Università di Pisa
Dipartimento di Informatica

Technical Report: TR-07-06

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March 6, 2007

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Abstract

The paper discusses a general framework for outer approximation type algorithms for the canonical DC optimization problem. A thorough analysis of properties which guarantee convergence is carried out: different sets of general conditions are proposed and compared. They are exploited to build six different algorithms, which include the first cutting plane algorithm proposed by Tuy but also new ones. Approximate optimality conditions are introduced to guarantee the termination of the algorithms and the relationships with the global optimal value are discussed.

Keywords: *DC problems, approximate optimality conditions, cutting plane algorithms*

1 Introduction

Nonconvex optimization problems often arise from applications in engineering, economics and other fields. A large number of them are actually DC optimization problems, that is nonconvex problems where the objective function is the difference of two convex functions and the constraint can be expressed as the set difference of two convex sets. In particular, the canonical DC (shortly CDC) problem has been investigated in many papers, as every DC optimization problem can be reduced to a CDC problem through standard transformations (see [10]). Several algorithms to solve it have been proposed (see, for instance, [4, 24, 20, 21, 26, 17]) and generally they are modifications of the first cutting plane algorithm proposed by Tuy in [4].

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In this paper, we consider the CDC problem relying on an alternative equivalent formulation based on a polar characterization of the constraint. The structure of this formulation allows to carry out a thorough analysis of convergence for cutting plane type algorithms. Different sets of conditions, which guarantee convergence, are proposed and exploited to build six algorithms, five of which can't be reduced to the original algorithm by Tuy. Furthermore, the alternative formulation allows to define proper approximate optimality conditions, which can be exploited to guarantee that the algorithms end after a finite number of iterations and provide approximate global optimal solutions.

The paper is organized as follows. In Section 2 the CDC problem and its polar based reformulation are introduced, and the well-known optimality conditions are recalled. In Section 3 we propose approximate optimality conditions and we investigate the relationship between the exact optimal value and the approximate optimal values. In Section 4 convergence analysis is carried out and six different algorithms are proposed and the corresponding proofs of finite termination are given. In the last section the connections of these results with the existing algorithms are outlined.

2 The Canonical DC Problem

Throughout all the paper we focus on the canonical DC minimization problem

$$(CDC) \qquad \min\{\ dx \mid x \in \Omega \setminus \text{int}\ C\ \}$$

where $\Omega \subseteq \mathbb{R}^n$ and $C \subseteq \mathbb{R}^n$ are full-dimensional closed convex sets, $d \in \mathbb{R}^n$ and dx denotes the scalar product between d and the vector of variables $x \in \mathbb{R}^n$.

The assumption on the dimension of the constraining sets is not restrictive. In fact, if Ω is not full-dimensional, the problem can be easily reformulated in the (affine) space generated by Ω. If C is not full-dimensional, then we have int $C = \emptyset$ and the problem is actually a convex minimization problem.

In order to avoid that (CDC) could be reduced to a convex minimization problem, we also suppose that the set C provides an essential constraint, i.e.,

$$\min\{dx \mid x \in \Omega\} < \min\{dx \mid x \in \Omega \setminus \text{int}\ C\}. \tag{1}$$

Relying on an appropriate translation, assumption (1) can be equivalently stated through the following two conditions

$$0 \in \Omega \cap \text{int}\ C \tag{2}$$

$$dx > 0 \quad \forall x \in \Omega \setminus \text{int}\ C. \tag{3}$$

Therefore, we assume that (2) and (3) hold. Notice that these assumptions guarantee that any feasible solution $x \in \Omega \setminus C$ provides a better feasible solution taking the unique intersection between the segment with 0 and x as end points and the boundary of C, i.e. $x' \in \delta C \cap (0, x)$ satisfies $dx' < dx$.

The constraint $x \notin \text{int } C$ is the source of nonconvexity in problem (CDC) and it is given just as a set relation. However, relying on the polarity between convex sets, we can express this nonconvex constraint in a different fashion. Let us recall that

$$C^* := \{w \in \mathbb{R}^n \mid xw \leq 1 \quad \forall x \in C\}$$

is the polar set of C and it is a closed convex set. Exploiting bipolarity relations (see, for instance, [30]), it is easy to check that the assumption $0 \in \text{int } C$ ensures that $x \notin \text{int } C$ if and only if $xw \geq 1$ for some $w \in C^*$. Therefore, problem (CDC) can be equivalently formulated in the following way

$$\min\{\ dx \mid x \in \Omega\ ,\ w \in C^*\ ,\ xw \geq 1\ \} \tag{4}$$

where polar variables w have been introduced and the nonconvexity is given by the inequality constraint, which asks for some sort of reverse polar condition.

Let $v(CDC)$ denote the optimal value of problem (CDC) and γ be any feasible value, i.e. $\gamma = dx$ for some $x \in \Omega \setminus \text{int } C$. In order to check whether a fesible value is optimal or not, we can consider also the set

$$D(\gamma) := \{x \in \Omega \mid dx \leq \gamma\}.$$

In fact, as an immediate consequence of the above definitions, if $\gamma = v(CDC)$, then the following equivalent inclusions hold

$$D(\gamma) \subseteq C \tag{5}$$

$$C^* \subseteq [D(\gamma)]^*. \tag{6}$$

Furthermore, it has been shown (see [11, Proposition 10]) that they are not only necessary but also sufficient optimality conditions when problem (CDC) is *regular*, i.e.

$$\min\{dx \mid x \in \Omega \setminus \text{int } C\} = \inf\{dx \mid x \in \Omega \setminus C\}. \tag{7}$$

3 Approximate Optimality Conditions

Given a feasible value γ, the optimality conditions (5) or (6) should be checked in order to recognize whether or not γ is the optimal value ($\bar{x}$ is optimal). Unfortunately, there is no known efficient way to check the inclusion between two convex sets. Yet, any exact algorithm for (CDC) must eventually cope with this problem.

In order to make this crucial step more readily approachable, we consider the following "optimization version" of the optimality conditions:

$$(OC_\gamma) \qquad \max\{\ vz - 1 \mid z \in D(\gamma)\ ,\ v \in C^*\ \} \tag{8}$$

It is trivial to show that (5) holds if and only if $v(OC_\gamma) \leq 0$, thus the above problem provides a way for checking optimality of a given value γ (solution

$\bar{x}$). Since the objective function of (8) is not concave, there are no known efficient approaches for this problem as well. However, checking (5) through the optimization problem (8) has the advantage of making it easy to define a proper notion of *approximate* optimality conditions.

A first way of approximating problem (8) is to replace Ω and C by two convex sets S and Q, respectively, satisfying

$$C^* \subseteq Q, \tag{9}$$

$$\Omega \subseteq S. \tag{10}$$

This is a standard step in cutting plane (outer approximation) approaches, where S and Q are chosen to be "easier" than the original sets (e.g., polyhedra with "few" vertices) and iteratively refined to become better and better approximations of Ω and C^* as needed. Hence, one considers the *relaxation* of (8)

$$(\overline{OC}_\gamma) \qquad \max\{\, vz - 1 \mid z \in S \;,\; v \in Q \;,\; dz \leq \gamma \,\} \tag{11}$$

whose optimal value provides an upper bound on $v(OC_\gamma)$; thus,

$$v(\overline{OC}_\gamma) \leq 0 \tag{12}$$

is a convenient *sufficient* optimality condition for (CDC). If (12) does not hold, then either γ is not the optimal value, or S and Q are not "good" approximations of Ω and C^*, respectively. All the cutting plane algorithms presented in this work follow the same basic scheme: (11) is solved, and its solution is used to improve S or Q or γ, in such a way to guarantee convergence of γ to the optimal value. The focus of the research is on devising a number of different ways to achieve this result, i.e., to obtain a convergent algorithm for (CDC) out of an "oracle" for (11). However, it is likely that in any such approach the solution of (11) is going to be the computational bottleneck; it therefore makes sense to consider solving (11) *only approximately.*

Approximately solving (11) may actually mean two different things:

1. computing a "large enough" lower bound on $v(\overline{OC}_\gamma)$, i.e., finding a feasible solution $(\bar{x}, \bar{w})$ "sufficiently close" to the optimal solution;

2. computing a "small enough" upper bound $l \geq v(\overline{OC}_\gamma)$.

Algorithmically, the two notions correspond to two entirely different classes of approaches: lower bounds are produced by *heuristics* computing feasible solutions, while upper bounds are produced by solving suitable *relaxations* of $(\overline{OC}_\gamma)$, e.g. replacing the non-concave objective function vz with a suitable concave upper approximation. Exact algorithms combining the two can then be used to push the lower bound and the upper bound arbitrarily close together. However, for the sake of our approaches only one of the two bounds is needed at any given time. In fact, $v(\overline{OC}_\gamma)$ is either positive or non-negative. To establish that the first case holds amounts at finding a feasible solution $(\bar{z}, \bar{v})$ to (11) such that $\bar{z}\bar{v} - 1 > 0$, while for the second case one needs an upper bound $l \leq 0$.

