Università di Pisa
Dipartimento di Informatica

Technical Report: TR-07-07

Outer Approximation Algorithms for Canonical Reverse-Polar Problems

Giancarlo Bigi Antonio Frangioni Qinghua Zhang

March 6, 2007

Address: Largo B. Pontecorvo 3, 56127 Pisa, Italy. Tel: +39 050 2212700 Fax: +39 050 2212726

Outer Approximation Algorithms for Canonical Reverse-Polar Problems

Giancarlo Bigi [*] Antonio Frangioni [†] Qinghua Zhang [‡]

March 6, 2007

Abstract

In this paper we propose a novel generalization of the canonical DC problem and we study the convergence of outer approximation (cutting planes) algorithms for its solution which use an "approximated" oracle for checking the global optimality conditions to the problem. Although the approximated optimality conditions are similar to those of the canonical DC problem, the new class of Canonical Reverse Polar (CRP) problems is shown to significantly differ from its special case. We also show that outer approximation approaches for DC problems need be substantially modified in order to cope with (CRP); interestingly, some outer approximation approaches for the latter cannot be applied to the formers, thus the more general problem allows for novel algorithms. We develop a hierarchy of conditions that guarantee the convergence of cutting plane algorithms; relying on these conditions, we build four cutting plane algorithms for solving (CRP), which seem to be new and cannot be reduced to each other.

Keywords: *Canonical reverse-polar problems, approximate optimality conditions, cutting plane algorithms*

1 Introduction

In the last decades, optimization techniques have been widely applied in engineering, economics and other fields. A large number of nonconvex optimization

[*]Università di Pisa, Dipartimento di Informatica, Largo B. Pontecorvo 3, 56127 Pisa – Italy, e-mail: bigi@di.unipi.it

[†]Università di Pisa, Dipartimento di Informatica, Largo B. Pontecorvo 3, 56127 Pisa – Italy, e-mail: frangio@di.unipi.it

[‡]Università di Pisa, Dipartimento di Matematica, Largo B. Pontecorvo 5, 56127 Pisa – Italy, e-mail: zhang@di.unipi.it

problem can be reduced to DC optimization problems. Furthermore, all DC optimization problems can be transformed to the canonical form

$$(CDC) \qquad\qquad \min\{\ dx \mid x \in \Omega \setminus \text{int}\ C\ \}$$

where Ω and C are convex sets. In turn, (CDC) can be rewritten as

$$\min\{\ dx \mid x \in \Omega\ ,\ w \in C^*\ ,\ xw \geq 1\ \} \tag{1}$$

i.e., a convex program with a single convex inequality constraint. Under mild assumption, the necessary and sufficient optimality conditions for (CDC) are

$$\{\ z \mid z \in \Omega\ ,\ dz \leq \gamma\ \} \subseteq C$$

which can be recasted in "optimization form" as

$$v(OC_\gamma) = \max\{\ vz - 1 \mid z \in \Omega\ ,\ v \in C^*\ ,\ dz \leq \gamma\ \} \leq 0 \quad . \tag{2}$$

In the previous paper [2], we have developed a family of outer approximation approaches for (1) which are based on an *approximated oracle* for the solution of (2). The latter problem has a convex feasible set and a nonconvex objective function, so there are no known efficient approaches for solving it; by allowing an approximate solution we relax the computational requirements of standard outer approximation algorithms, hopefully paving the way for more effective solution approaches to (CDC) in practice.

In this work, we explore about extending the canonical DC optimization problem to the (apparently, slightly) more general form

$$(CRP) \qquad\qquad \min\{\ dx + ew \mid x \in \Omega\ ,\ w \in C^*\ ,\ xw \geq 1\ \} \tag{3}$$

where $d \in \mathbb{R}^n$, $e \in \mathbb{R}^n$, Ω and C are closed convex sets in $\mathbb{R}^n$ and contain 0 (therefore, $\Omega = \Omega^{**}$ and $C = C^{**}$). This problem, which we call the *Canonical Reverse Polar* problem, differs from (CDC) because of the presence of the term "ey" in the objective function. The rationale behind this definition is that, under proper assumptions, an "optimization form" of the optimality conditions of (3) requires the solution of the problem

$$(OC_\gamma) \qquad\qquad \max\{\ vz - 1 \mid z \in \Omega\ ,\ v \in C^*\ ,\ dz + ev \leq \gamma\ \} \tag{4}$$

