots \pi_k \rhd \mu h.\, H_k\}]\!]^\pi_\rho &= \bigcup_{i\in 1..k} [\![\{\pi_i \rhd \mu h.\, H_i\}]\!]^\pi_\rho\end{aligned}$$

where $f(X) = [\![\{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}]\!]^\pi_{\rho\{X/h\}}$. Let $f_i(X) = [\![\{\pi_i \rhd H_i\}]\!]^\pi_{\rho\{X/h\}}$. Then, $f(X) = \bigcup_{i\in 1..k} f_i(X)$. Let $I \subseteq 1..k$ be the set of indexes i such that $\pi_i \sqsubseteq \pi$. Then, by Lemma 18 it follows that, for each $i \in I$:

$$\begin{aligned}[\![\{\pi_i \rhd H_i\}]\!]^\pi_{\rho\{X/h\}} &= [\![H_i]\!]^\pi_{\rho\{X/h\}} \\ [\![\{\pi_i \rhd \mu h.\, H_i\}]\!]^\pi_\rho = [\![\mu h.\, H_i]\!]^\pi_\rho \;&=\; \bigcup_{n\in\omega} g_i^n(\bot), \text{ where } g_i(X) = [\![H_i]\!]^\pi_{\rho\{X/h\}}\end{aligned}$$

By induction on n, we prove that $f_i^n(\bot) = g_i^n(\bot)$, for each $n \in \omega$ and $i \in I$.

$$f_i^{n+1}(\bot) = f_i(f_i^n(\bot)) = [\![H_i]\!]^\pi_{\rho\{f_i^n(\bot)/h\}} = [\![H_i]\!]^\pi_{\rho\{g_i^n(\bot)/h\}} = g_i^{n+1}(\bot)$$

Summing up, we have that:

$$\begin{aligned}
[\![\mu h.\,\{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}]\!]^\pi_\rho &= \bigcup_{n\in\omega}\bigcup_{i\in 1..k} f_i^n(\bot) \;=\; \bigcup_{n\in\omega}\bigcup_{i\in I} f_i^n(\bot) \\
&= \bigcup_{n\in\omega}\bigcup_{i\in I} g_i^n(\bot) \;=\; \bigcup_{i\in I}\bigcup_{n\in\omega} g_i^n(\bot) \\
&= \bigcup_{i\in I} [\![\{\pi_i \rhd \mu h.\, H_i\}]\!]^\pi_\rho \;=\; \bigcup_{i\in 1..k} [\![\{\pi_i \rhd \mu h.\, H_i\}]\!]^\pi_\rho \\
&= [\![\{\pi_1 \rhd \mu h.\, H_1 \cdots \pi_k \rhd \mu h.\, H_k\}]\!]^\pi_\rho
\end{aligned}$$

□

Lemma 28. $\{\pi_0 \rhd \{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}\} \equiv \{\pi_0 \mid \pi_1 \rhd H_1 \cdots \pi_0 \mid \pi_k \rhd H_k\}$

Proof. By induction on the structure of π_0. Let H (resp. H') be the left (resp. right) term in the equation above, and π be the evaluation plan. Then:

- if $\pi_0 = 0$, then:

$$\begin{aligned}
[\![H']\!]^\pi_\rho &= [\![\{0 \mid \pi_1 \rhd H_1 \cdots 0 \mid \pi_k \rhd H_k\}]\!]^\pi_\rho \;=\; \bigcup_{i\in 1..k} [\![\{0 \mid \pi_i \rhd H_i\}]\!]^\pi_\rho \\
&= \bigcup_{i\in 1..k} \Big([\![\{0 \rhd H_i\}]\!]^\pi_\rho \cup_\bot [\![\{\pi_i \rhd H_i\}]\!]^\pi_\rho \Big) \;=\; \bigcup_{i\in 1..k} [\![\{\pi_i \rhd H_i\}]\!]^\pi_\rho \\
&= [\![\{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}]\!]^\pi_\rho \;=\; [\![\{0 \rhd \{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}\}]\!]^\pi_\rho \\
&= [\![H]\!]^\pi_\rho
\end{aligned}$$

- if $\pi_0 = r[\ell]$, we have two subcases. If $r[\ell] \sqsubseteq \pi$, then:

$$\begin{aligned}
[\![H']\!]^\pi_\rho &= [\![\{r[\ell] \mid \pi_1 \rhd H_1 \cdots r[\ell] \mid \pi_k \rhd H_k\}]\!]^\pi_\rho \;=\; \bigcup_{i\in 1..k} [\![\{r[\ell] \mid \pi_i \rhd H_i\}]\!]^\pi_\rho \\
&= \bigcup_{i\in 1..k} \Big([\![\{r[\ell] \rhd H_i\}]\!]^\pi_\rho \cup_\bot [\![\{\pi_i \rhd H_i\}]\!]^\pi_\rho \Big) \;=\; \bigcup_{i\in 1..k} [\![\{\pi_i \rhd H_i\}]\!]^\pi_\rho \\
&= [\![\{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}]\!]^\pi_\rho \;=\; [\![\{r[\ell] \rhd \{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}\}]\!]^\pi_\rho \\
&= [\![H]\!]^\pi_\rho
\end{aligned}$$

  Otherwise, if π and π_0 are not compatible, then:

$$\begin{aligned}
[\![H']\!]^\pi_\rho &= \bigcup_{i\in 1..k} \Big([\![\{r[\ell] \rhd H_i\}]\!]^\pi_\rho \cup_\bot [\![\{\pi_i \rhd H_i\}]\!]^\pi_\rho \Big) \\
&= \bigcup_{i\in 1..k} \Big(\bot \cup_\bot [\![\{\pi_i \rhd H_i\}]\!]^\pi_\rho \Big) \;=\; \bigcup_{i\in 1..k} \bot \;=\; \bot \;=\; [\![H]\!]^\pi_\rho
\end{aligned}$$

- if $\pi_0 = \pi_0' \mid \pi_0''$, then, by the induction hypothesis:

$$\begin{aligned}
[\![H']\!]^\pi_\rho &= [\![\{(\pi_0' \mid \pi_0'') \mid \pi_1 \rhd H_1 \cdots (\pi_0' \mid \pi_0'') \mid \pi_k \rhd H_k\}]\!]^\pi_\rho \\
&= \bigcup_{i \in 1..k} [\![\{(\pi_0' \mid \pi_i) \mid (\pi_0'' \mid \pi_i) \rhd H_i\}]\!]^\pi_\rho \\
&= \bigcup_{i \in 1..k} \Big([\![\{(\pi_0' \mid \pi_i) \rhd H_i\}]\!]^\pi_\rho \cup_\perp [\![\{(\pi_0'' \mid \pi_i) \rhd H_i\}]\!]^\pi_\rho \Big) \\
&= \bigcup_{i \in 1..k} [\![\{(\pi_0' \mid \pi_i) \rhd H_i\}]\!]^\pi_\rho \cup_\perp \bigcup_{i \in 1..k} [\![\{(\pi_0'' \mid \pi_i) \rhd H_i\}]\!]^\pi_\rho \Big) \\
&= [\![\{\pi_0' \mid \pi_1 \rhd H_1 \cdots \pi_0' \mid \pi_k \rhd H_k\}]\!]^\pi_\rho \cup_\perp \\
&\qquad [\![\{\pi_0'' \mid \pi_1 \rhd H_1 \cdots \pi_0'' \mid \pi_k \rhd H_k\}]\!]^\pi_\rho \\
&= [\![\{\pi_0' \rhd \{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}\}]\!]^\pi_\rho \cup_\perp \\
&\qquad [\![\{\pi_0'' \rhd \{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}\}]\!]^\pi_\rho \\
&= [\![H]\!]^\pi_\rho
\end{aligned}$$

□

To prove Theorem 4, it suffices to generalize the above lemmas to the case where both the operand are planned selections with an arbitrary number of choices. To give the flavour of the proof, consider the case of a sequence of planned selections $\{\pi_i \rhd H_i\}_{i \in 1..k} \cdot \{\pi_j' \rhd H_j'\}_{j \in 1..p}$, with $k = p = 2$. Then:

$$\begin{aligned}
&\{\pi_1 \rhd H_1, \pi_2 \rhd H_2\} \cdot \{\pi_1' \rhd H_1', \pi_2' \rhd H_2'\} \\
\equiv &\{0 \rhd \{\pi_1 \rhd H_1, \pi_2 \rhd H_2\}\} \cdot \{\pi_1' \rhd H_1', \pi_2' \rhd H_2'\} \\
\equiv &\{0 \mid \pi_1' \rhd \{\pi_1 \rhd H_1, \pi_2 \rhd H_2\} \cdot H_1', 0 \mid \pi_2' \rhd \{\pi_1 \rhd H_1, \pi_2 \rhd H_2\} \cdot H_2'\} \\
\equiv &\{\pi_1' \rhd \{\pi_1 \rhd H_1, \pi_2 \rhd H_2\} \cdot \{0 \rhd H_1'\}, \pi_2' \rhd \{\pi_1 \rhd H_1, \pi_2 \rhd H_2\} \cdot \{0 \rhd H_2'\}\} \\
\equiv &\{\pi_1' \rhd \{\pi_1 \rhd H_1 \cdot H_1', \pi_2 \rhd H_2 \cdot H_1'\}, \pi_2' \rhd \{\pi_1 \rhd H_1 \cdot H_2', \pi_2 \rhd H_2 \cdot H_2'\}\} \\
\equiv &\{\pi_1' \mid \pi_1 \rhd H_1 \cdot H_1', \pi_1' \mid \pi_2 \rhd H_2 \cdot H_1', \pi_2' \rhd \pi_1 \rhd H_1 \cdot H_2', \pi_2' \rhd \pi_2 \rhd H_2 \cdot H_2'\}
\end{aligned}$$

Theorem 4. If $H = \{\pi_1 \rhd H_1 \cdots \pi_k \rhd H_k\}$ is linear, and H_i is valid for some $i \in 1..k$, then H is π_i-valid.

Proof. Let H_i be valid. Since for all $j \neq i$, π_i and π_j are independent, by Lemma 19 we have that $[\![H']\!]^{\pi_i} = [\![H_i]\!]^{\pi_i}$. Since H_i has no planned selections, then $[\![H_i]\!]^{\pi_i} = [\![H_i]\!]^0$. Since H_i is 0-valid, then H is π_i-valid. □

D Verification

Theorem 29. $H\!\downarrow$ is defined, for all H.

Proof. We prove a stronger result: for each history expression H, set of policies Φ and mapping Ω from variables to history expressions, $H\!\downarrow_{\Phi,\Omega}$ is defined. Let Φ_0 be the set of all policies contained in Φ, H and Ω. We proceed by induction on $|\Phi_0| - |\Phi|$. The base case is when $|\Phi| = |\Phi_0|$: since $\Phi \subseteq \Phi_0$, it must be $\Phi = \Phi_0$. By induction on the structure of H, we have the following cases:

- if $H = \varepsilon$, $H = h$, $H = \alpha$ are trivial.

- if $H = H_0 \cdot H_1$, then $H\downarrow_{\Phi_0,\Omega} = H_0\downarrow_{\Phi_0,\Omega} \cdot H_1\downarrow_{\Phi_0,\Omega}$. By the structural induction hypothesis, both $H_0\downarrow_{\Phi_0,\Omega}$ and $H_1\downarrow_{\Phi_0,\Omega}$ are defined.
- if $H = H_0 + H_1$, similar.
- if $H = \varphi[H']$, then $H\downarrow_{\Phi_0,\Omega} = H'\downarrow_{\Phi_0,\Omega}$, because $\varphi \in \Phi_0$. By the structural induction hypothesis, $H'\downarrow_{\Phi_0,\Omega}$ is defined.
- if $H = \psi\langle H'\rangle$, then $H\downarrow_{\Phi_0,\Omega} = H'\downarrow_{\Phi_0,\Omega}$, because $\psi \in \Phi_0$. By the structural induction hypothesis, $H'\downarrow_{\Phi_0,\Omega}$ is defined.
- if $H = \mu h.\, H'$, then $H\downarrow_{\Phi_0,\Omega} = \mu h.\, (H'\downarrow_{\Phi_0,\Omega})$, because there is no occurrence of $h \in fv(H')$ guarded by some $\varphi \notin \Phi_0$. By the structural induction hypothesis, $H'\downarrow_{\Phi_0,\Omega}$ is defined.