This is the rationale behind our definition of an *approximate oracle* for (11). In our development, we will assume availability of a procedure Θ which, given S, Q, γ, and two positive tolerances ε and ε'

- *either* produces an upper bound

$$\varepsilon v(\overline{OC}_\gamma) \leq l \qquad \text{such that} \qquad l \leq \varepsilon' \tag{13}$$

- *or* produces a feasible point $(\bar{z}, \bar{v})$ for (11) such that

$$\bar{z}\bar{v} - 1 \geq \varepsilon v(\overline{OC}_\gamma). \tag{14}$$

It is clear that (14) corresponds to a pretty weak requirement about the way in which (11) is solved: only an ε-approximate solution to (11) is needed, for *fixed but arbitrary* $\varepsilon > 0$. As for (13), it allows the lower bound to be "small enough" but positive, rather than non-negative; this is taken as the stopping condition of the approach, and we will show that the positive tolerance allows for finite termination of the algorithms even when γ is optimal. The drawback is that a feasible value γ needn't be optimal when (13) holds; clearly, the "quality" of γ has to be related somewhat with ε'. The remainder of this section is devoted to the study of this relationship.

Let h be a convex function such that

$$C = \{ \, x \in \mathbb{R}^n \mid h(x) \leq 0 \, \}. \tag{15}$$

Notice that if γ_C is the *gauge function* of C, i.e.

$$\gamma_C(x) := \inf\{ \, \lambda \in \mathbb{R}_+ \mid x \in \lambda C \, \},$$

then the convex function $h(x) = \gamma_C(x) - 1$ satisfies (15).

Our analysis uses the following three "approximated" problems

$$\begin{array}{lll} (CDC_\delta) & \phi(\delta) = & \min\{ \, dx \mid x \in \Omega \, , \, w \in C^* \, , \, xw \geq 1 + \delta \, \} \qquad (16) \\ (CDC'_\delta) & \psi(\delta) = & \min\{ \, dx \mid x \in \Omega \, , \, w \in C^*_\delta \, , \, xw \geq 1 \, \} \qquad (17) \\ (CDC''_\delta) & \varphi(\delta) = & \min\{ \, dx \mid x \in \Omega \, , \, h(x) \geq \delta \, \} \qquad (18) \end{array}$$

where $\delta \geq 0$ and $C_\delta = \{ \, x \mid h(x) \leq \delta \, \}$; clearly, $\psi(0) = \phi(0) = v(CDC)$, and ϕ, ψ, and φ are nondecreasing (the feasible set of all three problems shrinks as δ grows). For a proper choice of h, the three are equivalent.

Lemma 3.1 *If $h{+}1$ is the gauge function of C, then problems (CDC_δ), (CDC'_δ) and (CDC''_δ) are equivalent.*

Proof: The equivalence between (CDC'_δ) and (CDC''_δ) is trivial so we will concentrate upon (CDC_δ) and (CDC'_δ). If $h+1$ is the gauge function of C, then $h+1$ is the support function of C^*. For any feasible point (x, w) of (CDC_δ), $xw \geq 1+\delta$; this implies that

$$h(x) + 1 \geq xw \geq 1 + \delta,$$

thus $x \notin \text{int } C_\delta$ is feasible to (CDC'_δ). Vice-versa, given any feasible point x of problem (CDC'_δ) we have $x \notin \text{int } C_\delta$ and $h(x) \geq \delta$, that is,

$$h(x) + 1 \geq 1 + \delta.$$

Since $0 \in \text{int } C$, C^* is compact and thus there exists $w \in C^*$ such that $xw \geq 1+\delta$. $\boxdot$

Remark 3.1 When $h+1$ is not the gauge function of C, Problems (CDC_δ) and (CDC'_δ) may not be equivalent. Thus, (CDC_δ) can be viewed as a special case of (CDC'_δ) (equivalently, (CDC''_δ)) with $h+1$ being the gauge function of C. The gauge function is therefore a "preferred" way of expressing C via h in our setting.

The value δ in (CDC_δ) is strongly related with our approximate optimality conditions, as the following result shows:

Lemma 3.2 $\gamma \leq \psi(\delta) \quad \Leftrightarrow \quad D(\gamma) \subseteq C_\delta$

Proof: Using [11, Proposition 8], $\gamma \leq \psi(\delta) \Leftrightarrow D(\gamma) \subseteq \{\ x \mid h(x) \leq \delta\ \}$. $\boxdot$

Corollary 3.1 *If $h+1$ is the gauge function of C, then $\gamma \leq \phi(\delta) \Leftrightarrow z(OC_\gamma) \leq \delta$*

Proof: If $h+1$ is the gauge function of C, then $h+1$ is the support function of C^*; thus, $D(\gamma) \subseteq \{x \mid h(x) \leq \delta\}$ if and only if

$$\{\ (x, w) \in D(\gamma) \times C^*\ \} \quad \subseteq \quad \{\ (x, w) \mid xw \leq 1 + \delta\ \} \quad \boxdot$$

As a consequence, when (13) holds for some γ, one has

$$v(\overline{OC}_\gamma) \leq \varepsilon'/\varepsilon$$

and therefore $\gamma \leq \phi(\varepsilon'/\varepsilon)$. Thus, our stopping condition turns out to be that of the approximated problem (CDC_δ); one is then interested in the behavior of $\phi(\delta)$ as $\delta \to 0$ (remembering that $\delta = \varepsilon'/\varepsilon$). The first result is easy: ψ is continuous at 0.

Proposition 3.1 $\psi(\delta) \to \psi(0) = v(CDC)$ *when* $\delta \to 0$.

Proof: Given any $\delta^1 \geq \delta^2 \geq 0$, we clearly have $\psi(\delta^1) \geq \psi(\delta^2)$, i.e., ψ is nonincreasing and bounded below. Let $\bar{\gamma} = \lim_{\delta \to 0} \psi(\delta)$, we have that $\bar{\gamma} \geq \psi(0)$. Assume by contradiction that $\bar{\gamma} > \psi(0)$, by the definition of ψ we have

$$\max\{ \ h(z) \mid z \in D(\psi(\delta)) \ \} \ \leq \ \delta$$

for all $\delta > 0$. Therefore, we get that

$$\max\{ \ h(z) \mid z \in D(\bar{\gamma}) \ \} \ \leq \ 0$$

which contradicts $\psi(0) = \sup\{ \ \gamma \mid D(\gamma) \subseteq C \ \}$. $\boxdot$

Proposition 3.1 implies that $\phi(\delta) \to \phi(0) = v(CDC)$ when $\delta \to 0$. Although $\psi(\delta)$ and $\phi(\delta)$ converges to the right value as $\delta \to 0$, the rate of convergence may be less than linear, as the following example shows.

Example 3.1 Let

$$\begin{aligned} C &= \{ \ (x_1, x_2) \mid (x_2 - 1)^2 - x_1 - 2 \leq 0 \ \} \\ \Omega &= \{ \ (x_1, x_2) \mid x_2 \geq 0 \ , \ x_1 \geq -2 \ , \ x_1 + 2x_2 \geq 0 \ \} \end{aligned}$$

and $d = (0, 1)$. Let (x^*, w^*) be the optimal solution of (CDC_δ), it is easy to see that $x^* = -2$ for all $\delta \geq 0$. Moreover, $\frac{1}{1+\delta}(x^*, w^*) \in \partial C$, thus we get that $\phi(\delta) = w^* = 1 + \delta + \sqrt{2\delta(1+\delta)}$, thus $\lim_{\delta \to 0} \ (\phi(\delta) - \phi(0))/\delta = \lim_{\delta \to 0} \ 1 + \sqrt{2(1+\delta)/\delta} = +\infty$.

Moreover, let $h = (x_2 - 1)^2 - x_1 - 2$ and (x^*, w^*) the optimal value of problem (CDC'_δ), it is easy to see that $x^* = -2$ for all $\delta \geq 0$. Moreover, $h(x^*, w^*) = \delta$, thus we have $\psi(\delta) = w^* = 1 + \sqrt{\delta}$. Therefore, $\lim_{\delta \to 0} \ (\psi(\delta) - \psi(0))/\delta = \lim_{\delta \to 0} \ \sqrt{\delta}/\delta = +\infty$.

Thus, one would be interested in conditions ensuring that the value function ϕ is Lipschitz at 0.

Proposition 3.2 *Let x^* be an optimal point of problem (CDC). If there exists a ray $L = \{ \ x \mid x = x^* + \lambda u \ , \ \lambda > 0 \ \}$ such that*

$$L \cap \Omega \neq \emptyset \quad \text{and} \quad h'(x^*; u) > 0, \tag{19}$$

then the value function ψ is Lipschitz at 0.