which is a minimal modification of that in (2). In particular, the two problem share the same "difficult" part (the objective function), while the "easy" part (the feasible set) is very similar; only, in the more general case the single constraint $dz + ev \leq \gamma$ renders the feasible set nonseparable in z and v, while in the (CDC) case separability is retained. However, it is likely that this difference does not substantially impact the practical cost of the problems; thus, outer approximation approaches to (CDC) and (CRP) should have similar cost per iteration. Still, we will show that (CRP) is "substantially different" from (CDC), in the sense that several properties enjoyed be the latter are lost in the former. Since (CDC) is the special case of (CRP) with $e = 0$, it is not surprising that the outer approximation approaches for the former [2] need be substantially modified in order to cope with the latter. It is perhaps more surprising that some outer approximation approaches for (CRP) *require* $e \neq 0$, and therefore cannot be applied to (CDC); thus, broadening the class of problems also broadens the class of algorithms than can be applied to solve them. Our analysis of outer approximation algorithms for (CRP) also sheds some light on the algorithms for the original (CDC).

The paper is organized as follows. In Section 2 we describe analyze the main properties of Problem (CRP) and contrast them with those of its special case (CDC). Then, in Section 3 we extend our approximate optimality conditions for (CDC) [2] to the (CRP) case. In Section 4, we develop a hierarchy of conditions that guarantee the convergence of cutting plane algorithms; relying on these conditions, we build four cutting plane algorithms for solving (CRP), which seem to be new and cannot be reduced to each other.

2 Notations and Properties

Throughout the paper the following notation is used. The scalar product between two vectors v and w is denoted by vw. Given a function f, $\partial_\varepsilon f(x)$ is its *ε-subdifferential* at x, $\text{epi } f = \{(v, x) \mid v \geq f(x)\}$ is its *epigraph*, $\text{dom } f = \{x \mid f(x) < \infty\}$ is its *domain*, and $T(C, x) = \limsup_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}(C - x)$ is the tangent cone to a set C at a point x. Given a problem

$$(P) \qquad\qquad \inf_x[\sup]\{f(x) \mid x \in X\},$$

$v(\text{P})$ denotes the optimal value of f over X; as usual, $X = \emptyset \Rightarrow v(\text{P}) = +\infty[-\infty]$.

Problem-specific notations: x^* is an optimal solution of (CRP), $\gamma^* = dx^* = v(CRP)$ is the optimal value. h is a convex function representing C, i.e. such that $C = \{x \mid h(x) \leq 0\}$, $D(\gamma)$ is the level set $\{(x, w) \mid dx + ew \leq \gamma\}$, a value γ is feasible if there exists a feasible point (x, w) such that $dx + ew = \gamma$.

We assume that the following conditions hold in problem (CRP):

$$\inf\{dx + ey \mid x \in \Omega, y \in C^*\} < \inf\{dx + ey \mid x \in \Omega, y \in C^*, xy \geq 1\}, \tag{5}$$

$$dx + ey > 0 \text{ for all } (x, y) \text{ such that } x \in \Omega,\ y \in C^*,\ xy \geq 1, \tag{6}$$

$$\inf\{dx+ey \mid x \in \Omega, y \in C^*, xy \geq 1\} = \inf\{dx+ey \mid x \in \Omega, y \in C^*, xy > 1\}. \tag{7}$$

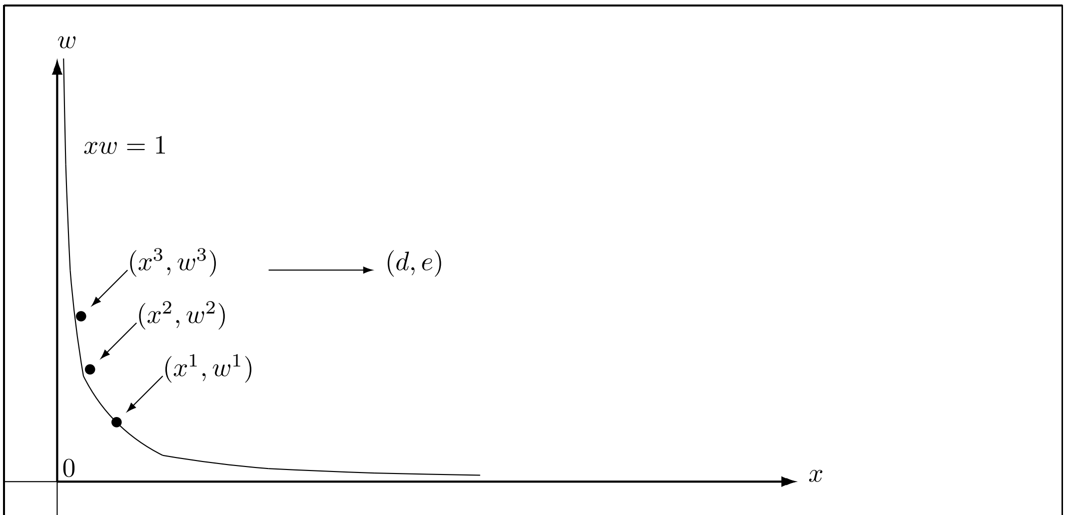
If condition (5) doesn't hold, then problem (CRP) can be reduced to a convex minimization problem. The role of regularity condition (7) will be discussed later, let's consider condition (6).