For the inductive case, let $|\Phi| = k$, and assume that the statement holds for sets of policies with size greater than k. We proceed by induction on the structure of H. The cases $\varepsilon, h, \alpha, H_0 \cdot H_1$ and $H_0 + H_1$ are treated similarly to the base case. The other cases follow:

- if $H = \varphi[H']$, then there are two further subcases. If $\varphi \in \Phi$, then $H\downarrow_{\Phi,\Omega} = H'\downarrow_{\Phi,\Omega}$, which is defined by the structural induction hypothesis. Otherwise, if $\varphi \notin \Phi$, then $H\downarrow_{\Phi,\Omega} = \varphi[H'\downarrow_{\Phi\cup\varphi,\Omega}]$. Since $|\Phi\cup\varphi| > |\Phi|$, then $H'\downarrow_{\Phi\cup\varphi,\Omega}$ is defined by the induction hypothesis on $|\Phi_0| - |\Phi|$.
- if $H = \psi\langle H'\rangle$, then there are two further subcases. If $\psi \in \Phi$, then $H\downarrow_{\Phi,\Omega} = H'\downarrow_{\Phi,\Omega}$, which is defined by the structural induction hypothesis. Otherwise, if $\psi \notin \Phi$, then $H\downarrow_{\Phi,\Omega} = \psi\langle H'\downarrow_{\Phi,\Omega}\rangle$, and $H'\downarrow_{\Phi,\Omega}$ is defined by structural the induction hypothesis.
- if $H = \mu h.\, H'$, then $H\downarrow_{\Phi,\Omega} = \mu h.(H''\sigma'\downarrow_{\Phi,\Omega\{(\mu h.H')\Omega/h\}})$, where H'', σ and σ' are as in the definition of regularization. When $\sigma'(h_i) = h$, the value of $\sigma(h_i)$ immaterial, thus $\sigma(h_i) = (\mu h.H')\downarrow_{\Phi\cup guard(H''),\Omega}$ needs to be computed only if $guard(H'') \not\subseteq \Phi$. Then, the induction hypothesis on $|\Phi_0|-|\Phi|$ ensures that $(\mu h.H')\downarrow_{\Phi\cup guard(H''),\Omega}$ is defined. By the structural induction hypothesis, $H'\downarrow_{\Phi,\Omega\{(\mu h.H')\Omega/h\}}$ is defined. Since the substitution σ' just replaces some free variable h_i for h, then also $H''\sigma'\downarrow_{\Phi,\Omega\{(\mu h.H')\Omega/h\}}$ is defined. □

The semantics of a closed history expression always contains histories with balanced framings. Also, the semantics of history expressions is preserved under substitution. Indeed, the following two lemmas hold. They can be proved by a straightforward structural induction.

Lemma 30. For each closed H and $\eta \in [\![H]\!]$, η has balanced framings.

Lemma 31 (Substitution lemma). Let H' be an history effect such that $[\![H']\!]_\rho$ is defined. Then, for each history effect H, $[\![H\{H'/h\}]\!]_\rho = [\![H]\!]_{\rho\{[\![H']\!]_\rho/h\}}$

We now characterize when a history expression has φ-framings. Note that the following definition in not a complete characterization, e.g. it does not tell us that $\varphi[\alpha]$ has φ-framings. However, this will suffice for the subsequent technical development. A complete characterization of φ-framings can be easily obtained by replacing variables for events in history expressions. For instance, $[\![\varphi[\alpha]]\!]_\rho$ has φ-framings if and only if $[\![\varphi[h]]\!]_{\rho\{\emptyset/h\}}$ has.

Lemma 32. $[\![H]\!]_\rho$ has φ-framings, if one of the following cases occurs:

(32a) for some $h \in fv(H)$, $\rho(h)$ has φ-framings.

(32b) some h is guarded by Φ in H, and $\varphi \in \Phi$.

Proof. We proceed by induction on the structure of H. The base cases $H = \varepsilon$, $H = \alpha$ and $H = h$ are trivial. The inductive cases follow:

- if $H = H_0 \cdot H_1$, then for (32a) assume that $h \in fv(H_0)$ (resp. $h \in fv(H_1)$), and $\rho(h)$ has φ-framings. By the induction hypothesis, $[\![H_0]\!]_\rho$ (resp. $[\![H_1]\!]_\rho$) has φ-framings. Then, $[\![H]\!]_\rho = [\![H_0]\!]_\rho [\![H_1]\!]_\rho$ has φ-framings. For (32b), assume that the occurrence of h is in H_0 (the other case is similar). Then, h is guarded by Φ in H_0. By the induction hypothesis, $[\![H_0]\!]_\rho$ has φ-framings. Then, $[\![H]\!]_\rho = [\![H_0]\!]_\rho [\![H_1]\!]_\rho$ has φ-framings.
- if $H' = H_0 + H_1$, similar.
- if $H = \varphi[H_0]$, then for (32a) assume that $\rho(h)$ has φ'-framings, for some $h \in fv(H) = fv(H_0)$. By the induction hypothesis, $[\![H_0]\!]_\rho$ has φ'-framings, and $[\![H]\!]_\rho = \varphi[[\![H_0]\!]_\rho]$ has φ'-framings. For (32b), assume that some h is guarded by Φ in H. Then, $\Phi = \{\varphi\} \cup \Phi_0$, and h is guarded by Φ_0 in H_0. By the induction hypothesis, $[\![H_0]\!]_\rho$ has φ'-framings for all $\varphi' \in \Phi_0$. Then, $[\![H]\!]_\rho = \varphi[[\![H_0]\!]_\rho]$ has φ-framings, and φ'-framings for all $\varphi' \in \Phi_0$. In conclusion, $[\![H]\!]_\rho$ has φ'-framings for all $\varphi' \in \Phi$.
- if $H = \mu h.H_0$, then $[\![H]\!]_\rho = \bigcup_{n \in \omega} X_n$, where $X_0 = \emptyset$, $X_{n+1} = [\![H_0]\!]_{\rho\{X_n/h\}}$. For (32a), let $h' \in fv(H) = fv(H_0) \setminus \{h\}$, and let $\rho(h')$ have φ-framings. By the induction hypothesis, $X_1 = [\![H_0]\!]_{\rho\{\emptyset/h\}}$ has φ-framings, as well as $[\![H]\!]_\rho \supseteq X_1$. For (32b), assume first that $h' \in fv(H)$ is guarded by Φ in H. Then, h' is guarded by Φ in H_0. By the induction hypothesis, $X_1 = [\![H_0]\!]_{\rho\{\emptyset/h\}}$ has φ-framings, for all $\varphi \in \Phi$, so it does $[\![H]\!]_\rho \supseteq X_1$. Now assume that $h' = h$ is guarded by Φ in H, and let $\varphi \in \Phi$. Then, there exists a free occurrence of h guarded by $\{\varphi\} \cup \Phi'$ in H_0, for some Φ'. By the induction hypothesis , $X_1 = [\![H_0]\!]_{\rho\{\emptyset/h\}}$ has φ-framings, and so it does $[\![H]\!]_\rho \supseteq X_1$. □

The previous lemma helps in establishing when a history expression has redundant framings. Again, we do not give a complete characterization (e.g. it does not tell us that $\varphi[\varphi[\alpha]]$ has redundant framings), because completeness is unneeded in the subsequent technical development.

Lemma 33. $[\![H]\!]_\rho$ has redundant framings, if one of the following cases occurs:

(33a) for some $h \in fv(H)$, $\rho(h)$ has redundant framings.

(33b) for some $h \in fv(H)$ guarded by $\{\varphi\} \cup \Phi$ in H, $\rho(h)$ has φ-framings.

(33c) $H = \mu h.H'$, and some free occurrence of h is guarded in H'

Proof. We proceed by induction on the structure of H. The base cases $H = \varepsilon$, $H = \alpha$ and $H = h$ are trivial. The inductive cases follow:

- if $H = H_0 \cdot H_1$, assume that the free occurrence of h is in H_0 (the other case is similar). For (33a), if $\rho(h)$ has redundant framings, then, by the induction hypothesis, $[\![H_0]\!]_\rho$ has redundant framings. For (33b)), if h is guarded by $\{\varphi\} \cup \Phi$ in H and $\rho(h)$ has φ-framings, then h is guarded by $\{\varphi\} \cup \Phi$ in H_0. By the induction hypothesis, $[\![H_0]\!]_\rho$ has redundant framings. In both cases, $[\![H]\!]_\rho = [\![H_0]\!]_\rho [\![H_1]\!]_\rho$ has redundant framings.

- if $H' = H_0 + H_1$, similar.

- if $H = \varphi[H_0]$, then $h \in fv(H)$ is also free in H_0. For (33a), if $\rho(h)$ has redundant framings, then, by the induction hypothesis, $[\![H_0]\!]_\rho$ has redundant framings. For (33b), let h be guarded by Φ in H. Then, $\Phi = \{\varphi\} \cup \Phi_0$, where h is guarded by Φ_0 in H. Let $\rho(h)$ have φ'-framings, for some $\varphi' \in \Phi$. If $\varphi' = \varphi$, then by (32a), $[\![H_0]\!]_\rho$ has φ-framings. Otherwise, if $\varphi' \in \Phi_0$, then, by the induction hypothesis, $[\![H_0]\!]_\rho$ has redundant framings. In both cases, $[\![H]\!]_\rho = \varphi[[\![H_0]\!]_\rho]$ has redundant framings.

- if $H = \mu h.H_0$, then $[\![H]\!]_\rho = \bigcup_{n\in\omega} X_n$, where $X_0 = \emptyset$, $X_{n+1} = [\![H_0]\!]_{\rho\{X_n/h\}}$. For (33a), let $h' \in fv(H) = fv(H_0) \setminus \{h\}$ be such that $\rho(h')$ has redundant framings. By the induction hypothesis, $X_1 = [\![H_0]\!]_{\rho\{\emptyset/h\}}$ has redundant framings, and so it does $[\![H]\!]_\rho \supseteq X_1$. For (33b), let h' be guarded by Φ in H_0 and let $\rho(h')$ have φ-framings, for some $\varphi \in \Phi$. By the induction hypothesis, $X_1 = [\![H_0]\!]_{\rho\{\emptyset/h\}}$ has redundant framings, and so it does $[\![H]\!]_\rho \supseteq X_1$. For (33c), let Φ be the union of all the sets $guard(H')$ such that $h \in fv(H')$. By (32a), $X_1 = [\![H_0]\!]_{\rho\{\emptyset/h\}}$ has framings in Φ. Since $\Phi \neq \emptyset$, let $\varphi \in \Phi$. Then, there exists an occurrence of h guarded in H_0 by $\{\varphi\} \cup \Phi'$, for some Φ'. By the induction hypothesis of (33b), $X_2 = [\![H_0]\!]_{\rho\{X_1/h\}}$ has redundant framings. Then, X_2 has redundant framings, and so $[\![H]\!]_\rho \supseteq X_2$. □

The following lemma, though a little contrived, will be very important while proving that our algorithm for regularizing history expressions does indeed produce histories with no redundant framings. It states that you can change the name of variables in history expressions and (under certain assumptions) the evaluation environment, while preserving the property of having φ-framings, and that of having no redundant framings.