Proof: For any $x \in L \cap \Omega$ (which exists for the hypothesis), all the segment $[x^*, x]$ is included in Ω. Define $x(\lambda) = x^* + \lambda u$; since h is convex, $h(x(\lambda)) \geq h(x^*) + \lambda h'(x^*; u)$ for any $\lambda \geq 0$. For $y(\delta) = x(\delta / h'(x^*; u))$ we therefore have

$$h(y(\delta)) \geq h(x^*) + \delta = \delta$$

which means that $y(\delta) \notin C_\delta$. For δ small enough, $y(\delta) \in \Omega$ and therefore $\psi(\delta) \leq dy(\delta)$. Hence,

$$\psi(\delta) - \psi(0) \leq d(y(\delta) - x^*) = (\delta / h'(x^*; u))du \ \ ,$$

i.e., ψ is Lipschitz at 0 with constant $du/h'(x^*;u)$. ⊡

Proposition 3.2 provides a condition that guarantees Lipschitz behavior of ψ; however, assumptions are required to ensure that condition (19) holds.

Definition 3.1 $T(C,x) = \{ \ u \in \mathbb{R}^n \mid \exists t^n \downarrow 0 \ , \ u^n \to u \ \text{ s.t. } \ x + t^n u^n \in C \ \}$ is called the tangent cone of C at x.

Corollary 3.2 *If there exists an optimal point x^* such that $T(\Omega, x^*) \nsubseteq T(C, x^*)$, then ψ is Lipschitz at* 0.

Proof: Take any $u \in T(\Omega, x^*) \setminus T(C, x^*)$; since $T(C, x^*) = \{ \ u \in \mathbb{R}^n \mid h'(x^*; u) \leq 0 \ \}$, then $h'(x^*; u) > 0$. As $u \in T(\Omega, x^*)$, there exist $t^n \downarrow 0$ and $u^n \to u$ such that $x + t^n u^n \in \Omega$. By continuity of $h'(x^*; \cdot)$, $h'(x^*; u^n) > 0$ for large enough n. Thus, u^n provides the ray L required by Proposition 3.2. ⊡

The following result provides an alternative form of the sufficient condition (19), which may be taken as a definition of a *strong regularity* condition which ensures Lipschitz behavior of ϕ.

Proposition 3.3 *Let $h+1$ be the gauge function of C and x^* be an optimal solution of problem (CDC), then there exists a ray L satisfying condition (19) if and only if $\partial h(x^*) \setminus \partial \Omega^* \neq \emptyset$.*

Proof: Assume $W = \partial h(x^*) \setminus \partial \Omega^* \neq \emptyset$; there exists $w \in W$ such that $w \notin \Omega^*$. Therefore, there exists $x \in \Omega$ such that $wx > 1$. Let $u = \frac{x - x^*}{\|x - x^*\|}$: we have $L \cap \Omega \neq \emptyset$ and $h'(x^*; u) \geq wu > 0$ [30, Theorem 23.2].

Vice-versa, assume there exists a ray L satisfying condition (19). Since $h'(x^*; u) = \max\{ \ uw \mid w \in \partial h(x^*) \ \}$ [30, Theorem 23.4], then there exists a point $w \in \partial h(x^*)$ such that $wu > 0$. Take $\lambda > 0$ small enough so that $x^* + \lambda u \in \Omega$; then $w(x^* + \lambda u) > 1$, which implies that $w \notin \Omega^*$. ⊡

The final result is that bounded polyhedra satisfy the strong regularity condition.

Lemma 3.3 *If C is a bounded polyhedron and the regularity condition holds, then ψ is Lipschitz at* 0.

Proof: By assumption, C is a full-dimensional polyhedron, thus all of its facets are $(n-1)$-dimensional polyhedral sets and there exists a unique hyperplane containing each facet.

For any facet F of C, let $H_F = \{x \mid tx\}$ be the unique hyperplane containing F and K_F be the cone generated by F. Moreover, let $H_F^- = \{x \mid tx \leq 1\}$ and $H_F^+ = \{x \mid tx \geq 1\}$. Given any point $x \in K_F$, there exists $y \in F$ such that $x = \mu y$ where $\mu \geq 0$. Then we have $x \in C$ if and only if $\mu \leq 1$, which is exactly $x \in H_F^-$. Therefore, $C \cap K_F = H_F^- \cap K_F$.

Lemmas $A.1$ and $A.2$ implies that there exists an optimal point $x^* \in \partial(\Omega \setminus C)$ and a sequence $\{x^i\}$ in $\Omega \setminus C$ such that $x^i \to x^*$. Let S be the set of facets of C that contain x^*, we have $x^* \in \operatorname{int} \ \bigcup_{F \in S} K_F$ follows by $0 \in \operatorname{int} \ C$. Therefore, there exists $I > 0$ such that $x^i \in \bigcup_{F \in S} K_F$ for all $i \geq I$. Thus there exists a facet $\hat{F} \in S$ such that $x^I \in K_{\hat{F}}$. Since $x^I \notin C$, we get that $x^I \notin H^-_{\hat{F}}$. The fact that $x^* \in K_{\hat{F}} \cap H_{\hat{F}}$ and $x^I \in K_{\hat{F}} \cap \text{int } H^+_{\hat{F}}$ implies that $(x^*, x^I] \subseteq K_{\hat{F}} \cap \text{int } H^+_{\hat{F}}$. Therefore, by $C \cap K_{\hat{F}} = H^-_{\hat{F}} \cap K_{\hat{F}}$ we have $(x^*, x^I] \cap C = \emptyset$.

Let $\bar{u} = \frac{x^I - x^*}{\|x^I - x^*\|}$ and $\hat{F} = \{x \mid \hat{t}x = 1\}$, we get that $\hat{t}\bar{u} > 0$. Since $\hat{t}x \leq 1$ for all $x \in C$, we have $\bar{u} \notin T(C, x^*)$; the thesis then follows by Corollary 3.2. $\boxdot$

4 Conditions and Algorithms

In this section we present several algorithms which, given an approximated oracle Θ, (approximately) solve the problem (CDC). In the presentation, we first establish a hierarchy of abstract conditions ensuring convergence, and then for each we propose actual implementatable procedures which realize the abstract conditions.

All these algorithms follow the generic cutting plane scheme sketched in the previous paragraph. More in details, a non decreasing sequence of feasible values $\{\gamma_k\}$ is produced, and for each γ_k the oracle Θ is called, thereby producing either a value l^k such that condition (13) holds, or points z^k and v^k satisfying conditions

$$(z^k, v^k) \in S \times Q \ , \ dz^k + ev^k \leq \gamma_k \tag{20}$$

and (14). By repeatedly calling the oracle, if necessary, we can construct a procedure which either proves that γ_k satisfies condition (13), or produces a better feasible value $\gamma_{k+1} < \gamma_k$. In the latter case, the algorithms produces points x^k and w^k such that

$$x^k \in C \ , \ w^k \in C^* \ , \ x^k w^k = 1 \tag{21}$$

and $\gamma_{k+1} = dx^k$.

By condition (3), we get that all optimal solutions (x, w) should satisfy $xw = 1$, otherwise there exists another point $(\bar{x}, \bar{w}) = \frac{(x,w)}{\sqrt{xw}}$ satisfying $d\bar{x} < dx$ and $\bar{x}\bar{w} = 1$. Moreover, by optimality condition (5) we know that all the optimal solutions (x, w) satisfy $(x, w) \in C \times C^*$. Relying on these observations, the sequence of points $\{(x^k, w^k)\}$ is produced to satisfy condition (21).

Given any point (x, w), there are two ways to compute the objective value. Let

$$\phi(x) = \begin{cases} dx & \text{if } x \in \Omega, \\ +\infty & \text{else.} \end{cases} \tag{22}$$

and

$$\varphi(w) = \min\{\phi(x) \mid xw \geq 1\}. \tag{23}$$

The objective function of a point (x, w) is given by $\gamma = \phi(x)$. Note that $\varphi(w) \leq \phi(x)$ for all (x, w) satisfying condition (21), we can also choose $\gamma = \varphi(w)$. In this case, if γ is optimal, then the optimal solution is $(\bar{x}, w)$ where $\bar{x} \in \text{argmin}\,\{dx \mid x \in \Omega, xw \geq 1\}$.

The basic idea of our algorithms is the following. Choose a point (x^0, w^0) satisfying condition (21). Set $k = 1$, then the incumbent value γ_k is given by $\gamma_k = \varphi(w^{k-1})$. If γ_k satisfies condition (8), then γ_k is the optimal value and w^{k-1} is an optimal solution; otherwise, find a point $w^k \in C^*$ such that $\varphi(w^k) < \gamma_k$ and then iterate. Under suitable assumptions, the sequence of points $\{w^k\}$ converges to an optimal solution.