Remark 2.1 If the set of optimal solutions of problem (CRP) is non-empty, then the optimal value must be positive, otherwise condition (6) is contradicted. When Ω and C^* are bounded, the set of optimal solutions of problem (CRP)

is non-empty follows by the feasible set is compact. Since the origin point is always feasible to problem $\min\{dx + ey \mid x \in \Omega, y \in C^*\}$, then we get that $\inf\{dx + ey \mid x \in \Omega, y \in C^*\} \leq 0$ and thus condition (5) is implied by condition (6).

However, when Ω and C^* are unbounded, condition (5) may not hold even if condition (6) holds.

Example 2.1 Let $\Omega = [0, +\infty)$ and $C = (-\infty, 0]$. Thus $\Omega^* = (-\infty, 0]$ and $C^* = [0, +\infty)$. Let $d = 1$ and $e = 0$. This problem has no optimal solution and there exists a sequence of feasible solutions $\{(x^k, w^k)\}$ such that $x^k = \frac{1}{k}$, $w^k = k$ and $dx^k + ew^k \to 0$.



the thick lines denote Ω and C^*, respectively.

Suppose condition (5) holds but condition (6) doesn't hold, an adequate translation is enough for (6) to hold: take $(x^0, y^0) \in \operatorname{argmin}\{dx + ey \mid x \in \Omega, y \in C^*\}$ as the new origin and let $x := x - x^0$, $y := y - y^0$, then condition (6) holds.

Lemma 2.1 *If condition* (6) *holds, then any optimal solution* (x, y) *of problem* (CRP) *satisfies* $xy = 1$.

Proof: Assume by contradiction that there exists an optimal solution (x, y) such that $xy > 1$. For $0 \in \Omega$ and $0 \in C^*$, we have $[0, x] \subseteq \Omega$ and $[0, y] \subseteq C^*$. Take $\lambda = \sqrt{\frac{1}{xy}} \in (0, 1)$, then by condition (6) we have $d(\lambda x) + e(\lambda y) < dx + ey$ and $(\lambda x, \lambda y)$ is feasible to problem (CRP), a contradiction. $\boxdot$

Remark 2.2 Lemma 2.1 states that condition (6) guarantees that a better feasible point $(x', y') \in (\Omega \times C^*) \cap \{(x, y) \mid xy = 1\}$ can be obtained from a feasible point $(x, y) \in (\Omega \times C^*) \cap \{(x, y) \mid xy > 1\}$. Therefore, when devising the algorithms, we always ask that each feasible point (x^k, w^k) satisfies $x^k w^k = 1$.

2.1 Relationship between Problems (CRP) and (CDC)

In this subsection, we discuss the relationship between problems (CRP) and (CDC). If problem (CDC) can be viewed as a special form of problem (CRP), then algorithms solving problem (CRP) also provide solution methods for all DC optimization problems.