Lemma 34. Let H, H' be history effects such that $H = H'\{h/h'\}$. Then:

(34a) if $[\![H]\!]_\rho$ has no framings in Φ for some ρ, then $[\![H']\!]_{\rho'}$ has no framings in Φ, for all ρ' such that $\rho'(h'')$ has no framings in Φ, for each $h'' \in fv(H')$.

(34b) if $[\![H]\!]_\rho$ has no redundant framings for some ρ, then $[\![H']\!]_{\rho'}$ has no redundant framings, for all ρ' such that $\rho'(h'')$ has no redundant framings, and no framings in $guard(H')$, for each $h'' \in fv(H')$.

Proof. By induction on the structure of H. The base cases $H = \varepsilon$ and $H = \alpha$ are trivial. If $H = h$, then $H' = h'$, and $[\![h']\!]_{\rho'} = \rho'(h')$ has neither framings in Φ nor redundant framings, by the hypotheses on ρ'. The inductive cases follow:

- if $H = H_0 \cdot H_1$, then $H = H'\{h/h'\}$ implies $H' = H'_0 \cdot H'_1$ for some H'_0, H'_1 such that $H_0 = H'_0\{h/h'\}$, $H_1 = H'_1\{h/h'\}$.

Assume that $[\![H]\!]_\rho = [\![H_0]\!]_\rho [\![H_1]\!]_\rho$ has no framings in Φ. This implies that neither $[\![H_0]\!]_\rho$ nor $[\![H_1]\!]_\rho$ have framings in Φ. Let ρ' be such that $\rho'(h)$ has no framings in Φ for all $h \in fv(H') = fv(H'_0) \cup fv(H'_1)$. By the induction hypothesis, neither $[\![H'_0]\!]_{\rho'}$ nor $[\![H'_1]\!]_{\rho'}$ have framings in Φ. Then, $[\![H']\!]_{\rho'} = [\![H'_0]\!]_{\rho'} [\![H'_1]\!]_{\rho'}$, has no framings in Φ. This proves (34a).

Now assume that $[\![H]\!]_\rho$ has no redundant framings. Then, neither $[\![H_0]\!]_\rho$ nor $[\![H_1]\!]_\rho$ have redundant framings. Let ρ' be such that $\rho'(h'')$ has no redundant framings and no framings in $guard(H')$ for all the occurrences of $h'' \in fv(H')$. If $h'' \in fv(H'_0)$ (resp. H'_1), then $guard(H') = guard(H'_0)$ (resp. H'_1). By the induction hypothesis, neither $[\![H'_0]\!]_{\rho'}$ nor $[\![H'_1]\!]_{\rho'}$ have redundant framings. Then, $[\![H']\!]_{\rho'} = [\![H'_0]\!]_{\rho'} [\![H'_1]\!]_{\rho'}$ has no redundant framings. This proves (34b).

- if $H = H_0 + H_1$, similar.

- if $H = \varphi[H_0]$, then $H' = \varphi[H'_0]$ for some H'_0 such that $H_0 = H'_0\{h/h'\}$.

  For (34a), assume that $[\![H]\!]_\rho = \varphi[[\![H_0]\!]_\rho]$ has no framings in Φ (clearly, $\varphi \notin \Phi$), and let ρ' be such that $\rho'(h'')$ has no framings in Φ for all $h'' \in fv(H') = fv(H'_0)$. By the induction hypothesis, $[\![H'_0]\!]_{\rho'}$ has no framings in Φ. Then, $[\![H']\!]_{\rho'} = \varphi[[\![H'_0]\!]_{\rho'}]$ has no framings in Φ.

  For (34b), assume that $[\![H]\!]_\rho$ has no redundant framings. Then, $[\![H_0]\!]_\rho$ has neither φ-framings nor redundant framings. Let ρ' be such that $\rho'(h'')$ has no redundant framings and no framings in $guard(H') = \{\varphi\} \cup guard(H'_0)$ for all $h'' \in fv(H') = fv(H'_0)$. By the induction hypothesis, $[\![H'_0]\!]_{\rho'}$ has neither φ-framings nor redundant framings. Then, $[\![H']\!]_{\rho'}$ has no redundant framings.

- if $H = \mu h_0.H_0$, then $H' = \mu h_0.H'_0$, for some H'_0 such that $H_0 = H'_0\{h/h'\}$. Let $X_0 = Y_0 = \emptyset$, $X_{n+1} = [\![H_0]\!]_{\rho\{X_n/h_0\}}$, and $Y_{n+1} = [\![H'_0]\!]_{\rho'\{Y_n/h_0\}}$. Then, $[\![H]\!]_\rho = \bigcup_{n\in\omega} X_n$, and $[\![H']\!]_{\rho'} = \bigcup_{n\in\omega} Y_n$.

  Assume that $[\![H]\!]_\rho$ has no framings in Φ: this means that all the approximants X_n have no framings in Φ. Let ρ' be such that $\rho'(h'')$ has no framings in Φ for all $h'' \in fv(H') = fv(H'_0) \setminus \{h_0\}$. We prove (34a) by induction on n. The base case $Y_0 = \emptyset$ is trivial. For the inductive case, assume that Y_n has no framings in Φ. By the structural induction hypothesis, the approximant $Y_{n+1} = [\![H'_0]\!]_{\rho'\{Y_n/h_0\}}$ has no framings in Φ. Since this holds for all the approximants Y_n, then $[\![H']\!]_{\rho'}$ has no framings in Φ.

  For (34b), assume that $[\![H]\!]_\rho$ has no redundant framings. Then, no approximant X_n has redundant framings. By lemma (33c), $guard(H) = \emptyset$, i.e. all the occurrences of h_0 in H'_0 are unguarded. Let ρ' be such that $\rho'(h'')$ has neither redundant framings nor framings in $guard(H') = guard(H'_0)$ for all $h'' \in fv(H') = fv(H'_0) \setminus \{h_0\}$. We prove that no approximant Y_n has redundant framings. The base case $Y_0 = \emptyset$ is trivial. For the inductive case, assume that Y_n has no redundant framings. By the structural induction hypothesis (recall that Y_n has no framings in $guard(H'_0) = \emptyset$), $Y_{n+1} = [\![H'_0]\!]_{\rho'\{Y_n/h_0\}}$ has no redundant framings. □

Theorem 35. For each closed H, $H\downarrow$ has no redundant safety framings.

Proof. We prove the following, stronger, result. Let H, Φ, Ω, ρ be such that (i) $[\![H]\!]_\rho$ is defined, (ii) $[\![\Omega(h)]\!]_\emptyset$ is defined for all $h \in \mathrm{dom}(\Omega) = fv(H)$, and (iii) $\rho(h)$ has no safety framings for each $h \in fv(H)$. Then, $[\![H \downarrow_{\Phi,\Omega}]\!]_\rho$ has neither safety framings in Φ, nor redundant safety framings.

Let Φ_0 be the set of policies contained in Φ, H and ρ. We proceed by induction on $|\Phi_0| - |\Phi|$. The base case is when $|\Phi| = |\Phi_0|$: since $\Phi \subseteq \Phi_0$, it must be $\Phi = \Phi_0$. In this case, we prove that $[\![H \downarrow_\Phi]\!]_\rho$ has no framings (and, consequently, no redundant framings). By induction on the structure of H, we have the following cases:

- if $H = \varepsilon$, then $[\![\varepsilon \downarrow_\Phi]\!]_\rho = [\![\varepsilon]\!]_\rho = \varepsilon$ has no framings. Similarly for $H = \alpha$.
- if $H = h$, then $[\![h \downarrow_\Phi]\!]_\rho = [\![h]\!]_\rho = \rho(h)$ has no framings by hypothesis (iii).
- if $H = H_0 \cdot H_1$, then:

  $$[\![(H_0 \cdot H_1) \downarrow_\Phi]\!]_\rho = [\![H_0 \downarrow_\Phi \cdot H_1 \downarrow_\Phi]\!]_\rho = [\![H_0 \downarrow_\Phi]\!]_\rho \, [\![H_1 \downarrow_\Phi]\!]_\rho$$

  By the structural induction hypothesis, $[\![H_0 \downarrow_\Phi]\!]_\rho$ and $[\![H_1 \downarrow_\Phi]\!]_\rho$ have no framings, as well as $[\![H_0 \downarrow_\Phi]\!]_\rho \, [\![H_1 \downarrow_\Phi]\!]_\rho$.
- if $H = H_0 + H_1$, similar.
- if $H = \varphi[H_0]$, since $\Phi = \Phi_0$, then $\varphi \in \Phi$. Thus, $\varphi[H_0] \downarrow_\Phi = H_0 \downarrow_\Phi$, and, by the structural induction hypothesis, $[\![H_0 \downarrow_\Phi]\!]_\rho$ has no framings.
- if $H = \psi\langle H_0 \rangle$, since $\Phi = \Phi_0$, then $\psi \in \Phi$. Thus, $\psi\langle H_0 \rangle \downarrow_\Phi = H_0 \downarrow_\Phi$, and, by the structural induction hypothesis, $[\![H_0 \downarrow_\Phi]\!]_\rho$ has no framings.
- if $H = \mu h.H_0$, then $(\mu h.\, H_0) \downarrow_{\Phi,\Omega} = \mu h.\, (H_0 \downarrow_{\Phi,\Omega\{H\Omega/h\}})$, since there is no $h \in fv(H_0)$ guarded by $\varphi \notin \Phi = \Phi_0$. Then, $[\![H \downarrow_{\Phi,\Omega}]\!]_\rho = \bigcup_{n\in\omega} X_n$, where $X_0 = \bot$, and $X_{n+1} = [\![H_0 \downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{X_n/h\}}$. We prove, by induction on n, that each X_n has no framings. The base case $X_0 = \bot$ is trivial. For the inductive case, assume that X_n has no framings. Then, by the structural induction hypothesis, $[\![H_0 \downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{X_n/h\}}$ has no framings.

For the inductive case, let $|\Phi| = k$, and assume that the statement holds for sets of policies with size greater than k. We proceed by induction on the structure of H. The cases $\varepsilon, \alpha, h, H_0 \cdot H_1$ and $H_0 + H_1$ are treated similarly to the base case. The other cases follow:

- if $H = \varphi[H_0]$, then we have the following two subcases. If $\varphi \in \Phi$, then $\varphi[H_0] \downarrow_\Phi = H_0 \downarrow_\Phi$. By the structural induction hypothesis, $[\![H_0 \downarrow_\Phi]\!]_\rho$ has neither framings in Φ, nor redundant framings. Otherwise, if $\varphi \notin \Phi$, then $\varphi[H_0] \downarrow_\Phi = \varphi[H_0 \downarrow_{\Phi\cup\{\varphi\}}]$. Since $|\Phi \cup \{\varphi\}| = |\Phi| + 1$, by induction on $|\Phi_0| - |\Phi|$, $[\![H_0 \downarrow_{\Phi\cup\{\varphi\}}]\!]_\rho$ has no φ-framings, no framings in Φ, and no redundant framings. Therefore, $[\![\varphi[H_0 \downarrow_{\Phi\cup\{\varphi\}}]]\!]_\rho = \varphi[[\![H_0 \downarrow_{\Phi\cup\{\varphi\}}]\!]_\rho]$ has neither framings in Φ, nor redundant framings.
- if $H = \psi\langle H_0 \rangle$, then we have the following two subcases. If $\psi \in \Phi$, then $\psi\langle H_0 \rangle \downarrow_\Phi = H_0 \downarrow_\Phi$. By the structural induction hypothesis, $[\![H_0 \downarrow_\Phi]\!]_\rho$ has neither safety framings in Φ, nor redundant safety framings. Otherwise, if $\psi \notin \Phi$, then $\psi\langle H_0 \rangle \downarrow_\Phi = \psi\langle H_0 \downarrow_\Phi \rangle$. By the induction hypothesis, $[\![H_0 \downarrow_\Phi]\!]_\rho$ has no safety framings and no redundant safety framings. Therefore,

$[\![\psi\langle H_0 \downarrow_\Phi \rangle]\!]_\rho = \psi\langle [\![H_0 \downarrow_\Phi]\!]_\rho \rangle$ has neither safety framings in Φ, nor redundant safety framings.