Algorithm 1 Prototype Algorithm

0. Let w^0 be the best available feasible solution, $\gamma_1 = \varphi(w^0)$.
(If no feasible solution is available, then set $\gamma_1 = +\infty$). $k = 1$.
1. If optimality condition (5) holds, then γ_k is the optimal value and stop;
2. Otherwise, select a feasible point $w^k \in C^*$ such that $\varphi(w^k) < \gamma_k$, set $\gamma_{k+1} = \varphi(w^k)$.
3. $k = k + 1$, goto 1.

An important feature for the convergence of Algorithm 1 is that $\{\gamma_k\}$ is a decreasing sequence and bounded below:

$$0 \leq \gamma_\infty < \cdots < \gamma_k < \gamma_{k-1} < \cdots < \gamma_1,$$

where $\gamma_\infty = \lim_{k\to\infty} \gamma_k$. Therefore, $\{D(\gamma_k)\}$ is a "non-increasing sequence", i.e.,

$$D(\gamma_\infty) \subseteq \cdots \subseteq D(\gamma_{k+1}) \subseteq D(\gamma_k) \subseteq \cdots \subseteq D(\gamma_1).$$

Algorithm 1 is too general to deduce any meaningful property. At least two important points are still unsaid:

Question 4.1 How to check optimality condition (5)?

Question 4.2 How to select (x^k, w^k) such that $\varphi(w^k) < \gamma_k$ once you know that condition (5) is not fulfilled?

Note that Question 4.1 and Question 4.2 are closely related to each other, i.e., if we can find a feasible point w^k such that $\varphi(w^k) < \gamma_k$ in Question 4.2, then Question 4.1 is answered at the same time. We start by answering Question 4.2. Assume that we have any constructive procedure that answers Question 4.1 by eventually producing a point $z^k \in D(\gamma_k) \setminus C$, then there exists $w^k \in C^*$ such that $w^k z^k > 1$. Then we have

$$\varphi(w^k) < dz^k \leq \gamma_k.$$

So the question is: does this method provide a convergent algorithm? the answer is no.

Example 4.1 Let $d = (0, 1)$ and $e = (0, 0)$; $\Omega = \{(x_1, x_2) \mid -1.8 \leq x_1 \leq 1.96, x_2 \geq 0\}$, $C = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 4\}$. Therefore, $C^* = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 1/4\}$.

In this problem, we can find a sequence of points $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ converging to a non-optimal point (x, w) where $x = (-1.8, 0.87)$ and $w = \frac{1}{4}(-1.8, 0.87)$. However, the optimal point is $(\bar{x}, \bar{w})$ where $\bar{x} = (-1.96, 0.4)$ and $\bar{w} = \frac{1}{4}(-1.96, 0.4)$.

Example 4.1 shows that if there is no further restriction for the way to select $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$, Algorithm 1 may not converge to an optimal solution. We aim at providing general and weak assumptions under which convergence can be proved. We propose the following conditions:

$$\liminf v^k z^k \leq 1, \tag{24}$$

$$v^k z^k - 1 \geq \varepsilon \max\{vz - 1 \mid (z, v) \in D(\gamma_k) \times C^*\}, \tag{25}$$

where $\varepsilon \in (0, 1)$.

Proposition 4.1 *If conditions* (24) *and* (25) *hold, then the sequence of feasible values* $\{\gamma_k\}$ *converges to the optimal value.*

Proof: Since the sequence $\{\gamma_k\}$ is non-increasing and has a lower bound, then there exists a limit $\bar{\gamma}$ of $\{\gamma_k\}$. Let $\hat{\gamma}$ be the optimal value of problem (CDC), we get that $\hat{\gamma}$ is not greater than γ_k for all k, which implies that $\hat{\gamma} \leq \bar{\gamma}$ and thus $\bar{\gamma}$ is a feasible value.

Since $\bar{\gamma} \leq \gamma_k$ for all k, then condition (25) implies that

$$v^k z^k - 1 \geq \varepsilon \max\{vz - 1 \mid (z, v) \in D(\bar{\gamma}) \times C^*\}$$

for all k. It follows from condition (24) that

$$\max\{vz - 1 \mid (z, v) \in D(\bar{\gamma}) \times C^*\} \leq 0,$$

which means that the feasible value $\bar{\gamma}$ is optimal. ⊡

Conditions (24) and (25) provide a weak and general assumptions under which the sequence of feasible values $\{\gamma_k\}$ converges to the optimal value. Note that condition (24) is difficult to check. In the following paragraphs, we aim to construct sequences of points $\{(z^k, v^k)\}$ satisfying condition (24).

Let $\{\sigma_1^k\}$, $\{\sigma_2^k\}$, $\{\sigma_3^k\}$ and $\{\sigma_4^k\}$ be four positive sequences such that $\sigma_1^k \to 0$, $\sigma_2^k \to 0$, $\sigma_3^k \to 0$ and $\sigma_4^k \to 0$. The following conditions are proposed.

$$v^k z^k \leq v^k x^k + \sigma_1^k, \tag{26}$$

$$v^k x^k \leq 1 + \sigma_2^k. \tag{27}$$

$$v^k z^k \leq z^k w^k + \sigma_3^k, \tag{28}$$

$$z^k w^k \leq 1 + \sigma_4^k. \tag{29}$$

Lemma 4.1 *If conditions* (26) *and* (27) *(*(28) *and* (29)*) hold, then condition* (24) *holds.*

Proof: By conditions (26) and (27), we get that $v^k z^k \leq 1 + \sigma_1^k + \sigma_2^k$. $\boxdot$

Remark 4.1 Let A be the set of conditions (24) and (25); B_1 the set of conditions (25), (26) and (27); and B_2 the set of conditions (25), (28) and (29). Lemma 4.1 shows that A is implied by both B_1 and B_2. Therefore, both B_1 and B_2 guarantee that a sequence of feasible values converges to the optimal value. In the following sections, we give some sub-procedures to produce points satisfying B_1 or B_2.

4.1 Algorithms Exploiting Conditions B_1

In order that condition (26) holds, we propose the following condition.

$$z^k = \mu_1^k x^k, \tag{30}$$

where $\mu_1^k > 0$.

Lemma 4.2 *Suppose that* $\{v^k x^k\}$ *is bounded. If conditions* (29) *and* (30) *hold, then condition* (26) *holds.*

Proof By conditions (29) and (30), we get that $\mu_1^k \leq 1 + \sigma_3^k$ for all k, thus $\limsup \mu_1^k \leq 1$. Therefore, when $\{v^k x^k\}$ is bounded we have

$$\limsup v^k(z^k - x^k) = \limsup v^k x^k(\mu_1^k - 1) \leq 0.$$

$\boxdot$

There are many ways to get conditions (29) and (30). In many existing algorithms, d is used to get these conditions. The following conditions are proposed.

$$dz^k \leq dx^{k-1}, \tag{31}$$

$$x^k \in (0, z^k) \cap \Omega \cap \partial C. \tag{32}$$

Condition (32) implies that the sequence of points $\{x^k\}$ is feasible and condition (30) holds. Furthermore, we have $\mu_1^k > 1$ in condition (30).

Lemma 4.3 *Suppose that the optimal value of problem* (4) *is positive. If conditions* (31) *and* (32) *hold, then condition* (29) *holds.*

Proof: Condition (32) implies that x^k is feasible, then we have $dx^k > 0$. Therefore, $dz^k > dx^k$ for all k since $\mu_1^k > 1$. Conditions (31) and (32) imply $dx^{k-1} \geq dz^k > dx^k$. By this inequality, it follows that

$$dx^{k-1} - dx^k \geq dz^k - dx^k > 0.$$

Let γ^* be the optimal value, we get that

$$dx^h = dx^0 - \sum_{k=1}^{h}(dx^{k-1} - dx^k) \geq \gamma^*.$$

This means that

$$dx^0 - \gamma^* \geq \sum_{k=1}^{h}(dx^{k-1} - dx^k) \geq \sum_{k=1}^{h}(dz^k - dx^k).$$

Taking the limits as $h \to +\infty$, we get $\lim_{h\to\infty}\sum_{k=1}^{h}(dz^k - dx^k) \leq dx^0 - \gamma^*$. It follows from the fact $dz^k - dx^k$ is non-negative that $dz^k - dx^k \downarrow 0$ and thus $(\mu_1^k - 1)dx^k \downarrow 0$, which implies that $\lim_{k\to\infty}\mu_1^k = 1$ since $\lim_{k\to\infty} dx^k \geq \gamma^* > 0$. Therefore,

$$\limsup z^k w^k = \limsup \mu_1^k x^k w^k = \limsup \mu_1^k \leq 1.$$

$\boxdot$

Lemma 4.3 implies that conditions (29) and (30) are implied by conditions (31) and (32), thus condition (26) is also implied by (31) and (32) when $\{v^k x^k\}$ is bounded. Therefore, B_1 are implied by conditions (25), (27), (31) and (32).

Let C_1 be the set of conditions (25), (27), (31) and (32), C_1 guarantees that a sequence of feasible values $\{\gamma_k\}$ converges to the optimal value. Before giving a sub-procedure to produce points satisfying these conditions, we introduce a theorem about the outer approximation method.