It is obvious that the objective functions of these two problems are the same when $e = 0$. Let's consider the relationship between their feasible sets. It follows from conditions $x \in \Omega$, $y \in C^*$ and $xy \geq 1$ that $x \in \Omega \backslash \text{int } C$ and $y \in C^* \backslash \text{int } \Omega^*$, i.e.,

$$\{x \in \Omega \mid y \in C^*, xy \geq 1\} \subseteq \Omega \backslash \text{int } C,$$

$$\{y \in C^* \mid x \in \Omega, xy \geq 1\} \subseteq C^* \backslash \text{int } \Omega^*.$$

However, we don't have

$$\{x \in \Omega \mid y \in C^*, xy \geq 1\} = \Omega \backslash \text{int } C$$

and

$$\{y \in C^* \mid x \in \Omega, xy \geq 1\} = C^* \backslash \text{int } \Omega^*$$

in some cases, i.e., when $0 \notin \text{int } C$, we have $0 \in \Omega \backslash \text{int } C$. For $\sup\{0y \mid y \in C^*\} = 0$, 0 doesn't belong to the set $\{x \in \Omega \mid y \in C^*, xy \geq 1\}$, which means that $\{x \in \Omega \mid y \in C^*, xy \geq 1\} \subsetneq \Omega \backslash \text{int } C$, problem (CRP) and problem (CDC) have the different feasible region.

In order that problem (CRP) is equivalent to problem (CDC) when they have the same objective function, i.e., $e = 0$, we assume that the following two conditions hold:

$$0 \in \text{int } C, \tag{8}$$

$$0 \in \text{int } \Omega^*. \tag{9}$$

Lemma 2.2 *[2, Lemma 2.2] If condition* (8) *holds, then problems* (CDC) *and* (CRP) *are equivalent.*

In the same way, we get that when condition (9) holds, then problems (CRP) is equivalent to the following problem

$$\min ew \quad s.t. \; w \in C^* \backslash \text{int } \Omega^*.$$

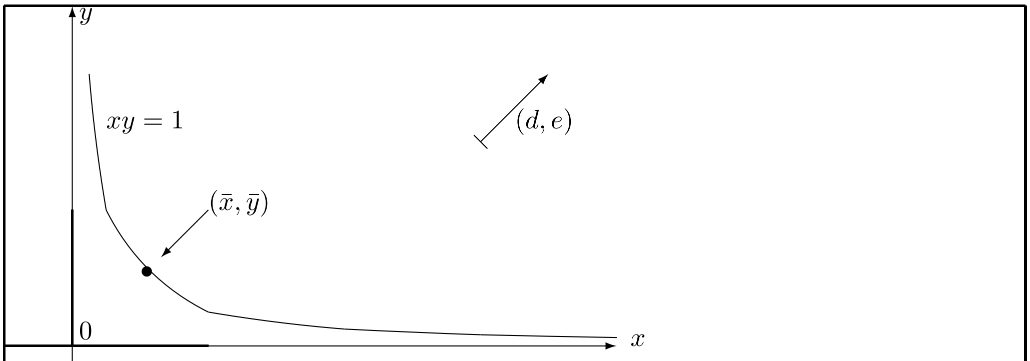
2.2 Properties of Optimal Solutions

As it has been shown in Remark 2.1, when Ω and C^* are bounded, the feasible set of problem (CRP) is compact and its set of optimal solutions is non-empty. However, as it has been pointed out in Example 2.1, when Ω or C^* is unbounded, there may exist no optimal solution of problem (CRP). In the following sections, we always assume that the set of optimal solutions of problem (CRP) is non-empty, otherwise it is meaningless to discuss the solution methods.

We already know that, when condition (6) holds, the optimal solution is always located in $\Omega \times C^* \cap \{(x, y) \mid xy = 1\}$. In this subsection, we try to explore more properties of the optimal solutions.

In problem (CDC), there exists at least one optimal solution in $\partial\Omega \cap \partial C$ when $\partial\Omega \cap \partial C \neq \emptyset$, we aim to get the same property in problem (CRP). Let (x, y) be any optimal solution, we find that x and y may be in the interior of Ω and C^*, respectively. [Example 2.2]

Example 2.2 Let $\Omega = [0, 2]$, $C = [-\frac{1}{2}, \frac{1}{2}]$, $d = 1$ and $e = 1$. Thus $\Omega^* = (-\infty, \frac{1}{2}]$ and $C^* = [-2, 2]$. The optimal solution is $(\bar{x}, \bar{y}) = (1, 1)$. Here $0 \in \text{int } C$, $0 \in \text{int } \Omega^*$, $\partial\Omega \cap \partial C = \emptyset$, $\bar{x} \in \text{int } \Omega$ and $\bar{y} \in \text{int } C^*$.



the thick lines denote Ω and C^*, respectively.

When condition (8) holds and $e = 0$, problem (CRP) is equivalent to problem (CDC), thus we can get the following property.