- if $H = \mu h.H_0$, let $H_0 = H'\{h/h_i\}_i$, where the h_i are fresh and $h \notin fv(H')$. Let $\Phi_i = guard(H')$, $\sigma(h_i) = H\Omega \downarrow_{\Phi\cup\Phi_i,\Omega}$, and $\sigma'(h_i) = h$ if $\Phi_i \subseteq \Phi$, otherwise $\sigma'(h_i) = h_i$. Then, $H \downarrow_{\Phi,\Omega} = \mu h.\,(H'\sigma' \downarrow_{\Phi,\Omega\{H\Omega/h\}} \sigma)$. Consider the history effect $H\Omega$. By hypothesis, $[\![H]\!]_\rho$ is defined, and $[\![\Omega(h')]\!]_\emptyset$ is defined for all $h' \in \mathrm{dom}(\Omega)$. Then, lemma 31 ensures that $[\![H\Omega]\!]_\rho$ is defined. For all i such that $h_i \in \mathrm{dom}(\sigma)$, let $\mathcal{H}_i = [\![H\Omega \downarrow_{\Phi\cup\Phi_i,\Omega}]\!]_\rho$. By lemma 31:

$$\begin{aligned}[\![H \downarrow_{\Phi,\Omega}]\!]_\rho &= [\![\mu h.\, H'\sigma' \downarrow_{\Phi,\Omega\{H\Omega/h\}} \sigma]\!]_\rho \\ &= [\![\mu h.\, H'\sigma' \downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{\mathcal{H}_i/h_i\}_i} \\ &= \bigcup_{n\in\omega} X_n\end{aligned}$$

where $X_0 = \bot$, and $X_{n+1} = [\![H'\sigma' \downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{\mathcal{H}_i/h_i, X_n/h\}_i}$. We proceed by induction on n. The base case $X_0 = \bot$ is trivial. For the inductive case, assume that X_n has neither framings in Φ, nor redundant framings. By hypothesis, $\rho(h')$ has no framings for all $h' \in fv(H) = fv(H_0) \setminus \{h\}$. Since $\Omega(h')$ is closed for all $h' \in \mathrm{dom}(\Omega)$, and $fv(H) = \mathrm{dom}(\Omega)$, then $H\Omega$ is closed, and $fv(H_0) = fv(H) \cup \{h\} = \mathrm{dom}(\Omega) \cup \{h\} = \mathrm{dom}(\Omega\{H\Omega/h\})$. Therefore, by the structural induction hypothesis, $[\![H_0 \downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{\emptyset/h\}}$ has neither framings in Φ, nor redundant framings.

Consider the histories $\mathcal{H}_i = [\![H\Omega \downarrow_{\Phi\cup\Phi_i,\Omega}]\!]_\rho$, and let $h' \in fv(H\Omega)$. Since $\Omega(h')$ is closed, then $h' \in fv(H) \setminus \mathrm{dom}(\Omega)$, and $\rho(h')$ has no framings by hypothesis. Since $\Phi_i \not\subseteq \Phi$, then $|\Phi \cup \Phi_i| > |\Phi|$, and, by the induction hypothesis on $|\Phi_0| - |\Phi|$, $\mathcal{H}_i$ has neither framings in $\Phi \cup \Phi_i$, nor redundant framings. Note that $H_0 = (H'\sigma')\{h/h_i\}_i$: then, by lemma (34a), X_{n+1} has no framings in Φ. Let $\rho' = \rho\{\mathcal{H}_i/h_i, X_n/h\}_i$. Consider a free occurrence of some $h' \in fv(H'\sigma')$ guarded by Φ'. If $h' = h$, then $\rho'(h) = X_n$ has neither redundant framings nor framings in Φ. Since $\Phi' \subseteq \Phi$ by definition of σ', then $\rho'(h)$ has no framings in Φ'. If $h' = h_i$, then $\rho'(h') = \mathcal{H}_i$ has neither redundant framings, nor framings in $\Phi \cup \Phi_i$, and then no framings in Φ_i. If $h' \notin \{h, h_i\}_i$, then $\rho'(h') = \rho(h')$ has no framings by hypothesis. By lemma (34b), X_{n+1} has no redundant framings. □

Regularization preserves validity. To show that, it is convenient to introduce a *normal form* for histories. It permits to compare the histories produced by an expression H with those of the regularization of H. Note that normalization is a non-regular transformation: constructing the normal form of a history requires counting the framing openings and closings.

Normalization of histories

$$\varepsilon \Downarrow_\Phi = \varepsilon \qquad \alpha \Downarrow_\Phi = (\bigwedge \Phi)\,[\alpha]$$

$$(\mathcal{H}\mathcal{H}') \Downarrow_\Phi = \mathcal{H} \Downarrow_\Phi \mathcal{H}' \Downarrow_\Phi \qquad (\mathcal{H} \cup \mathcal{H}') \Downarrow_\Phi = \mathcal{H} \Downarrow_\Phi \cup \mathcal{H}' \Downarrow_\Phi$$

$$\varphi[\mathcal{H}] \Downarrow_\Phi = \mathcal{H} \Downarrow_{\Phi\cup\{\varphi\}} \qquad \psi\langle\mathcal{H}\rangle \Downarrow_\Phi = \psi\langle\mathcal{H} \Downarrow_\Phi\rangle$$

Intuitively, normalization transforms safety framings into local checks. Indeed, $\eta\Downarrow_\Phi$ is intended to record that each event in η must obey to *all* the policies in Φ. This is apparent in the second and in the last equation above. We abbreviate $\mathcal{H}\Downarrow_\emptyset$ with $\mathcal{H}\Downarrow$. Note that $\mathcal{H}\Downarrow_\emptyset$ is defined if and only if $\mathcal{H}$ has balanced framings.

Example 18. Consider the history $\eta = \alpha\varphi[\alpha'\varphi'[\alpha'']]$. Its normal form is:

$$\begin{aligned}\eta\Downarrow &= \alpha\Downarrow\,(\varphi[\alpha'\varphi'[\alpha'']])\Downarrow \;=\; \alpha\,(\alpha'\varphi'[\alpha''])\Downarrow_\varphi \;=\; \alpha\,(\alpha'\Downarrow_\varphi)\,(\varphi'[\alpha''])\Downarrow_\varphi \\ &= \alpha\,\varphi[\alpha']\,(\alpha''\Downarrow_{\varphi,\varphi'}) \;=\; \alpha\,\varphi[\alpha']\,(\varphi\wedge\varphi')[\alpha'']\end{aligned}$$

□

Lemma 36. $\mathcal{H}\Downarrow_\Phi\Downarrow_\Phi = \mathcal{H}\Downarrow_\Phi$, for all $\mathcal{H}$ and Φ (i.e. normalization is idempotent).

Proof. By induction on the structure of $\mathcal{H}$. The base case ε is trivial, as well as the cases $\mathcal{H} = \mathcal{H}_0\,\mathcal{H}_1$, $\mathcal{H} = \mathcal{H}_0 \cup \mathcal{H}_1$ and $\mathcal{H} = \psi\langle\mathcal{H}_0\rangle$, which are dealt with a straightforward use of the induction hypothesis. If $\mathcal{H} = \alpha$, then:

$$\alpha\Downarrow_\Phi\Downarrow_\Phi = (\bigwedge\Phi)[\alpha]\Downarrow_\Phi = \alpha\Downarrow_{\Phi\cup\{\bigwedge\Phi\}} = (\bigwedge\Phi\wedge\bigwedge\Phi)[\alpha] \;=\; (\bigwedge\Phi)[\alpha] \;=\; \alpha\Downarrow_\varphi$$

Otherwise, if $\mathcal{H} = \varphi[\mathcal{H}']$, then:

$$\varphi[\mathcal{H}]\Downarrow_\Phi\Downarrow_\Phi = \mathcal{H}\Downarrow_{\Phi\cup\{\varphi\}}\Downarrow_\Phi = \mathcal{H}\Downarrow_\Phi\Downarrow_{\{\varphi\}}\Downarrow_\Phi = \mathcal{H}\Downarrow_\Phi\Downarrow_\Phi\Downarrow_{\{\varphi\}} = \mathcal{H}\Downarrow_\Phi\Downarrow_{\{\varphi\}} = \varphi[\mathcal{H}]\Downarrow_\Phi$$

where the next-to-last equality is implied by the induction hypothesis. □

The following technical lemma helps in proving Theorem 38. Roughly, it states that normalization is preserved by substitution of a history variable, provided that the new evaluation enviroment maps that variable to a suitably normalized set of histories.

Lemma 37. Let $h \in fv(H)$. Then, for all Φ, ρ and $\Phi' \subseteq \Phi$:

$$[\![H]\!]_\rho\Downarrow_\Phi \;=\; [\![H]\!]_{\rho\{\rho(h)\Downarrow_{\Phi'}/h\}}\Downarrow_\Phi \;=\; [\![H]\!]_{\rho\{\rho(h)\Downarrow_{\Phi\cup guard(H)}/h\}}\Downarrow_\Phi$$

Proof. By induction on the structure of H. The base cases $H = \varepsilon$ and $H = \alpha$ are trivial. If $H = h$, then $guard(h) = \emptyset$, and:

$$\begin{aligned}[\![h]\!]_\rho\Downarrow_\Phi \;&=\; \rho(h)\Downarrow_\Phi \;=\; \rho(h)\Downarrow_{\Phi'}\Downarrow_\Phi \;=\; [\![h]\!]_{\rho\{\rho(h)\Downarrow_{\Phi'}/h\}}\Downarrow_\Phi \\ [\![h]\!]_\rho\Downarrow_\Phi \;&=\; \rho(h)\Downarrow_\Phi \;=\; \rho(h)\Downarrow_\Phi\Downarrow_\Phi \;=\; [\![h]\!]_{\rho\{\rho(h)\Downarrow_\Phi/h\}}\Downarrow_\Phi\end{aligned}$$

The inductive cases follow.