Theorem 4.1 *[11, Proposition 17] Let Ω be a convex set in $\mathbb{R}^n$ such that $\Omega = \{x : \tilde{g}(x) \leq 0\}$, $\tilde{g}$ is a convex function. Assume that $0 \in \operatorname{int}\,\Omega$ and let S_k, $k = 1, 2, \ldots$ be a sequence of polyhedrons satisfying*

1) $z^k \in S_k \setminus \Omega$;

2) $S_{k+1} = S_k \cap \{x \mid p^k, x - y^k + \alpha_k \leq 0\}$, where $y^k \in [0, z^k) \setminus \operatorname{int}\,\Omega$, $0 \leq \alpha_k \leq \tilde{g}(y^k)$, $p^k \in \partial\tilde{g}(y^k)$ and $\alpha_k - \tilde{g}(y^k) \to 0$.

Then any cluster point $\bar{z}$ of the sequence $\{z^k\}$ belongs to $\partial\Omega$.

Theorem 4.1 gives an outer approximation method and proves that all the cluster points of of this method belong to Ω. Replying on the method provided by this theorem, we get Sub-procedure 4.1.

Subprocedure 4.1

1. Ω is a closed convex set such that $\Omega = \{g(x) \leq 0\}$ and $0 \in \operatorname{int}\,\Omega$, $S \supseteq \Omega$ is a closed convex set.

2. Given a point $x \in S \setminus \Omega$, select a point $y \in (0, x) \cap \partial\Omega$ and a sub-gradient $p \in \partial g(y)$. Set $S = S \cap \{x \mid p(x - y) + g(y) \leq 0\}$

It is obvious that Sub-procedure 4.1 is a simplified form of the method provided by Theorem 4.1. Although the proof of Theorem 4.1 has shown that Sub-procedure 4.1 cut off x from Ω without cutting any point from Ω, we can still use a more simple way to explain it.

Remark 4.2 Let $H = \{z \mid p(z-y)+g(y) = 0\}$ and $H^+ = \{z \mid p(z-y)+g(y) \geq 0\}$. Note that for any point $z \in \Omega$, we have

$$p(z - y) + g(y) \leq g(z) \leq 0$$

Therefore, no point in Ω is cut away form S. In fact, it follows from the definition of the hyperplane H that it is a tangent hyperplane of Ω at y and $0 \in \text{int } H^-$ since $0 \in \text{int } \Omega$.

Consider the point $x \in S \setminus \Omega$ in Sub-procedure 4.1, it is easy to see that x and Ω are separated by H strictly: Assume by contradiction that $x \in H^-$, then $z \in \text{int } H^-$ for all $z \in (0, x)$ [30, Theorem 6.1], this contradicts the fact that $y \in (0, x)$ and $y \in H$.

Remark 4.2 shows that Sub-procedure 4.1 constructs a hyperplane separating Ω and x strictly. x is removed from S and Ω is still included in S. Note that the condition $0 \in \text{int } \Omega$ is required, otherwise Sub-procedure 4.1 may not be able to construct a hyperplane separating Ω and x strictly [Example 4.2].

Example 4.2 Let $\Omega \subseteq \mathbb{R}^2$ such that $\Omega = \{(u, v)|(u + 1)^2 + v^2 \leq 1\}$, $S = \{(u, v)| - 1 \leq u \leq 1, -1 \leq v \leq 1\}$ and $x = (1, 0)$.

Apply Sub-procedure 4.1 to Ω and x, we get that $y = (0, 0)$ and the separating hyperplane is $\{(u, v)|u = 0\}$. It's easy to see that x still belongs to $S = S \cap \{(u, v)|u \leq 0\}$, i,e, Sub-procedure 4.1 can not remove x from S.$\boxdot$

Let's give a sub-procedure that finds points satisfying conditions B_1.

Subprocedure 4.2 a) Let S and Q be the closed convex sets satisfying conditions (10) and (9), respectively. Let $S^1 = S$ and $Q^1 = Q$. Set $i = 1$.
b) Use the oracle Θ to select l^i satisfying condition (13).
If $l^i \leq \varepsilon'$, then $l' = l^i$ and stop.
c) Use the oracle Θ to select (z^i, v^i) satisfying (20) and (14), choose $(x^i, w^i) \in \partial C \times \partial C^*$ such that $z^i = \mu_1^i x^i$ and $x^i w^i = 1$, where $\mu_1^i > 0$.
If $z^i \notin \Omega$, then use Sub-procedure 4.1 with S^i and z^i to get a convex set S^{i+1}; else, $S^{i+1} = S^i$.
If $v^i \notin C^*$, then use Sub-procedure 4.1 with Q^i and v^i to get a convex set Q^{i+1}; else, $Q^{i+1} = Q^i$.
If $x^i \in \Omega$ and $x^i v^i \leq 1 + \sigma$, then $x' = x^i$, $w' = w^i$, $l' = l^i$, $z' = z^i$, $v' = v^i$, $Q' = Q^{i+1}$ and $S' = S^{i+1}$, stop;
d) Else, set $i = i + 1$, return to b).

Remark 4.3 In Sub-procedure 4.2, ε is the parameter defined in condition (25). Since $\varepsilon \in (0, 1)$, we can always find a value l^i satisfying condition (13). ε' and σ are two small enough positive values such that $\sigma < \varepsilon'$. In this case, we have $v'z' > v'x'$, which implies that $\mu_1' > 1$ and thus $z' \notin C$. Therefore, $x' \in (0, z') \cap \partial C$.

Remark 4.4 In Sub-procedure 4.2, it follows from conditions (10) and (9) that Ω and C^* are included in S^1 and Q^1, respectively. As it has been shown in Remark 4.2, Sub-procedure 4.1 constructs a hyperplane separating z^i and Ω strictly. By using this hyperplane, Sub-procedure 4.1 cuts off z^i from S^i without removing any point in Ω, thus we get a "non-increasing" sequence $\{S^i\}$:

$$\Omega \subseteq \cdots \subseteq S^i \subseteq S^{i-1} \subseteq \cdots \subseteq S^1. \tag{33}$$

In the same way, we have

$$C^* \subseteq \cdots \subseteq Q^i \subseteq Q^{i-1} \subseteq \cdots \subseteq Q^1. \tag{34}$$

When we start Sub-procedure 4.2 with a feasible value γ, we hope that Sub-procedure 4.2 can produce a better feasible value or prove that γ is optimal. Let's consider the case that γ is not optimal.

Proposition 4.2 *Suppose the set S and Q are bounded. If $D(\gamma) \nsubseteq C$, then Sub-procedure* 4.2 *ends in a finite number of steps and it either reports l' such that $l' \leq \varepsilon'$ or reports (z', v') and (x', w') satisfying conditions* (25)*,* (27)*,* (31) *and* (32)*.*

Proof: Assume by contradiction that Sub-procedure 4.2 never ends. Sub-procedure 4.2 generates two sequences of points $\{(z^i, v^i)\}$ and $\{(x^i, w^i)\}$. (z^i, v^i) is contained in $S^i \times Q^i$ for all i. As it has been shown in (33) and (34), $S^i \subseteq S$ and $Q^i \subseteq Q$ for all i, thus the sequence $\{(z^i, v^i)\}$ is bounded since $S \times Q$ is bounded. Theorem 4.1 guarantees that all the cluster points of $\{(z^i, v^i)\}$ are in $\Omega \times C^*$. Moreover, $dz^i \leq \gamma$ for all i, we get that all the cluster points of $\{(z^i, v^i)\}$ are in $D(\gamma) \times C^*$.

If there exists a cluster point of $\{(z^i, v^i)\}$ in $C \times C^*$, then we have $\liminf v^i z^i \leq 1$ and thus there exists l^I such that $l^I < \varepsilon'$, a contradiction. In this case Sub-procedure 4.2 ends and outputs $l' = l^I$.

Let's consider the case that all the cluster points of $\{z^i\}$ are not in C. Theorem 4.1 guarantees that all the cluster points of $\{(z^i, v^i)\}$ are in $\Omega \times C^*$, thus $\limsup v^i x^i \leq 1$ since $x^i \in C$ for all i. Therefore, there are only a finite number of i such that $v^i x^i > 1 + \sigma$.

Let $\bar{z}$ be any cluster point of $\{z^i\}$, we have $\bar{z} \in \Omega \setminus C$, thus there exists $\bar{x} \in (0, \bar{z})$ such that $\bar{x}$ is a cluster point of $\{x^i\}$. The fact that $0 \in \text{int } \Omega$ and $\bar{z} \in \Omega$ implies that $\bar{x} \in \text{int } \Omega$. Therefore, there exists $I > 0$ such that $x^i \in \Omega$ for all $i \geq I$, a contradiction.