Lemma 2.3 *Suppose that condition* (8) *holds. Let C be the closed convex set in problem (CRP). If $e = 0$, then all the optimal points (x, y) satisfy $x \in \partial C$.*

Proof: Lemma 2.2 states that there is no point y in C^* satisfying $xy \geq 1$ for any point $x \in \text{int } C$, this implies that all the optimal points (x, y) satisfy $x \notin \text{int } C$.

Assume by contradiction that there exists an optimal solution (x, y) such that $x \notin \partial C$, then we get that $x \notin C$. Since dx is the objective function of problem (CRP) when $e = 0$, it follows from condition (6) that there exists $\bar{x} \in (0, x) \cap \partial C$ such that $d\bar{x} < dx$. Corollary **??** states that, there exists $\bar{y} \in C^*$ such that $\bar{x}\bar{y} = 1$ since $\bar{x} \in \partial C$, which implies that $(\bar{x}, \bar{y})$ is feasible. This contradicts the assumption that (x, y) is optimal. ⊡

In order to explore more properties of the optimal sets, let's give the following lemma.

Lemma 2.4 *Suppose that C, D are closed convex sets and* $\text{int } C \cap \text{int } D \neq \emptyset$. *Let $d \in \mathbb{R}^n$ and $h(x) = dx + \delta(x \mid C)$. If $x^* \in$* int *C is the minimum point of*

$h(x)$ on the set D, then we have $dx^* \leq dx$ for all $x \in D$, i.e., $-d$ is normal to the set D at x^*.

Furthermore, if $D = \{x \mid xy^* \geq 1\}$ and $x^*y^* = 1$, then $y^* = \frac{d}{dx^*}$.

Proof: Assume by contradiction that there exists $\hat{x} \in D$ such that $d\hat{x} < dx^*$, then we have $[x^*, \hat{x}] \in D$ since D is convex. It follows from $x^* \in \text{int } C$ that there exists a point $\bar{x} \in (x^*, \hat{x}) \cap C$ such that $d\bar{x} < dx^*$. Therefore, we have $h(\bar{x}) < h(x^*)$ and $\bar{x} \in D$. A contradiction.

Since $-d$ is normal to the set $\{x \mid xy^* \geq 1\}$, then we get that $d = \lambda y^*$. By $x^*y^* = 1$ we have $\lambda = dx^*$, i.e., $y^* = \frac{d}{dx^*}$. $\boxdot$

Proposition 2.1 *Suppose that condition* (8) *holds. In problem* (CRP)*, if $\partial\Omega \cap \partial C \neq \emptyset$ and $e = 0$, then there exists at least one optimal solution (x, y) satisfying $x \in \partial\Omega \cap \partial C$.*

Proof: Let (x^*, y^*) be an optimal solution and consider the case that $x^* \in \text{int } \Omega$. Let $h(x) = dx + \delta(x \mid \Omega)$ and $D = \{x \mid xy^* \geq 1\}$, we get that x^* is a minimum point of $h(x)$ on D. Lemma 2.4 guarantees that $y^* = \frac{d}{dx^*}$.

Since $y^* \in C^*$, we have $dx \leq dx^*$ for all $x \in C$. Take any point $\hat{x} \in \partial\Omega \cap \partial C$, we get that $d\hat{x} \leq dx^*$. Moreover, the fact that $\hat{x} \in \partial C$ implies that there exists $\hat{y} \in C^*$ such that $\hat{x}\hat{y} = 1$, so $(\hat{x}, \hat{y})$ is feasible and hence is optimal. $\boxdot$

In the same way, when $d = 0$ and condition (9) holds, if $\partial\Omega^* \cap \partial C^* \neq \emptyset$, then there exists at least one optimal solution (x, y) such that $y \in \partial\Omega^* \cap \partial C^*$. Let's consider a more general case where e need not be 0.

Corollary 2.1 *Suppose that condition* (8) *holds. In problem* (CRP)*, if there exists no optimal solution (x, y) satisfying $x \in \partial\Omega$, then Ω can be expressed by the following form:*

$$\Omega = \{x \mid \gamma_1 \leq dx \leq \gamma_2\}$$

where $\gamma_1 \leq 0$ and $\gamma_2 \geq 0$.