- if $H = H_0 \cdot H_1$, let $\rho' = \rho\{\rho(h)\Downarrow_{\Phi'}/h\}$. Then, for the first equation:

$$\begin{aligned}[\![H_0 \cdot H_1]\!]_\rho\Downarrow_\Phi &= ([\![H_0]\!]_\rho\,[\![H_1]\!]_\rho)\Downarrow_\Phi \\ &= [\![H_0]\!]_\rho\Downarrow_\Phi\;[\![H_1]\!]_\rho\Downarrow_\Phi \\ &= [\![H_0]\!]_{\rho'}\Downarrow_\Phi\;[\![H_1]\!]_{\rho'}\Downarrow_\Phi \\ &= ([\![H_0]\!]_{\rho'}\,[\![H_1]\!]_{\rho'})\Downarrow_\Phi \\ &= [\![H_0 \cdot H_1]\!]_{\rho\{\rho(h)\Downarrow_\Phi/h\}}\Downarrow_\Phi\end{aligned}$$

  where the third step is justified by the induction hypothesis. For the second equation, let $h \in fv(H_0)$ (the case $h \in fv(H_1)$ is similar). Then,

$guard(H) = guard(H_0)$. Let $\rho' = \rho\{\rho(h)\Downarrow_{\Phi\cup guard(H_0)}/h\}$. By the induction hypothesis, we have that:

$$\begin{aligned}
[\![H_0 \cdot H_1]\!]_\rho \Downarrow_\Phi &= ([\![H_0]\!]_\rho\, [\![H_1]\!]_\rho)\Downarrow_\Phi \\
&= [\![H_0]\!]_\rho \Downarrow_\Phi\ [\![H_1]\!]_\rho \Downarrow_\Phi \\
&= [\![H_0]\!]_{\rho'} \Downarrow_\Phi\ [\![H_1]\!]_\rho \Downarrow_\Phi \\
&= [\![H_0]\!]_{\rho'} \Downarrow_\Phi\ [\![H_1]\!]_{\rho'} \Downarrow_\Phi \\
&= ([\![H_0]\!]_{\rho'}\, [\![H_1]\!]_{\rho'})\Downarrow_\Phi \\
&= [\![H_0 \cdot H_1]\!]_{\rho'} \Downarrow_\Phi \\
&= [\![H_0 \cdot H_1]\!]_{\rho\{\rho(h)\Downarrow_{\Phi\cup guard(H)}/h\}} \Downarrow_\Phi
\end{aligned}$$

- if $H' = H_0 + H_1$, similar.
- if $H = \varphi[H_0]$, then, for the first equation:

$$\begin{aligned}
[\![\varphi[H]]\!]_\rho \Downarrow_\Phi &= \varphi[[\![H]\!]_\rho]\Downarrow_\Phi = [\![H]\!]_\rho \Downarrow_{\Phi\cup\{\varphi\}} \\
&= [\![H]\!]_{\rho\{\rho(h)\Downarrow_{\Phi'}/h\}} \Downarrow_{\Phi\cup\{\varphi\}} \\
&= \varphi[[\![H]\!]_{\rho\{\rho(h)\Downarrow_{\Phi'}/h\}}]\Downarrow_\Phi \\
&= [\![\varphi[H]]\!]_{\rho\{\rho(h)\Downarrow_{\Phi'}/h\}} \Downarrow_\Phi
\end{aligned}$$

For the second equation, $guard(\varphi[H]) = \{\varphi\} \cup guard(H)$. Then:

$$\begin{aligned}
[\![\varphi[H]]\!]_\rho \Downarrow_\Phi &= \varphi[[\![H]\!]_\rho]\Downarrow_\Phi = [\![H]\!]_\rho \Downarrow_{\Phi\cup\{\varphi\}} \\
&= [\![H]\!]_{\rho\{\rho(h)\Downarrow_{\Phi\cup\{\varphi\}\cup guard(H)}/h\}} \Downarrow_{\Phi\cup\{\varphi\}} \\
&= \varphi[[\![H]\!]_{\rho\{\rho(h)\Downarrow_{\Phi\cup\{\varphi\}\cup guard(H)}/h\}}]\Downarrow_\Phi \\
&= [\![\varphi[H]]\!]_{\rho\{\rho(h)\Downarrow_{\Phi\cup guard(\varphi[H])}/h\}} \Downarrow_\Phi
\end{aligned}$$

- if $H = \mu h'.H'$, then $guard(H) = guard(H')$ (note that $h \neq h'$). Let $\rho' = \rho\{\rho(h)\Downarrow_{\Phi'}/h\}$, $\rho'' = \rho\{\rho(h)\Downarrow_{\Phi\cup guard(H')}/h\}$, $[\![\mu h'.H']\!]_\rho \Downarrow_\Phi = \bigcup X_n$, $[\![\mu h'.H']\!]_{\rho'} \Downarrow_\Phi = \bigcup Y_n$, and $[\![\mu h'.H']\!]_{\rho''} \Downarrow_\Phi = \bigcup Z_n$, where $X_0 = Y_0 = Z_0 = \bot$, $X_{n+1} = [\![H']\!]_{\rho\{X_n/h'\}} \Downarrow_\Phi$, $Y_{n+1} = [\![H']\!]_{\rho'\{Y_n/h'\}} \Downarrow_\Phi$, and $Z_{n+1} = [\![H']\!]_{\rho''\{Z_n/h'\}} \Downarrow_\Phi$. We prove by induction on n that $X_n = Y_n = Z_n$, for all n. By the structural induction hypotheses, we have that:

$$\begin{aligned}
X_{n+1} &= [\![H']\!]_{\rho\{X_n/h'\}} \Downarrow_\Phi \\
&= [\![H']\!]_{\rho\{Y_n/h'\}} \Downarrow_\Phi \\
&= [\![H']\!]_{\rho\{Y_n/h', \rho(h)\Downarrow_{\Phi'}/h\}} \Downarrow_\Phi \\
&= Y_{n+1} \\
X_{n+1} &= [\![H']\!]_{\rho\{X_n/h'\}} \Downarrow_\Phi \\
&= [\![H']\!]_{\rho\{Z_n/h'\}} \Downarrow_\Phi \\
&= [\![H']\!]_{\rho\{Z_n/h', \rho(h)\Downarrow_{\Phi\cup guard(H')}/h\}} \Downarrow_\Phi \\
&= Z_{n+1}
\end{aligned}$$

□

The following theorem establishes that a history expression H and its regularization $H\downarrow$ have the same normal form. Together with Theorem 39, this will lead to proving that validity is preserved by regularization.

Theorem 38. $[\![H\downarrow]\!]\Downarrow = [\![H]\!]\Downarrow$, for each H closed.

Proof. We prove a stronger result, i.e.: $[\![H]\!]_\rho\Downarrow_\Phi \subseteq [\![H\downarrow_{\Phi,\Omega}]\!]_\rho\Downarrow_\Phi \subseteq [\![H\Omega]\!]_\rho\Downarrow_\Phi$ for all H, Φ, ρ and Ω such that $\rho(h) \subseteq [\![\Omega(h)]\!]$ for all $h \in fv(H) \cap \mathrm{dom}(\Omega)$. Let Φ_0 be the set of all policies contained in Φ, H and ρ. We proceed by induction on $|\Phi_0| - |\Phi|$. The base case is when $|\Phi| = |\Phi_0|$: since $\Phi \subseteq \Phi_0$, it must be $\Phi = \Phi_0$. By induction on the structure of H, we have the following cases:

- if $H \in \{\varepsilon, \alpha\}$, then $H\downarrow_{\Phi,\Omega} = H = H\Omega$, and so the three terms are equal.
- if $H = h$, then $h\downarrow_{\Phi,\Omega} = h$ proves the first inclusion. For the second one, let $h \in \mathrm{dom}(\Omega)$ (otherwise $h\Omega = h$ and the statement holds trivially). Then, $[\![h]\!]_\rho = \rho(h) \subseteq [\![\Omega(h)]\!] = [\![h\Omega]\!]$.
- if $H = H_0 \cdot H_1$, then, by the induction hypotheses:

$$\begin{aligned}
[\![H_0 \cdot H_1]\!]_\rho\Downarrow_\Phi &= ([\![H_0]\!]_\rho\ [\![H_1]\!]_\rho)\Downarrow_\Phi \\
&= [\![H_0]\!]_\rho\Downarrow_\Phi\ [\![H_1]\!]_\rho\Downarrow_\Phi \\
&\subseteq [\![H_0\downarrow_\Phi]\!]_\rho\Downarrow_\Phi\ [\![H_1\downarrow_\Phi]\!]_\rho\Downarrow_\Phi \\
&= ([\![H_0\downarrow_\Phi]\!]_\rho\ [\![H_1\downarrow_\Phi]\!]_\rho)\Downarrow_\Phi \\
&= [\![H_0\downarrow_\Phi \cdot H_1\downarrow_\Phi]\!]_\rho\Downarrow_\Phi \\
&= [\![(H_0 \cdot H_1)\downarrow_\Phi]\!]_\rho\Downarrow_\Phi
\end{aligned}$$

$$\begin{aligned}
[\![(H_0 \cdot H_1)\downarrow_\Phi]\!]_\rho\Downarrow_\Phi &= [\![H_0\downarrow_\Phi \cdot H_1\downarrow_\Phi]\!]_\rho\Downarrow_\Phi \\
&= ([\![H_0\downarrow_\Phi]\!]_\rho\ [\![H_1\downarrow_\Phi]\!]_\rho)\Downarrow_\Phi \\
&= [\![H_0\downarrow_\Phi]\!]_\rho\Downarrow_\Phi\ [\![H_1\downarrow_\Phi]\!]_\rho\Downarrow_\Phi \\
&\subseteq [\![H_0\Omega]\!]_\rho\Downarrow_\Phi\ [\![H_1\Omega]\!]_\rho\Downarrow_\Phi \\
&= ([\![H_0\Omega]\!]_\rho\ [\![H_1\Omega]\!]_\rho)\Downarrow_\Phi \\
&= [\![(H_0 \cdot H_1)\Omega]\!]_\rho\Downarrow_\Phi
\end{aligned}$$

- if $H = H_0 + H_1$, similar.
- if $H = \varphi[H']$, then $\varphi \in \Phi = \Phi_0$, and, by the induction hypotheses:

$$[\![\varphi[H']\downarrow_\Phi]\!]_\rho\Downarrow_\Phi = [\![H'\downarrow_\Phi]\!]_\rho\Downarrow_\Phi \supseteq [\![H']\!]_\rho\Downarrow_\Phi = \varphi[[\![H']\!]_\rho]\Downarrow_\Phi = [\![\varphi[H']]\!]_\rho\Downarrow_\Phi$$
$$[\![\varphi[H']\downarrow_\Phi]\!]_\rho\Downarrow_\Phi = [\![H'\downarrow_\Phi]\!]_\rho\Downarrow_\Phi \subseteq [\![H'\Omega]\!]_\rho\Downarrow_\Phi = \varphi[[\![H'\Omega]\!]_\rho]\Downarrow_\Phi = [\![\varphi[H']\Omega]\!]_\rho\Downarrow_\Phi$$

- if $H = \psi\langle H'\rangle$, similar to the previous case.
- if $H = \mu h.H_0$, then $(\mu h.\, H_0)\downarrow_{\Phi,\Omega} = \mu h.\,(H_0\downarrow_{\Phi,\Omega\{H\Omega/h\}})$, since there is no $h \in fv(H_0)$ guarded by $\varphi \notin \Phi = \Phi_0$, and $H\Omega = \mu h.H_0\Omega_0$, where $\Omega_0(h') = \Omega(h')$ if $h' \neq h$. Let:

$$\begin{array}{llll}
[\![H]\!]_\rho = \bigcup X_n & X_0 = \bot & X_{n+1} = [\![H_0]\!]_{\rho\{X_n/h\}} & X_n\Downarrow_\Phi = X'_n \\
[\![H\downarrow_{\Phi,\Omega}]\!]_\rho = \bigcup Y_n & Y_0 = \bot & Y_{n+1} = [\![H_0\downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{Y_n/h\}} & Y_n\Downarrow_\Phi = Y'_n \\
[\![H\Omega]\!]_\rho = \bigcup Z_n & Z_0 = \bot & Z_{n+1} = [\![H_0\Omega_0]\!]_{\rho\{Z_n/h\}} & Z_n\Downarrow_\Phi = Z'_n
\end{array}$$