In this case, Sub-procedure 4.2 ends. Conditions (13), (14), (33), and (34) imply that condition (25) holds. The stopping criteria $v'x' \leq 1 + \sigma$ implies that condition (27) holds. (31) is implied by condition (20) when $\gamma = \gamma_k$. As it has been shown in Remark 4.3, $x' \in (0, z') \cap \partial C$, thus condition (32) holds since $x' \in \Omega$. $\boxdot$

Proposition 4.2 shows that Sub-procedure 4.2 can produce a better feasible value or prove that γ is an approximate optimal value in a finite number of steps. Then let's consider the case that γ is optimal.

Proposition 4.3 *Suppose the set S and Q are bounded. Given any $\varepsilon' > 0$, if $D(\gamma) \subseteq C$, then Sub-procedure* 4.2 *ends in a finite number of steps and reports l' such that $l' \leq \varepsilon'$.*

Proof: Assume by contradiction that Sub-procedure 4.2 never ends. As it has been shown in the proof of Proposition 4.2, Sub-procedure 4.2 generates a bounded sequence of points $\{(z^i, v^i)\}$. Theorem 4.1 guarantees that all the cluster points of $\{(z^i, v^i)\}$ are in $\Omega \times C^*$. Moreover, $dz^i \leq \gamma$ implies that all the cluster points of $\{(z^i, v^i)\}$ are in $D(\gamma) \times C^*$. Since $D(\gamma) \subseteq C$, then we get that all the cluster points of $\{(z^i, v^i)\}$ are in $C \times C^*$. Then we have $\liminf v^i z^i \leq 1$, which implies that there exists $I > 0$ such that $l^I < \varepsilon'$, a contradiction. Sub-procedure 4.2 ends and outputs $l' = l^I$. $\boxdot$

Remark 4.5 When γ is optimal and $\varepsilon' = 0$, it is obvious that Sub-procedure 4.2 can not find a better feasible value. Sub-procedure 4.2 may never stop since we can only get that $\limsup l^i \leq 0$, which doesn't guarantee that there exists an I such that $l^I \leq 0$. If $l^i > 0$ for all i, then Sub-procedure 4.2 never stops. Therefore, Sub-procedure 4.2 can not prove that the optimal value is optimal. We will consider the relationship between approximated optimal values and the optimal value later.

C_1 is not the only set of conditions implying B_1. Let's show that condition (27) is implied by condition

$$v^k x^i \leq 1 \text{ for all } i < k. \tag{35}$$

Lemma 4.4 *Suppose that* $\{v^k\}$ *and* $\{x^k\}$ *are bounded. If condition* (35) *holds, then condition* (27) *holds.*

Proof: By taking subsequences if necessary, let $v^k \to \bar{v}$ and $x^k \to \bar{x}$. Since $v^k x^i \leq 1$ for all $i < k$, we get that $\bar{v} x^i \leq 1$ for all i and thus $\bar{v}\bar{x} \leq 1$. $\boxdot$

Let C_2 be the set of conditions (25), (31), (32) and (35), then C_2 also guarantees that a sequence of feasible values $\{\gamma_k\}$ converges to the optimal value. Let's show how to construct a sub-procedure to obtain points satisfying the set of conditions C_2.

Subprocedure 4.3 a) Let S and Q be the closed convex sets satisfying conditions (10) and (9).
Let $S^1 = S$ and $Q^1 = Q$. Set $i = 1$.
b) Use the oracle Θ to select l^i satisfying condition (13), if $l^i \leq \varepsilon'$, then $l' = l^i$ and stop;
c) Use the oracle Θ to choose (z^i, v^i) satisfying (14) and (20), choose $(x^i, w^i) \in \partial C \times \partial C^*$ such that $z^i = \mu_1^i x^i$ and $x^i w^i = 1$, where $\mu_1^i > 0$.
Set $Q^{i+1} = Q^i \cap \{v \mid v x^i \leq 1\}$.
If $z^i \notin \Omega$, use Sub-procedure 4.1 with S^i and z^i to get a convex set S^{i+1}, else, $S^{i+1} = S^i$.
d) If $x^i \in \Omega$ and $z^i \notin C$, then $x' = x^i$, $z' = z^i$, $v' = v^i$, $w' = w^i$, $l' = l^i$, $Q' = Q^{i+1}$ and $S' = S^{i+1}$, stop;
Else, set $i = i + 1$, return to b).

Remark 4.6 In Sub-procedure 4.3, ε' is a small enough positive value. Let $\tilde{g}$ be the gauge function of C^*, then $\tilde{g}$ is also the support function of $C^{**} = C$

[30, Theorem 14.5]. This implies that $C^* = \{v \mid \tilde{g}(v) - 1 \leq 0\}$ and $\tilde{g}(v) = \max\{vx \mid x \in C\}$. Note that $x^i \in \partial C$ and $x^i w^i = 1$, then we have $\tilde{g}(w^i) = w^i x^i$. Therefore, $x^i \in \partial(\tilde{g}(w^i) - 1)$. Given any sequence $\{v^i\}$ such that $v^i \notin C^*$ for all i, $Q^{i+1} = Q^i \cap \{v \mid vx^i \leq 1\}$ satisfies conditions 1) and 2) in Theorem 4.1. As it has been shown in Remark 4.4, the sequence of sets $\{Q^i\}$ satisfies condition (34).

Proposition 4.4 *Suppose the set S and Q are bounded. If $D(\gamma) \nsubseteq C$, then Sub-procedure* 4.3 *ends in a finite number of steps and it either reports l' such that $l' \leq \varepsilon'$ or reports (z', v') and (x', w') satisfying conditions* (25)*,* (31) *and* (32).

Proof: Assume by contradiction that Sub-procedure 4.3 never ends. Sub-procedure 4.3 generates two sequences of points $\{(z^i, v^i)\}$ and $\{(x^i, w^i)\}$. (z^i, v^i) is contained in $S^i \times Q^i$ for all i. As it has been shown in (33) and (34), $S^i \subseteq S$ and $Q^i \subseteq Q$ for all i, thus the sequence $\{(z^i, v^i)\}$ is bounded since $S \times Q$ is bounded. Theorem 4.1 guarantees that all the cluster points of $\{(z^i, v^i)\}$ are in $\Omega \times C^*$. Moreover, $dz^i \leq \gamma$ for all i, we get that all the cluster points of $\{(z^i, v^i)\}$ are in $D(\gamma) \times C^*$.

If there exists a cluster point of $\{(z^i, v^i)\}$ in $C \times C^*$, then we have $\liminf v^i z^i \leq 1$ and thus there exists l^I such that $l^I < \varepsilon'$, a contradiction. In this case Sub-procedure 4.3 ends and outputs $l' = l^I$.

Let's consider the case that all the cluster points of $\{z^i\}$ are not in C, then there are only a finite number of $z^i \in C$. Let $\bar{z}$ be any cluster point of $\{z^i\}$, we have $\bar{z} \in \Omega \setminus C$, thus there exists $\bar{x} \in (0, \bar{z})$ such that $\bar{x}$ is a cluster point of $\{x^i\}$. The fact that $0 \in \text{int } \Omega$ and $\bar{z} \in \Omega$ implies that $\bar{x} \in \text{int } \Omega$. Therefore, there exists $I > 0$ such that $x^i \in \Omega$ for all $i \geq I$, a contradiction.

In this case, Sub-procedure 4.3 ends. Conditions (13), (14), (33), and (34) imply that condition (25) holds. $x' \in \Omega$ and $z' \notin C$ imply that condition (32) holds. (31) is implied by condition (20) when $\gamma = \gamma_k$. ⊡

As it has been shown in the proof of Proposition 4.3, when $D(\gamma) \subseteq C$, Theorem 4.1 guarantees that all the cluster points of $\{(z^i, v^i)\}$ in $D(\gamma) \times C^* \subseteq C \times C^*$. Thus Sub-procedure 4.3 always ends in a finite number of steps since $\limsup v^i z^i \leq 0$.

We aim to find points satisfying conditions C_2, however, Proposition 4.4 can only guarantee that the produced points satisfy conditions (25), (31) and (32). Obviously this result is not enough, in order to check whether condition (35) holds or not, we should consider the relationship between points generated by Sub-procedure 4.3 with different sets of S_k, γ_k and Q_k. When we start Sub-procedure 4.3 with γ_k and Q_k, we assume that Q^k satisfy the following condition

$$Q_k \subseteq \bigcap_{i=1,\ldots,k-1} \{v \mid vx^i \leq 1\}. \tag{36}$$

It is trivial to show that any point of $\{v^i\}$ generated by Sub-procedure 4.3 with Q_k satisfies condition (35).

Lemma 4.4 states that condition (27) is implied by condition (35) when $\{v^k\}$ and $\{x^k\}$ are bounded. In the same way, we can prove that condition (29) is implied by

$$z^k w^i \leq 1 \text{ for all } i < k, \tag{37}$$

when $\{z^k\}$ and $\{w^k\}$ are bounded.