Proof: Let (x^*, y^*) be an optimal solution of problem (CRP). Lemma 2.4 states that $y^* = \frac{d}{dx^*}$. Since $y^* \in C^*$, we have $dx \leq dx^*$ for all $x \in C$. If $\{x \mid dx = dx^*\} \cap \partial\Omega \neq \emptyset$, take any point $\bar{x} \in \{x \mid dx = dx^*\} \cap \partial\Omega$, we have $\bar{x}y^* = 1$ and hence $(\bar{x}, y^*)$ is an optimal solution, a contradiction.

Then we get that $\{x \mid dx = dx^*\} \cap \partial\Omega = \emptyset$ and the hyperplane $\{x \mid dx = 0\}$ is included in the recession cone of Ω. Let γ_1 and γ_2 be the lower bound and upper bound of dx on Ω, respectively, we get that $\gamma_1 \leq 0$ and $\gamma_2 \geq 0$ since $0 \in \Omega$. $\boxdot$

2.3 Optimality Condition

How to recognize an optimal solution is important in studying optimization problems. In global optimization problems, the optimality criterion should be based on the information of the global behavior. In problem (CRP), when regularity condition (7) holds, we get the following optimality condition:

$$D(\gamma) \subseteq \{(x, y) \mid xy \leq 1\}. \tag{10}$$

Proposition 2.2 *Let γ be a feasible value of problem (CRP). Then γ is optimal if and only if condition* (10) *holds.*

Proof: When γ is optimal, assume by contradiction that there exists $(x^1, y^1) \in \Omega \times C^*$ such that $x^1y^1 > 1$ and $dx^1 + ey^1 \leq \gamma$. Take $\lambda = \frac{1}{\sqrt{x^1y^1}}$, we get that $(\lambda x^1, \lambda y^1)$ is a feasible point and $\lambda(dx^1 + ey^1) < \gamma$, a contradiction.

Vice versa, assume by contradiction that γ is not optimal. Then there exists another feasible value $\gamma^1 < \gamma$. By condition (7) we know that there exists a sequence $\{(x^k, y^k)\}$ such that $dx^k + ey^k \downarrow \gamma^1$ where $x^k \in \Omega$, $y^k \in C^*$ and $x^ky^k > 1$. Therefore, there exists a number $K > 0$ such that $x^K \in \Omega$, $y^K \in C^*$, $dx^K + ey^K < \gamma$ and $x^Ky^K > 1$, which contradicts condition (10). $\boxdot$

Now we have discussed the properties of problem (CRP). In the following sections, we try to give approximate optimality condition.

3 Approximate Optimality Conditions

Given a feasible value γ ($\gamma = d\bar{x} + e\bar{w}$ for a feasible $(\bar{x}, \bar{w})$), the optimality conditions (10) should be checked in order to recognize whether or not γ is the optimal value ($(\bar{x}, \bar{w})$ is optimal). Unfortunately, there is no known efficient way to check the inclusion between two sets. Yet, any exact algorithm for (CRP) must eventually cope with this problem.

In order to make this crucial step more readily approachable, we consider the "optimization version" (4) of the optimality. It is trivial to show that (10) holds if and only if $v(OC_\gamma) \leq 0$, thus the above problem provides a way for checking optimality of a given value γ (solution $(\bar{x}, \bar{w})$). Since the objective function of (4) is not concave, there are no known efficient approaches for this problem as well. However, checking (10) through the optimization problem (4) has the advantage of making it easy to define a proper notion of *approximate* optimality conditions.

A first way of approximating problem (4) is to replace Ω and C by two convex sets S and Q, respectively, satisfying

$$\Omega \subseteq S. \tag{11}$$

$$C^* \subseteq Q, \tag{12}$$

This is a standard step in cutting plane (outer approximation) approaches, where S and Q are chosen to be "easier" than the original sets (e.g., polyhedra with "few" vertices) and iteratively refined to become better and better approximations of Ω and C^* as needed. Hence, one considers the *relaxation* of (4)

$$(\overline{OC}_\gamma) \qquad \max\{\ vz - 1 \mid z \in S\ ,\ v \in Q\ ,\ dz + ev \leq \gamma\ \} \tag{13}$$

whose optimal value provides an upper bound on $v(OC_\gamma)$; thus,

$$v(\overline{OC}_\gamma) \leq 0 \tag{14}$$

is a convenient *sufficient* optimality condition for (CRP). If (14) does not hold, then either γ is not the optimal value, or S and Q are not "good" approximations of Ω and C^*, respectively. All the cutting plane algorithms presented in this work follow the same basic scheme: (13) is solved, and its solution is used to improve S or Q or γ, in such a way to guarantee convergence of γ to the optimal value. The focus of the research is on devising a number of different ways to achieve this result, i.e., to obtain a convergent algorithm for (CDC) out of an "oracle" for (13). However, it is likely that in any such approach the solution of (13) is going to be the computational bottleneck; it therefore makes sense to consider solving (13) *only approximately.*