We prove, by induction on n, that $X'_n \subseteq Y'_n \subseteq \bigcup Z'_n$ for all n. The base case is trivial. For the inductive case, by the induction hypotheses:

$$\begin{aligned}
Y'_{n+1} &= [\![H_0 \downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{Y'_n/h\}} \Downarrow_\Phi \\
&\supseteq [\![H_0 \downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{X'_n/h\}} \Downarrow_\Phi && \text{(ind. hyp.)} \\
&= [\![H_0 \downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{X_n/h\}} \Downarrow_\Phi && \text{(Lemma 37)} \\
&\supseteq [\![H_0]\!]_{\rho\{X_n/h\}} \Downarrow_\Phi && \text{(ind. hyp.)} \\
&= X'_{n+1}
\end{aligned}$$

$$\begin{aligned}
Y'_{n+1} &= [\![H_0 \downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{Y'_n/h\}} \Downarrow_\Phi \\
&\subseteq [\![H_0 \downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{[\![H\Omega]\!]_{\rho^{\Downarrow_\Phi}/h\}}} \Downarrow_\Phi && \text{(ind. hyp.)} \\
&= [\![H_0 \downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{[\![H\Omega]\!]_{\rho}/h\}} \Downarrow_\Phi && \text{(Lemma 37)} \\
&\subseteq [\![H_0\Omega\{H\Omega/h\}]\!]_{\rho\{[\![H\Omega]\!]_{\rho}/h\}} \Downarrow_\Phi && \text{(ind. hyp.)} \\
&= [\![H_0\Omega_0]\!]_{\rho\{[\![H\Omega]\!]_{\rho}/h\}} \Downarrow_\Phi && \text{(ind. hyp.)} \\
&= [\![H\Omega]\!]_\rho \Downarrow_\Phi
\end{aligned}$$

For the inductive case, let $|\Phi| = k$, and assume that the statement hold for sets of policies with size greater than k. We proceed by induction on the structure of H. The cases $\varepsilon, \alpha, h, H_0 \cdot H_1$ and $H_0 + H_1$ are treated similarly to the base case. The inductive cases follow:

- if $H = \varphi[H_0]$, and $\varphi \notin \Phi$, then, by the induction hypothesis:

$$\begin{aligned}
[\![\varphi[H'] \downarrow_{\Phi,\Omega}]\!]_\rho \Downarrow_\Phi = [\![\varphi[H' \downarrow_{\Phi\cup\{\varphi\},\Omega}]]\!]_\rho \Downarrow_\Phi &= \varphi[[\![H' \downarrow_{\Phi\cup\{\varphi\},\Omega}]\!]_\rho] \Downarrow_\Phi \\
&= [\![H' \downarrow_{\Phi\cup\{\varphi\},\Omega}]\!]_\rho \Downarrow_{\Phi\cup\{\varphi\}} \supseteq [\![H']\!]_\rho \Downarrow_{\Phi\cup\{\varphi\}} \\
&= \varphi[[\![H']\!]_\rho] \Downarrow_\Phi = [\![\varphi[H']]\!]_\rho \Downarrow_\Phi
\end{aligned}$$

$$\begin{aligned}
[\![\varphi[H'] \downarrow_{\Phi,\Omega}]\!]_\rho \Downarrow_\Phi = [\![\varphi[H' \downarrow_{\Phi\cup\{\varphi\},\Omega}]]\!]_\rho \Downarrow_\Phi &= \varphi[[\![H' \downarrow_{\Phi\cup\{\varphi\},\Omega}]\!]_\rho] \Downarrow_\Phi \\
&= [\![H' \downarrow_{\Phi\cup\{\varphi\},\Omega}]\!]_\rho \Downarrow_{\Phi\cup\{\varphi\}} \subseteq [\![H'\Omega]\!]_\rho \Downarrow_{\Phi\cup\{\varphi\}} \\
&= \varphi[[\![H'\Omega]\!]_\rho] \Downarrow_\Phi = [\![\varphi[H'\Omega]]\!]_\rho \Downarrow_\Phi = [\![\varphi[H']\Omega]\!]_\rho \Downarrow_\Phi
\end{aligned}$$

  The case $\varphi \in \Phi$ is similar to the base case.

- if $H = \psi\langle H_0 \rangle$, similar to the previous case.

- if $H = \mu h.H_0$, then $H\Omega = \mu h.H_0\Omega_0$, where $\Omega_0(h') = \Omega(h')$ if $h' \neq h$. Let:

$$\begin{array}{llll}
[\![H]\!]_\rho = \bigcup X_n & X_0 = \bot & X_{n+1} = [\![H_0]\!]_{\rho\{X_n/h\}} & X_n \Downarrow_\Phi = X'_n \\
[\![H \downarrow_{\Phi,\Omega}]\!]_\rho = \bigcup Y_n & Y_0 = \bot & Y_{n+1} = [\![H'\sigma' \downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{Y_n/h,\mathcal{H}_i/h_i\}} & Y_n \Downarrow_\Phi = Y'_n \\
[\![H\Omega]\!]_\rho = \bigcup Z_n & Z_0 = \bot & Z_{n+1} = [\![H_0\Omega_0]\!]_{\rho\{Z_n/h\}} & Z_n \Downarrow_\Phi = Z'_n
\end{array}$$

  where $H = H'\{h/h_i\}_i$, h_i fresh, $\Phi_i = \mathit{guard}(H')$, $\mathcal{H}_i = H\Omega \downarrow_{\Phi\cup\Phi_i,\Omega}$, and $\sigma'(h_i) = h$ if $\Phi_i \subseteq \Phi$, otherwise $\sigma'(h_i) = h_i$. We prove by induction

that $X'_n \subseteq Y'_n \subseteq \bigcup Z'_n$ for all n (this works because normalization $\Downarrow$ is continuous on $(2^{\mathsf{Ev}^*}, \subseteq)$).

$$
\begin{aligned}
Y'_{n+1} &= [\![H'\sigma'\downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{Y'_n/h,\mathcal{H}_i/h_i\}}\Downarrow_\Phi \\
&\supseteq [\![H'\sigma'\downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{X'_n/h,\mathcal{H}_i/h_i\}}\Downarrow_\Phi && \text{(ind. hyp.)} \\
&= [\![H'\sigma'\downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{X'_n/h,\mathcal{H}_i\Downarrow_{\Phi\cup\Phi_i}/h_i\}}\Downarrow_\Phi && \text{(Lemma 37)} \\
&= [\![H'\sigma'\downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{X'_n/h,[\![H\Omega\downarrow_{\Phi\cup\Phi_i}]\!]_\rho\Downarrow_{\Phi\cup\Phi_i}/h_i\}}\Downarrow_\Phi && \text{(def. } \mathcal{H}_i\text{)} \\
&\supseteq [\![H'\sigma'\downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{X'_n/h,[\![H\Omega]\!]_\rho\Downarrow_{\Phi\cup\Phi_i}/h_i\}}\Downarrow_\Phi && \text{(ind. hyp.)} \\
&= [\![H'\sigma'\downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{X_n/h,[\![H\Omega]\!]_\rho/h_i\}}\Downarrow_\Phi && \text{(Lemma 37)} \\
&\supseteq [\![H'\sigma'\Omega\{H\Omega/h\}]\!]_{\rho\{X_n/h,[\![H\Omega]\!]_\rho/h_i\}}\Downarrow_\Phi && \text{(ind. hyp.)} \\
&= [\![H'\sigma'\Omega_0]\!]_{\rho\{[\![H\Omega]\!]_\rho/h,[\![H\Omega]\!]_\rho/h_i\}}\Downarrow_\Phi && \text{(def. } \Omega_0\text{)} \\
&= [\![H_0\Omega_0]\!]_{\rho\{[\![H\Omega]\!]_\rho/h\}}\Downarrow_\Phi && \text{(def. } \sigma'\text{)} \\
&= [\![H_0\Omega_0]\!]_{\rho\{[\![H]\!]_\rho/h\}}\Downarrow_\Phi && \text{(hyp. on } \rho\text{)} \\
&\supseteq [\![H_0\Omega_0]\!]_{\rho\{X_n/h\}}\Downarrow_\Phi \\
&= X_{n+1}
\end{aligned}
$$

$$
\begin{aligned}
Y'_{n+1} &= [\![H'\sigma'\downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{Y'_n/h,\mathcal{H}_i/h_i\}}\Downarrow_\Phi \\
&\subseteq [\![H'\sigma'\downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{[\![H\Omega]\!]_\rho\Downarrow_\Phi/h,\mathcal{H}_i/h_i\}}\Downarrow_\Phi && \text{(ind. hyp.)} \\
&= [\![H'\sigma'\downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{[\![H\Omega]\!]_\rho\Downarrow_\Phi/h,[\![H\Omega\downarrow_{\Phi\cup\Phi_i}]\!]_\rho/h_i\}}\Downarrow_\Phi && \text{(def. } \mathcal{H}_i\text{)} \\
&= [\![H'\sigma'\downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{[\![H\Omega]\!]_\rho\Downarrow_\Phi/h,[\![H\Omega\downarrow_{\Phi\cup\Phi_i}]\!]_\rho\Downarrow_{\Phi\cup\Phi_i}/h_i\}}\Downarrow_\Phi && \text{(Lemma 37)} \\
&\subseteq [\![H'\sigma'\downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{[\![H\Omega]\!]_\rho\Downarrow_\Phi/h,[\![H\Omega\Omega]\!]_\rho\Downarrow_{\Phi\cup\Phi_i}/h_i\}}\Downarrow_\Phi && \text{(ind. hyp.)} \\
&= [\![H'\sigma'\downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{[\![H\Omega]\!]_\rho\Downarrow_\Phi/h,[\![H\Omega]\!]_\rho\Downarrow_{\Phi\cup\Phi_i}/h_i\}}\Downarrow_\Phi && (\Omega \text{ closed}) \\
&= [\![H'\sigma'\downarrow_{\Phi,\Omega\{H\Omega/h\}}]\!]_{\rho\{[\![H\Omega]\!]_\rho/h,[\![H\Omega]\!]_\rho/h_i\}}\Downarrow_\Phi && \text{(Lemma 37)} \\
&\subseteq [\![H'\sigma'\Omega\{H\Omega/h\}]\!]_{\rho\{[\![H\Omega]\!]_\rho/h,[\![H\Omega]\!]_\rho/h_i\}}\Downarrow_\Phi && \text{(ind. hyp.)} \\
&= [\![H'\sigma'\Omega_0]\!]_{\rho\{[\![H\Omega]\!]_\rho/h,[\![H\Omega]\!]_\rho/h_i\}}\Downarrow_\Phi && \text{(def. } \Omega_0\text{)} \\
&= [\![H_0\Omega_0]\!]_{\rho\{[\![H\Omega]\!]_\rho/h\}}\Downarrow_\Phi && \text{(def. } \sigma'\text{)} \\
&= [\![H\Omega]\!]_\rho\Downarrow_\Phi && \square
\end{aligned}
$$

The next theorem states that normalization preserves the validity of histories. Summing up, we can conclude that a history expression H is valid if and only if its regularization $H\downarrow$ is valid. To do that, it is convenient to restate the notion of validity. Let $\Phi(\eta)$ be the *set* of policies φ such that the number of $[_\varphi$ is greater than the number of $]_\varphi$ in η. We say that a history $\eta = \beta_1 \cdots \beta_n$ is *safe* when $(\beta_1 \cdots \beta_k)^\flat \models \bigwedge \Phi(\beta_1 \cdots \beta_k)$, for all $k \in 1..n$.

Example 19. Consider the history $\eta = \alpha_r[_\varphi \alpha_c]_\varphi$, where φ is the property saying that no α_c occurs after α_r. Then, η is *not* safe, because $(\alpha_r[_\varphi \alpha_c)^\flat = \alpha_r\alpha_c$ does not satisfy $\bigwedge \Phi(\alpha_r[_\varphi \alpha_c) = \bigwedge\{\varphi\} = \varphi$.

Equivalently, a history η is safe if and only if $\eta^{(k)} \models \varphi^{(k)}$, for all $k \in 1..n$, where $\eta^{(k)}$ is sequence of the first k access events in η (i.e. the first k events in $\eta^\flat$), and $\varphi^{(k)} = \bigwedge\{\, \Phi(\beta_1 \cdots \beta_i) \mid (\beta_1 \cdots \beta_i)^\flat = \eta^{(k)} \,\}$.