Let C_3 be the set of conditions (25), (27), (30), and (37), then C_3 also guarantees that a sequence of feasible values $\{\gamma_k\}$ converges to the optimal value. Let's show how to construct a sub-procedure to obtain points satisfying conditions C_3.

Subprocedure 4.4 Let S and Q be the closed convex sets satisfying conditions (10) and (9).
Let $S^1 = S$ and $Q^1 = Q$. Set $i = 1$.
b) Use the oracle Θ to select l^i satisfying (13);
If $l^i \leq \varepsilon'$, then $l' = l^i$ and stop.
c) Use the oracle Θ to select (z^i, v^i) satisfying (14) and (20), choose $(x^i, w^i) \in \partial C \times \partial C^*$ such that $z^i = \mu_1^i x^i$ and $x^i w^i = 1$, where $\mu_1^i > 0$.
Use Sub-procedure 4.1 with S^i and z^i to get a new set S^{i+1}; Set $Q^{i+1} = Q^i \cap \{v \mid vx^i \leq 1\}$.
If $\min\{dx \mid x \in \Omega, xw^i \geq 1\} \geq \gamma$ and $v^i k x^i \geq 1 + \sigma$, then goto d).
Else, set $x' = x^i$, $w' = w^i$, $z' = z^i$, $v' = v^i$, $l' = l^i$, $Q' = Q^{i+1}$ and $S' = S^{i+1} \cap \{z \mid zw^i \leq 1\}$, stop.
d) Set $i = i + 1$, return to b).

Remark 4.7 When we start Sub-procedure 4.4 with γ_k, Q_k and S_k, we aim to find points satisfying condition (37). As it has been shown in Sub-procedure 4.3, we need only make sure that S_k satisfies condition

$$S_k \subseteq \bigcap_{i=1,\ldots,k-1} \{z \mid zw^i \leq 1\}. \tag{38}$$

Remark 4.8 Let C_4 be the set of conditions (25), (30), (35) and (37), since condition (35) implies condition (27) when $\{v^k\}$ and $\{x^k\}$ are bounded, C_4 guarantees that a sequence of feasible values $\{\gamma_k\}$ converges to the optimal value. In order that condition (35) holds for points produced by Sub-procedure 4.4, we need only make sure that Q_k satisfies condition (36). Moreover, the stopping criteria $v^k x^k \geq 1 + \sigma$ is not needed.

Proposition 4.5 *Suppose the set S and Q are bounded. If $D(\gamma) \nsubseteq C$, then Sub-procedure* 4.4 *ends in a finite number of steps and it either reports l' such that $l' \leq \varepsilon'$ or reports (z', v') and (x', w') satisfying conditions* (25) *and* (30).

Proof: Assume by contradiction that Sub-procedure 4.4 never ends. Sub-procedure 4.4 generates two sequences of points $\{(z^i, v^i)\}$ and $\{(x^i, w^i)\}$. (z^i, v^i) is contained in $S^i \times Q^i$ for all i. As it has been shown in (33) and (34), $S^i \subseteq S$

and $Q^i \subseteq Q$ for all i, thus the sequence $\{(z^i, v^i)\}$ is bounded since $S \times Q$ is bounded. Theorem 4.1 guarantees that all the cluster points of $\{(z^i, v^i)\}$ are in $\Omega \times C^*$. Moreover, $dz^i \leq \gamma$ for all i, we get that all the cluster points of $\{(z^i, v^i)\}$ are in $D(\gamma) \times C^*$. Therefore, there are only finite number of i such that $v^i x^i \geq 1 + \sigma$ for any $\sigma > 0$.

If there exists a cluster point of $\{(z^i, v^i)\}$ in $C \times C^*$, then we have $\liminf v^i z^i \leq 1$ and thus there exists l^I such that $l^I < \varepsilon'$, a contradiction. In this case Sub-procedure 4.4 ends and outputs $l' = l^I$.

Let's consider the case that all the cluster points of $\{z^i\}$ are not in C, then there are only a finite number of $z^i \in C$. Let $\bar{z}$ be any cluster point of $\{z^i\}$, we have $\bar{z} \in \Omega \setminus C$, thus there exists $\bar{x} \in (0, \bar{z})$ such that $\bar{x}$ is a cluster point of $\{x^i\}$. The fact that $0 \in \text{int } \Omega$ and $\bar{z} \in \Omega$ implies that $\bar{x} \in \text{int } \Omega$. Therefore, there exists $I > 0$ such that $x^i \in \Omega$ for all $i \geq I$, which implies that

$$\varphi(w^i) \leq dx^i < dz^i \leq \gamma$$

for all $i \geq I$, a contradiction.

In this case, Sub-procedure 4.4 ends. Conditions (13), (14), (33), and (34) imply that condition (25) holds. That $z^i = \mu_1^i x^i$ for all i implies that condition (30) holds. ⊡

As it has been shown in the proof of Proposition 4.3, when $D(\gamma) \subseteq C$, Theorem 4.1 guarantees that all the cluster points of $\{(z^i, v^i)\}$ in $D(\gamma) \times C^* \subseteq C \times C^*$. Thus Sub-procedure 4.4 always ends in a finite number of steps since $\limsup v^i z^i \leq 0$.

By now we have given three working sub-procedures for conditions C_1, C_2, C_3 and C_4, respectively. We also show that B_1 is implied by C_1. When the sequence of points $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ are bounded, C_2, C_3 and C_4 imply B_1, too.

4.2 Algorithms Exploiting Conditions B_2

In order that condition (28) holds, we propose the following conditions.

$$v^k = \mu_2^k w^k, \tag{39}$$

where $\mu_2^k > 0$ for all k.

Lemma 4.5 *Suppose that $\{z^k w^k\}$ is bounded. If conditions conditions* (27) *and* (39) *hold, then condition* (28) *holds.*

Proof By conditions (27) and (39), we get that $\mu_2^k \leq 1 + \sigma_2^k$ for all k, thus $\limsup \mu_2^k \leq 1$. Therefore,

$$\limsup z^k(v^k - w^k) = \limsup z^k w^k(\mu_2^k - 1) \leq 0.$$

⊡

Let D_1 be the set of conditions (25), (27), (37) and (39), then D_1 implies B_2 when the sequence of points $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ are bounded. Note that condition (39) is "symmetric" to condition (30), we view D_1 as the "symmetric version" of C_3.

Remark 4.9 In order to give the "symmetric conditions" of C_1 and C_2, we should find "symmetric conditions" of (31) and (32). Let's replace Ω, C, d, z^k and x^k by C^*, Ω^*, e, v^k and w^k in (31) and (32), respectively. Then we have the following conditions.

$$ev^k \leq ew^{k-1}, \tag{40}$$

$$w^k \in (0, v^k) \cap C^* \cap \partial\Omega^* \tag{41}$$

However, e always equals 0 in Problem (CDC), thus (31) is trivial. Moreover, condition (32) implies that there exists at least one optimal solution w in the set $C^* \cap \partial\Omega^*$, which is contradicted by Example 4.3.

Example 4.3 Let $d = 1$ and $e = 0$, $\Omega = C^* = [-\frac{1}{2}, 4]$. Then the optimal point is $(x, w) = (-\frac{1}{4}, 4)$ where $w \notin C^* \cap \partial\Omega^*$.

Let's construct a sub-procedure to obtain points satisfying conditions D_1.

Subprocedure 4.5 Let S and Q be the closed convex sets satisfying conditions (10) and (9).
Let $S^1 = S$ and $Q^1 = Q$. Set $i = 1$.
b) Use the oracle Θ to select l^i satisfying (13);
If $l^i \leq \varepsilon'$, then $l' = l^i$ and stop.
c) Use the oracle Θ to select (z^i, v^i) satisfying (20) and (14), choose $(x^i, w^i) \in \partial C \times \partial C^*$ such that $v^i = \mu_2^i w^i$ and $x^i w^i = 1$, where $\mu_2^i > 0$.
Use Sub-procedure 4.1 with S^i and z^i to get a new set S^{i+1}; Set $Q^{i+1} = Q^i \cap \{v \mid vx^i \leq 1\}$.
If $\min\{dx \mid x \in \Omega, xw^i \geq 1\} \geq \gamma$ and $v^i x^i \geq 1 + \sigma$, then goto d).
Else, set $x' = x^i$, $w' = w^i$, $z' = z^i$, $v' = v^i$, $Q' = Q^{i+1}$ and $S' = S^{i+1} \cap \{z \mid zw^i \leq 1\}$, stop.
d) Set $i = i + 1$, return to b).

Remark 4.10 As it has been shown in Remark 4.7, when we start Sub-procedure 4.5 with γ_k, Q_k and S_k, we aim to find points satisfying condition (37). All we need to do is to make sure that S_k satisfies condition (38).