Approximately solving (13) may actually mean two different things:

1. computing a "large enough" lower bound on $v(\overline{OC}_\gamma)$, i.e., finding a feasible solution $(\bar{x}, \bar{w})$ "sufficiently close" to the optimal solution;
2. computing a "small enough" upper bound $l \geq v(\overline{OC}_\gamma)$.

Algorithmically, the two notions correspond to two entirely different classes of approaches: lower bounds are produced by *heuristics* computing feasible solutions, while upper bounds are produced by solving suitable *relaxations* of $(\overline{OC}_\gamma)$, e.g. replacing the non-concave objective function vz with a suitable concave upper approximation. Exact algorithms combining the two can then be used to push the lower bound and the upper bound arbitrarily close together. However, for the sake of our approaches only one of the two bounds is needed at any given time. In fact, $v(\overline{OC}_\gamma)$ is either positive or non-negative. To establish that the first case holds amounts at finding a solution

$$(\bar{z}, \bar{v}) \in \{(z, v) \in S \times Q \mid dz + ev \leq \gamma\} \tag{15}$$

such that $\bar{z}\bar{v} - 1 > 0$, while for the second case one needs an upper bound $l \leq 0$.

This is the rationale behind our definition of an *approximate oracle* for (13). In our development, we will assume availability of a procedure Θ which, given S, Q, γ, and two positive tolerances ε and ε'

- *either* produces an upper bound

$$\varepsilon v(\overline{OC}_\gamma) \leq l \qquad \text{such that} \qquad l \leq \varepsilon' \tag{16}$$

- *or* produces a point $(\bar{z}, \bar{v})$ satisfying condition (15) such that

$$\bar{z}\bar{v} - 1 \geq \varepsilon v(\overline{OC}_\gamma). \tag{17}$$

It is clear that (17) corresponds to a pretty weak requirement about the way in which (13) is solved: only an ε-approximate solution to (13) is needed, for *fixed but arbitrary* $\varepsilon > 0$. As for (16), it allows the lower bound to be "small enough" but positive, rather than non-negative; this is taken as the stopping condition of the approach, and we will show that the positive tolerance allows for finite termination of the algorithms even when γ is optimal. The drawback is that a feasible value γ needn't be optimal when (16) holds; clearly, the "quality" of γ has to be related somewhat with ε'. The remainder of this section is devoted to the study of this relationship.

An important object in our analysis is the "approximated" problem

$$(CRP_\delta) \qquad \min\{\ dx + ew \mid x \in \Omega\ ,\ w \in C^*\ ,\ xw \geq 1 + \delta\ \} \tag{18}$$

where $\delta \geq 0$. Let $\phi(\delta) = v(CRP_\delta)$ be the value function of (18); clearly, $\phi(0) = v(CRP)$, and $\phi(\delta) \geq v(CRP)$ for each $\delta \geq 0$ as (CRP_δ) is a restriction of (CRP). We assume this problem to be regular. The value δ in (CDC_δ) is strongly related with our approximate optimality conditions, as the following result shows:

Lemma 3.1 $\gamma \leq \phi(\delta) \quad \Leftrightarrow \quad z(OC_\gamma) \leq \delta$

Proof: Using [10, Proposition 8], $\gamma \leq \phi(\delta)$ if and only if

$$D(\gamma) \quad \subseteq \quad \{\ (x, w) \mid xw \leq 1 + \delta\ \} \quad \boxdot$$

As a consequence, when (16) holds for some γ, one has

$$v(\overline{OC}_\gamma) \leq \varepsilon'/\varepsilon$$

and therefore $\gamma \leq \phi(\varepsilon'/\varepsilon)$. Thus, our stopping condition turns out to be that of the approximated problem (CRP_δ); one is then interested in the behavior of $\phi(\delta)$ as $\delta \to 0$ (remembering that $\delta = \varepsilon'/\varepsilon$). The first result is easy: ϕ is continuous at 0.