Example 20. Let φ be the policy of the previous example. Then, the history $\eta = [_\varphi \alpha_r]_\varphi \alpha_c$ is valid: $\eta^{(1)} = \alpha_r$ satisfies $\varphi^{(1)} = \bigwedge(\Phi([_\varphi \alpha_r) \cup \Phi([_\varphi \alpha_r]_\varphi)) = \varphi$, and $\eta^{(2)} = \alpha_r\alpha_c$ satisfies $\bigwedge \Phi([_\varphi \alpha_r]_\varphi \alpha_c) = \bigwedge \emptyset = tt$.

Validity of a history η can then be restated, by requiring that η is safe, and that, for each live set $\psi\langle\mathcal{H}\rangle$ of η, there exists $\eta' \in \mathcal{H}$ such that $\eta' \models \psi$.

Theorem 39. A history η is valid if and only if $\eta\Downarrow$ is valid.

Proof. We prove the following, stronger result: for each η and Φ such that $\eta \Downarrow_\Phi$ is defined, $(\eta \Downarrow_\Phi)^{(k)} = \eta^{(k)}$ and $\varphi^{(k)}_{\eta\Downarrow_\Phi} = (\bigwedge \Phi) \wedge \varphi^{(k)}_\eta$. We proceed by induction on the structure of η. The base case $\eta = \varepsilon$ is trivial. If $\eta = \alpha$, then $\eta \Downarrow_\Phi = (\bigwedge \Phi)[\alpha]$, $\eta \Downarrow^{(1)}_\Phi = \alpha = \eta^{(1)}$, $\varphi^{(1)}_\eta = tt$, and $\varphi^{(1)}_{\eta\Downarrow_\Phi} = \bigwedge \Phi$. The inductive cases follow.

- if $\eta = \eta_0\eta_1$, let η_0, η_1 have balanced framings (otherwise $\eta\Downarrow_\Phi$ is undefined). Then, $\eta\Downarrow_\Phi = (\eta_0\Downarrow_\Phi)(\eta_1\Downarrow_\Phi)$. Let k be the number of access events in η_0.

$$\varphi^{(i)}_\eta \;=\; \begin{cases} \varphi^{(i)}_{\eta_0} & \text{if } i \in 1..k \\ \varphi^{(i)}_{\eta_1} & \text{otherwise} \end{cases} \qquad \varphi^{(i)}_{\eta\Downarrow_\Phi} \;=\; \begin{cases} \varphi^{(i)}_{\eta_0\Downarrow_\Phi} & \text{if } i \in 1..k \\ \varphi^{(i)}_{\eta_1\Downarrow_\Phi} & \text{otherwise} \end{cases}$$

  By the induction hypothesis, $(\eta_0 \Downarrow_\Phi)^{(i)} = \eta_0^{(i)}$ and $\varphi^{(i)}_{\eta_0\Downarrow_\Phi} = (\bigwedge \Phi) \wedge \varphi^{(i)}_{\eta_0}$ for $i \in 1..k$, while $(\eta_1 \Downarrow_\Phi)^{(i)} = \eta_1^{(i)}$ and $\varphi^{(i)}_{\eta_0\Downarrow_\Phi} = (\bigwedge \Phi) \wedge \varphi^{(i)}_{\eta_1}$ for $i > k$. Then, $(\eta\Downarrow_\Phi)^{(i)} = \eta^{(i)}$, and $\varphi^{(i)}_{\eta\Downarrow_\Phi} = (\bigwedge \Phi) \wedge \varphi^{(i)}_\eta$.

- if $\eta = \varphi[\eta']$, then $\eta\Downarrow_\Phi = \eta'\Downarrow_{\Phi\cup\{\varphi\}}$ and $\varphi^{(i)}_\eta = \varphi \wedge \varphi^{(i)}_{\eta'}$ for $i < |\eta| - 1$. By the induction hypothesis, $(\eta'\Downarrow_\Phi)^{(i)} = \eta'^{(i)}$, and $\varphi^{(i)}_{\eta'\Downarrow_{\Phi\cup\{\varphi\}}} = \bigwedge(\Phi \cup \{\varphi\}) \wedge \varphi^{(i)}_{\eta'}$. Then, $(\eta\Downarrow_\Phi)^{(i)} = \eta^{(i)}$, and:

$$\varphi^{(i)}_{\eta\Downarrow_\Phi} \;=\; \varphi^{(i)}_{\eta'\Downarrow_{\Phi\cup\{\varphi\}}} \;=\; (\bigwedge \Phi) \wedge \varphi \wedge \varphi^{(i)}_{\eta'} \;=\; (\bigwedge \Phi) \wedge \varphi^{(i)}_\eta$$

- if $\eta = \psi\langle\eta'\rangle$, then $\eta\Downarrow_\Phi = \psi\langle\eta'\Downarrow_\Phi\rangle$ and $\varphi^{(i)}_\eta = \varphi^{(i)}_{\eta'}$ for $i < |\eta| - 1$. By the induction hypothesis, $(\eta'\Downarrow_\Phi)^{(i)} = \eta'^{(i)}$, and $\varphi^{(i)}_{\eta'\Downarrow_\Phi} = (\bigwedge \Phi) \wedge \varphi^{(i)}_{\eta'}$. Then, $(\eta\Downarrow_\Phi)^{(i)} = \eta^{(i)}$, and:

$$\varphi^{(i)}_{\eta\Downarrow_\Phi} \;=\; \varphi^{(i)}_{\eta'\Downarrow_\Phi} \;=\; (\bigwedge \Phi) \wedge \varphi^{(i)}_{\eta'} \;=\; (\bigwedge \Phi) \wedge \varphi^{(i)}_\eta$$

□

In the following, we say that a history η is *safe* when, for all $\varphi[\mathcal{H}] \in S(\eta)$, each $\eta' \in \mathcal{H}$ obeys φ. Similarly, η is *live* when, for all $\psi\langle\mathcal{H}\rangle \in L(\eta)$, there exists $\eta' \in \mathcal{H}$ such that $\eta' \models \psi$. The proof of Theorem 9 is then split in two parts. First, we show that η is safe iff $\eta \in \mathcal{L}(A_{\varphi[\,]})$ for all φ occurring in η. Second, we show that η is live iff $\eta \in \mathcal{L}(A_{\psi\langle\,\rangle})$ for all ψ occurring in η.

Lemma 40. A history η with no redundant safety framings is safe if and only if $\eta \models \varphi_{[\,]}$, for all φ such that $[_\varphi \in \eta$.

Proof. The statement holds trivially for $\eta = \varepsilon$, because $S(\varepsilon) = \emptyset$ and $\varepsilon \models \varphi_{[\,]}$. So, let $\eta = \eta'\beta$, where $\eta' = \beta_1 \cdots \beta_n$.

For the "if" part, we proceed by induction on the length of η. Assume that $\eta \models \varphi_{[\,]}$ for each φ present in η. Since the only non-final state in $A_{\psi_{[\,]}}$ is the sink, then also $\eta' \models \psi_{[\,]}$. Therefore, by the induction hypothesis, η' is safe. We have the following cases:

- if $\beta \in \mathsf{Ev}$, then consider a safe-set $\varphi[\mathcal{H}] \in S(\eta)$. If $\varphi[\mathcal{H}]$ corresponds to a safety framing that is already balanced in η, i.e. $\eta = \eta_0\varphi[\eta_1]\eta_2$ and $\mathcal{H} = \eta_0^\flat(\eta_1^\flat)^\partial$, then $\varphi[\mathcal{H}] \in S(\eta')$. Since η' is safe, then all the histories in $\mathcal{H}$ obey φ. Otherwise, $\varphi[\mathcal{H}]$ corresponds to a non-balanced safety framing, i.e. $\eta = \eta_0[_\varphi\eta_1\beta$ and $\mathcal{H} = \eta_0^\flat(\eta_1^\flat\beta)^\partial$. Then, $S(\eta')$ comprise the safe-set $\varphi[\mathcal{H}']$, where $\mathcal{H}' = \eta_0^\flat(\eta_1^\flat)^\partial$. Let $\bar\eta = \eta_0^\flat\eta_1^\flat\beta$. Since η' is safe, it suffices to prove that $\bar\eta \models \varphi$. By hypothesis, $\eta = \eta_0[_\varphi\eta_1\beta \models \varphi_{[\,]}$, i.e. the automaton $A_{\varphi_{[\,]}}$ has an accepting run $q^{(0)} \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} q^{(n)} \xrightarrow{\beta} q$. A run in A_φ on $\bar\eta$ can be constructed from the run in $A_{\varphi_{[\,]}}$ on η. Since the scope of φ has been entered but not exited, $q^{(n)}$ is in the second layer of $A_{\varphi_{[\,]}}$. Then, the transition $(q^{(n)}, \beta, q)$ in $A_{\varphi_{[\,]}}$ requires a corresponding transition in A_φ. The states in the second layer in $A_{\varphi_{[\,]}}$ are final in A_φ. Thus, we have found an accepting run in A_φ on $\bar\eta$, and so $\bar\eta \models \varphi$. Since this can be done for all φ occurring in η, then η is safe.
- if $\beta = [_\varphi$, then $S(\eta) = S(\eta') \cup \varphi[(\eta')^\flat]$. Since η' is safe, it suffices to prove that $(\eta')^\flat \models \varphi$. Since $\eta \models \varphi_{[\,]}$, the automaton $A_{\varphi_{[\,]}}$ has an accepting run $q^{(0)} \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} q^{(n)} \xrightarrow{[_\varphi} q$, for some state q that is the second-layer counterpart of a final state in A_φ. An accepting run of A_φ on $(\eta')^\flat$ can be constructed as in the previous case.
- if $\beta =]_\varphi$, then η is safe if η' is such.

For the "only if" part it suffices to prove that, if η is safe and $\beta = [_\varphi$ for some φ, then there is a run of $A_{\varphi_{[\,]}}$ on η' that leads to the first-layer counterpart of some final state in A_φ. The remaining event $[_\varphi$ can be then consumed by $A_{\varphi_{[\,]}}$, which has a transition from the final states of the first layer to their counterparts in the second layer. If φ occurs in η', then an accepting run $q^{(0)} \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} q^{(n)}$ of $A_{\varphi_{[\,]}}$ on η' can be easily extended to accept η. Instead, if φ does not occur in η', then $S(\eta) = S(\eta')[_\varphi = S(\eta')[_\varphi\,]_\varphi = S(\eta') \cup \varphi[(\eta')^\flat]$. Since η is safe, then $(\eta')^\flat$ obeys φ, i.e. there exists an accepting run of A_φ on $(\eta')^\flat$. It is possible to construct an accepting run $s^{(0)} \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} s^{(n)}$ of $A_{\varphi_{[\,]}}$ on η' as follows.