Remark 4.11 Let D_2 be the set of conditions (25), (35), (37) and (39). When $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ are bounded, D_2 implies B_2 and thus D_2 guarantees the convergence of a sequence of feasible values $\{\gamma_k\}$. As it has been shown in Remark 4.8, in order that condition (35) holds for points produced by Sub-procedure 4.5, we need only make sure that Q_k satisfies condition (36). Moreover, the stopping criteria $v^k x^k \geq 1 + \sigma$ is not needed.

Proposition 4.6 *Suppose the sets S and Q are bounded. If $D(\gamma) \nsubseteq C$, then Sub-procedure* 4.5 *ends in a finite number of steps and it either reports l' such that $l' \leq \varepsilon'$ or reports (z', v') and (x', w') satisfying conditions* (25) *and* (39).

Proof: Assume by contradiction that Sub-procedure 4.5 never ends. Sub-procedure 4.5 generates two sequences of points $\{(z^i, v^i)\}$ and $\{(x^i, w^i)\}$. (z^i, v^i) is contained in $S^i \times Q^i$ for all i. As it has been shown in (33) and (34), $S^i \subseteq S$ and $Q^i \subseteq Q$ for all i, thus the sequence $\{(z^i, v^i)\}$ is bounded since $S \times Q$ is bounded. Theorem 4.1 guarantees that all the cluster points of $\{(z^i, v^i)\}$ are in $\Omega \times C^*$. Moreover, $dz^i \leq \gamma$ for all i, we get that all the cluster points of $\{(z^i, v^i)\}$ are in $D(\gamma) \times C^*$. Therefore, there are only a finite number of i such that $v^i x^i \geq 1 + \sigma$ for any $\sigma > 0$.

Since $w^i \in \partial C^*$ for all i, we get that $\limsup \mu_2^i \leq 1$, which implies that $\liminf z^i(w^i - v^i) \geq 0$. Therefore, there exists $I > 0$ such that $z^i w^i - 1 \geq \frac{\varepsilon'}{2}$ for all $i \geq I$ since $v^i z^i - 1 \geq \varepsilon'$ for all i. Hence we have $\varphi(w^i) \leq \phi(\frac{z^i}{1+\frac{\varepsilon'}{2}})$ for all $i \geq I$.

Let $\bar{z}$ be any cluster point of $\{z^i\}$, then $\frac{\bar{z}}{1+\frac{\varepsilon'}{2}} \in \text{int }\Omega$ since $0 \in \text{int }\Omega$ and $\bar{z} \in \Omega$. Thus there exists $I^2 > 0$ such that $\frac{z^i}{1+\frac{\varepsilon'}{2}} \in \Omega$ for all $i \geq I^2$. Therefore,

$$\varphi(w^i) \leq d\frac{z^i}{1 + \frac{\varepsilon'}{2}} < dz^i \leq \gamma$$

for all $i \geq \max\{I, I^2\}$, a contradiction.

In this case, Sub-procedure 4.5 ends. Conditions (13), (14), (33), and (34) imply that condition (25) holds. That $v^i = \mu_2^i w^i$ for all i implies that condition (39) holds. ⊡

As it has been shown in the proof of Proposition 4.3, when $D(\gamma) \subseteq C$, Theorem 4.1 guarantees that all the cluster points of $\{(z^i, v^i)\}$ in $D(\gamma) \times C^* \subseteq C \times C^*$. Thus Sub-procedure 4.5 always ends in a finite number of steps since $\limsup v^i z^i \leq 0$.

5 Comparison with Existing Algorithms

In these two decades, different algorithms have been proposed for solving the canonical DC Programming problem and related problems, i.e., other forms of DC problems, concave problems and reverse convex problems. Generally, these algorithms are based either on the cutting plane method or on the branch and bound method. In this paper, we discuss only the cutting plane algorithms for solving the canonical DC problem and related works. In order to make the algorithms more clear, we use the notations of our paper rather than the original ones.

The first cutting plane algorithm for problem (CDC) was proposed by Tuy [4, 10]. Tuy introduced the canonical DC problem and show how any DC problem can be reduced to this canonical form. This algorithm always cuts off the feasible point x such that $dx > \gamma_k - \alpha$, then it either finds that γ_k is optimal or finds a better feasible point x^k and iterate. This algorithm either terminates at an α-optimal solution x^k or converges to an optimal solution. A variant [12] of this algorithm is proposed by Tuy for solving a more general reverse convex problem where dx is replaced by a convex finite function $f(x)$. A modified algorithm is given in Tuy[5]. Here γ_1 can be $+\infty$ when there is no available feasible point and the algorithm cuts the feasible points such that $dx > \gamma_k$.

Nghia and Hieu [20] proposed an algorithm for solving the reverse convex problem. This algorithm finds an interval (γ_1, γ_2) including the optimal value, then it checks whether the mean value $\gamma = \frac{\gamma_1+\gamma_2}{2}$ is optimal or not and iterate.

Another attempt to solve (CDC) problem is given by Thoai [21]. The algorithm in [21] is a modified form of the ones in [4, 10]. However, this algorithm as well as its modified form [28] are not guaranteed to converge [5]. Ben Saad and Jacobsen have also proposed cutting plane algorithms for problem (CDC) [31, 32], and their counter example was given later in [33].

Strekalovsky and Tsevendori [2] proposed an algorithm for solving general reverse convex problem. This algorithm use an optimality condition

$$\sup_{x\in\partial C}\ \sup_{y\in D(\gamma)} \nabla h(x)(y-x) \le 0\ , \tag{42}$$

which is equivalent to the classical optimality condition $D(\gamma) \subseteq C$. However, Tuan [3] shows that its implementation doesn't guarantee a correct solution and (42) is not easier to check than $D(\gamma) \subseteq C$.

On the other hand, Tuy [7, 8] have proposed a polyhedral annexation method for a special type of (CDC) problem where Ω is a polyhedron. This algorithm finds sequence of points $\{(z^k, v^k)\}$ and uses the following optimality condition

$$v^k z^k \ge \max\{\ vz \mid (z, v) \in D(\gamma_k) \times C^*\ \}.$$

In [6, 11] it is shown that this algorithm can be extended to the problem (CDC).

The points and functions produced by these algorithms satisfy C_1 and C_2 when h is the gauge function of C. According to our knowledge, there is no algorithm satisfying the set of conditions C_3, C_4, D_1 and D_2. And there also lacks work exploring the hierarchy of conditions guaranteeing the convergence of cutting plane algorithms for problem (CDC) and the Lipschitz property of value function γ_δ. Our contribution is to give a more general framework of cutting plane algorithms for problem (CDC), incorporating all the existing outer approximation algorithms and polyhedral annexation algorithms. Then we build six algorithms for conditions C_1, C_2, C_3, C_4, D_1 and D_2 and prove that these algorithm can generate an approximate optimal value in a finite number of steps. Moreover, the algorithms for C_3, C_4, D_1 and D_2 can not be reduced to any existing algorithm. We also give the conditions that guarantee the Lipschitz property of the value function, thus the error can be managed and controlled.

A Appendix

Lemma A.1 *Suppose the set Ω is bounded. If the regularity condition is satisfied, then there exists a global optimal point $x^* \in \partial(\Omega \setminus C)$.*

Proof: Since the regularity condition is satisfied, we have for the optimal value

$$\gamma^* = \min\{\ dx \mid x \in \Omega \setminus \mathrm{int}\ \ C\ \} = \inf\{\ dx \mid x \in \Omega \setminus C\ \} \quad . \tag{43}$$

Thus, there exists a sequence $\{x^k \in \Omega \setminus C\}$ such that $dx^k \to \gamma^*$. Since Ω is bounded, the sequence $\{x^k\}$ is also bounded; hence, there exists at least one cluster point x^* such that $x^* \in \mathrm{cl}\,(\Omega \setminus C)$ and $dx^* = \gamma^*$.

Since the set Ω is closed and $x^k \notin C$ for all k, we have $x^* \in \Omega$ and $x^* \notin \mathrm{int}\ C$. This implies that x^* is feasible and hence x^* is optimal. From (3) $x^* \notin C$, thus we have $x^* \in \partial(\Omega \setminus C)$. $\boxdot$

Lemma A.2 *Suppose the set C is bounded. If the regularity condition is satisfied, then there exists a global optimal point $x^* \in \partial(\Omega \setminus C)$.*

Proof: Let $\{x^k\}$ be a feasible sequence such that $dx^k \to \gamma^*$. It is sufficient to show that the sequence $\{x^k\}$ is bounded when C is bounded.

Let $H = \{x \mid dx = 0\}$; since C is bounded, then $\Omega \cap H$ is also bounded, otherwise there exists a point $\hat{x}$ in $(\Omega \cap H) \setminus C$ such that $d\hat{x} = 0$. This implies that the set $\Omega \cap \{x \mid 0 \le dx \le dx^1\}$ is bounded, too. Since $dx^k \to \gamma^*$, there exists $K > 0$ such that $dx^k \le dx^1$ for all $k \ge K$, hence $\{x^k\}$ is bounded. $\boxdot$

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