At the first step, let $s^{(0)} = r^{(0)}$. For the remaining steps, we will scan the history η' with the index i, $(\eta')^\flat$ with the index j, and at each step we will define $s^{(i)}$. Let Q and $\dot Q$ be respectively the states in the first and in the second layer of $A_{\varphi_{[\,]}}$. For each state q in $A_{\varphi_{[\,]}}$, let $q@Q$ and $q@\dot Q$ be respectively the projections of q on the first and on second layer of $A_{\varphi_{[\,]}}$ (recall that the first layer of $A_{\varphi_{[\,]}}$ contains all the states of A_φ). For the i-th step, there are the following exhaustive cases:

- if $\beta_i \in \mathsf{Ev}$, there are the following subcases:
  - if $s^{(i)} \in Q$, then $s^{(i+1)} = r^{(j+1)}$.
  - if $s^{(i)} \in \dot{Q}$, then $r^{(j)}$ must be final in A_φ: otherwise, we have found a safe-set $\varphi[\mathcal{H}']$ for which some $\bar{\eta} \in \mathcal{H}'$ is not accepted by $A_{\varphi_{[\,]}}$. Therefore, A_φ has a transition $(r^{(j)}, \beta_i, r^{(j+1)})$, that is rendered in $A_{\varphi_{[\,]}}$ by the transition $(s^{(i)}, \beta_i, s^{(i+1)})$, where $s^{(i+1)} = r^{(j+1)}@\dot{Q}$.
- if $\beta_i = [_\varphi$, then it must be $s^{(i)} \in Q$ because η has no redundant safety framings. Moreover, $r^{(j)}$ must be final in A_φ: otherwise, we have found a safe-set $\varphi[\mathcal{H}']$ for which some $\bar{\eta} \in \mathcal{H}'$ is not accepted by $A_{\varphi_{[\,]}}$. Therefore, $A_{\varphi_{[\,]}}$ has a transition $(s^{(i)}, [_\varphi, s^{(i+1)})$, where $s^{(i+1)} = s^{(i)}@\dot{Q}$.
- if $\beta_i =]_\varphi$, then it must be $s^{(i)} \in \dot{Q}$. Therefore, $A_{\varphi_{[\,]}}$ has a transition $(s^{(i)},]_\varphi, s^{(i+1)})$, where $s^{(i+1)} = s^{(i)}@Q$.
- if $\beta_i = [_{\varphi'}$ or $\beta_i =]_{\varphi'}$ for some $\varphi' \neq \varphi$, then $s^{(i+1)} = s^{(i)}$.
- if $\beta_i = \langle_\psi$ or $\beta_i = \rangle_\psi$ for some ψ, then $s^{(i+1)} = s^{(i)}$. □

Lemma 41. A history η is live if and only if $\eta \models \psi_{\langle\,\rangle}$, for all ψ such that $\langle_\psi \in \eta$.

Proof. The statement holds trivially for $\eta = \varepsilon$, because $L(\varepsilon) = \emptyset$ and $\varepsilon \models \psi_{\langle\,\rangle}$. So, let $\eta = \eta'\beta$, where $\eta' = \beta_1 \cdots \beta_n$

For the "if" part, we proceed by induction on the length of η. Assume that $\eta \models \psi_{\langle\,\rangle}$ for each ψ occurring in η. Since the only non-final state in $A_{\psi_{\langle\,\rangle}}$ is the sink, then also $\eta' \models \psi_{\langle\,\rangle}$. Therefore, by the induction hypothesis, η' is live. We have the following cases:

- if $\beta \in \mathsf{Ev}$, then $L(\eta) = L(\eta'\beta) = L(\eta')$. Let $\psi\langle\mathcal{H}\rangle \in L(\eta)$. Since η' is live by the induction hypothesis, then there exists $\eta'' \in \mathcal{H}$ such that $\eta'' \models \psi$.
- if $\beta = \langle_\psi$, then, similarly to the previous case, $L(\eta) = L(\eta'\langle_\psi) = L(\eta')$.
- if $\beta = \rangle_\psi$, then, since η is well-formed, it follows that η is of the form $\eta_0\langle_\psi\eta_1\rangle_\psi = \eta_0\psi\langle\eta_1\rangle$ for some well-formed η_0 and balanced η_1. Thus, $L(\eta) = L(\eta_0\eta_1) \cup \psi\langle\eta_0^\flat(\eta_1^\flat)^\partial\rangle$. Since $\eta' = \eta_0\langle_\psi\eta_1$, then $L(\eta') = L(\eta_0\eta_1)$. Let $\mathcal{H} = \eta_0^\flat(\eta_1^\flat)^\partial$. We have to prove that there exists some $\eta'' \in \mathcal{H}$ such that $\eta'' \models \psi$. Let $q^{(0)} \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} q^{(n)} \xrightarrow{\rangle_\psi} q$ be an accepting run of $A_{\psi_{\langle\,\rangle}}$ on η. Then, $q^{(n)}$ cannot be in the third layer: by contradiction, if $q^{(n)}$ were such, then no transition would exist from $q^{(n)}$ under the input symbol $\rangle_\psi$ – apart from that falling into the sink. Therefore, $q^{(n)}$ is in the first or in the second layer.

  If $q^{(n)}$ is in the second layer, let k be the greatest index such that $q^{(k)}$ is not in the second layer and $q^{(i)}$ is in the second layer for all $i \in k+1..n$. There are two further subcases:
  - if $q^{(k)}$ is in the first layer, then $\beta_{k+1} = \langle_\psi$ and $q^{(k)}$ is final in A_ψ. Let $\eta'' = (\beta_1 \cdots \beta_{k+1})^\flat$. We can then determine an accepting run of A_ψ on η'', i.e. $\eta'' \models \psi$. Clearly, $\eta'' \in \mathcal{H}$, because $q^{(i)}$ stays in the second layer for all $i > k$ (so you do not leave the scope of ψ).

– if $q^{(k)}$ is in the third layer, then $\beta_{k+1} \in \mathsf{Ev}$ and $q^{(k+1)}$ is final in A_ψ. Similarly to the previous case, there exists an accepting run A_ψ on $(\beta_1 \cdots \beta_{k+1})^\flat$.

If $q^{(n)}$ is in the first layer, let k be the greatest index such that $q^{(k)}$ is not in the second layer and $q^{(k+1)}$ is in the second layer. Such an index must exist, because, upon $\rangle_\psi$, you can only reach the first layer by passing through the second layer. Then, an accepting run of A_ψ can be determined as above.

For the "only if" part, it suffices to prove that, if η is live and $\beta = \rangle_\psi$ for some ψ, then there is a run of $A_{\psi\langle\rangle}$ on η' that does not lead to a state in the third layer of $A_{\psi\langle\rangle}$. Indeed, this would be the only way for making η not accepted by $A_{\psi\langle\rangle}$ – recall that the copies of the sink state of A_ψ comply with all the events for which there is no explicit transition in A_ψ. So, let $\eta = \eta_0\psi\langle\eta_1\rangle$, and $\eta' = \eta_0\langle_\psi\eta_1$, with η_0 and η_1 well-formed.

If η_1 is balanced, then let $\mathcal{H} = \eta_0^\flat(\eta_1^\flat)^\partial$. Otherwise, consider the rightmost and innermost framing of ψ in η', i.e. $\eta' = \eta_0\langle_\psi\eta_2\langle_\psi\eta_3$, where $\langle_\psi$ does not occur in η_3, and let $\mathcal{H} = \eta_0^\flat\eta_2^\flat(\eta_3^\flat)^\partial$. In both cases, the live-set $\psi\langle\mathcal{H}\rangle$ belongs to $L(\eta)$. Since η is live, then η' is live (as this property is prefix-closed), and so there exists $\bar{\eta} = \alpha_1 \cdots \alpha_k \in \mathcal{H}$ such that $\bar{\eta} \models \psi$. In other words, there exists an accepting run $r^{(0)} \xrightarrow{\alpha_1} \cdots \xrightarrow{\alpha_k} r^{(k)}$ of A_ψ on $\bar{\eta}$. We construct a run $s^{(0)} \xrightarrow{\beta_1} \cdots \xrightarrow{\beta_n} s^{(n)}$ of $A_{\psi\langle\rangle}$ on η' as follows.

At the first step, let $s^{(0)} = r^{(0)}$. For the remaining steps, we will scan the history η' with the index i, $\bar{\eta}$ with the index j, and at each step we will define $s^{(i)}$. Let $Q, \dot{Q}, \ddot{Q}$ be respectively the states in the first, second and third layer of $A_{\psi\langle\rangle}$. For each state q in $A_{\psi\langle\rangle}$, let $q@Q, q@\dot{Q}$ and $q@\ddot{Q}$ be respectively the projections of q on the first, second and third layer of $A_{\psi\langle\rangle}$. For the i-th step, there are the following exhaustive cases:

- if $\beta_i \in \mathsf{Ev}$, there are the following subcases:
  - if $s^{(i)} \in Q$, then $s^{(i+1)} = r^{(j+1)}@Q$.
  - if $s^{(i)} \in \dot{Q}$, then $s^{(i+1)} = r^{(j+1)}@\dot{Q}$.
  - if $s^{(i)} \in \ddot{Q}$, then $s^{(i+1)} = r^{(j+1)}@\dot{Q}$ if $r^{(j+1)}$ is final. Otherwise, $s^{(i+1)} = r^{(j+1)}@\ddot{Q}$.
- if $\beta_i = \langle_\psi$, there are the following subcases:
  - if $s^{(i)} \in Q$, then $s^{(i+1)} = s^{(i)}@\dot{Q}$ if $r^{(j)} \in F$, o.w. $s^{(i+1)} = s^{(i)}@\ddot{Q}$.
  - if $s^{(i)} \in \dot{Q}$, then $s^{(i+1)} = s^{(i)}$ if $r^{(j)} \in F$, otherwise $s^{(i+1)} = s^{(i)}@\ddot{Q}$.
  - if $s^{(i)} \in \ddot{Q}$, then $s^{(i+1)} = s^{(i)}$.
- if $\beta_i = \rangle_\psi$, there are the following subcases:
  - if $s^{(i)} \in Q$, then $s^{(i+1)} = s^{(i)}$.
  - if $s^{(i)} \in \dot{Q}$, then $s^{(i+1)} = s^{(i)}@Q$.
  - if $s^{(i)} \in \ddot{Q}$, then we have a contradiction, because $\rho(s^{(i)}, \rangle_\psi)$ would lead to the non-final sink state, and so we would have found a live-set $\psi\langle\mathcal{H}'\rangle$ for which no history in $\mathcal{H}'$ has acceptings runs in $A_{\psi\langle\rangle}$.

- if $\beta_i = \langle_{\psi'}$ or $\beta_i = \rangle_{\psi'}$ for some $\psi' \neq \psi$, then $s^{(i+1)} = s^{(i)}$.
- if $\beta_i = [_{\varphi}$ or $\beta_i =]_{\varphi}$ for some φ, then $s^{(i+1)} = s^{(i)}$.

Summing up, we have constructed an run of $A_{\psi_{\langle\rangle}}$ on a history η'' such that $(\eta'')^\flat = \bar{\eta}$. Since the event α_k is final in A_ψ, then the last state $s^{(i)}$ defined above is in the first or in the second layer. The copies of the sink state of A_ψ ensure that all the remaining access events can be consumed by $A_{\psi_{\langle\rangle}}$. For the remaining framing events, recall that η_3 does not contain further $\langle_\psi$ or $\rangle_\psi$. Therefore, the remaining states $s^{(i)}$ cannot lead to the third layer. □

Lemma 7. $[\![H]\!]^\partial = [\![BPA(H)]\!]$.

Theorem 9. A history expression H with no planned selections is valid iff:

$$[\![BPA(H\downarrow)]\!] \models \bigwedge_{\varphi\in H} \varphi_{[\,]} \wedge \bigwedge_{\psi\in H} \psi_{\langle\rangle}$$

Proof. By Theorem 39, $[\![H]\!]$ is valid if and only if $[\![H]\!] \Downarrow$ is valid. By Theorem 38, $[\![H]\!]\Downarrow = [\![H\downarrow]\!]\Downarrow$. By Theorem 39, $[\![H\downarrow]\!]\Downarrow$ is valid if and only if $[\![H\downarrow]\!]$ is valid. By Theorem 35, $[\![H\downarrow]\!]$ has no redundant safety framings. By definition, $[\![H\downarrow]\!]$ is valid if and only if $[\![H\downarrow]\!]^\partial$ is valid. By Lemma 7, $[\![H\downarrow]\!]^\partial = [\![BPA(H\downarrow)]\!]$. By Lemma 40 and Lemma 41, $[\![BPA(H\downarrow)]\!]$ is valid if and only if $[\![BPA(H\downarrow)]\!] \models \bigwedge_{\varphi\in H} \varphi_{[\,]} \wedge \bigwedge_{\psi\in H} \psi_{\langle\rangle}$. □