Proposition 3.1 $\phi(\delta) \to \phi(0) = v(CRP)$ *when* $\delta \to 0$.

Proof: Given any $\delta^1 \geq \delta^2 \geq 0$, we clearly have $\phi(\delta^1) \geq \phi(\delta^2)$, i.e., ϕ is nonincreasing and bounded below. Let $\bar{\gamma} = \lim_{\delta \to 0} \phi(\delta)$, we have that $\bar{\gamma} \geq \phi(0)$. Assume by contradiction that $\bar{\gamma} > \phi(0)$, by the definition of ϕ we have

$$\max\{\ zv - 1 \mid (z, v) \in D(\phi(\delta))\ \} \ \leq \ \delta$$

for all $\delta > 0$. Therefore, we get that

$$\max\{\ zv - 1 \mid (z, v) \in D(\bar{\gamma})\ \} \ \leq \ 0$$

which contradicts $\phi(0) = \sup\{\gamma \mid D(\gamma) \subseteq \{(x, w) \mid xw \leq 1\}\}$. $\boxdot$

Although $\phi(\delta)$ converges to the right value as δ, the rate of convergence may be less than linear, as the following example shows.

Example 3.1 Let

$$C = \{ \ (x_1, x_2) \mid (x_2 - 1)^2 - x_1 - 2 \leq 0 \ \}$$
$$\Omega = \{ \ (x_1, x_2) \mid x_2 \geq 0 \ , \ x_1 \geq -2 \ , \ x_1 + 2x_2 \geq 0 \ \}$$

and $d = (0, 1)$. Let (x^*, w^*) be the optimal value of problem (CDC_δ), it is easy to see that $x^* = -2$ for all $\delta \geq 0$. Moreover, $\frac{1}{1+\delta}(x^*, w^*) \in \partial C$, thus we get that $\phi(\delta) = w^* = 1 + \delta + \sqrt{2\delta(1+\delta)}$, thus $\lim_{\delta \to 0} \ (\phi(\delta) - \phi(0))/\delta$ $= \lim_{\delta \to 0} \ 1 + \sqrt{2(1+\delta)/\delta} = +\infty$.

Moreover, let $h = (x_2 - 1)^2 - x_1 - 2$ and (x^0, w^0) the optimal value of problem $(\overline{CDC_\delta})$, it is easy to see that $x^0 = -2$ for all $\delta \geq 0$. Moreover, $h(x^0, w^0) = \delta$, thus we have $\psi(\delta) = w^0 = 1 + \sqrt{\delta}$. Therefore, $\lim_{\delta \to 0} \ (\psi(\delta) - \psi(0))/\delta$ $= \lim_{\delta \to 0} \ \sqrt{\delta}/\delta = +\infty$.

Thus, one would be interested in conditions ensuring that the value function ϕ is Lipschitz at 0.

Proposition 3.2 *If there exists an optimal solution (x^0, w^0) of problem (CRP) such that $x^0 \notin \partial C$ or $w^0 \notin \partial\Omega^*$, then then the value function γ_δ satisfies the Lipschitz condition at 0.*

Proof: Without loss of generality, let $x^0 \notin \partial C$, then there exists $w^1 \in C^*$ such that $x^0 w^1 > 1$. Since (x^0, w^1) is feasible, so we have $dw^1 \geq dw^0$.

If $dw^1 = dw^0$, then (x^0, w^1) is also an optimal point, which contradicts the optimality condition since $x^0 w^1 > 1$.

Therefore, $dw^1 > dw^0$. Let $u = \frac{w^1 - w^0}{\|w^1 - w^0\|}$. Take $\delta \leq x^0 w^1 - 1$ and λ such that $x^0(w^0 + \lambda u) = 1 + \delta$, we get that $\lambda = \frac{\delta}{x^0 u}$. Therefore,

$$dx^0 + ew^1 - (dx^0 + ew^0) = e\lambda u = \delta \frac{eu}{x^0 u} \leq M\delta$$

where $M = |\frac{eu}{x^0 u}|$. ⊡

Lemma 3.2 *Suppose that Ω and C^* are both bounded. Let (x^0, w^0) be an optimal point of problem (CRP). If γ_δ does not satisfy the Lipschitz condition at 0, then we have $x^0 u^k \to 0$ and $w^0 y^k \to 0$.*

Proof: Since γ_δ does not satisfy the Lipschitz condition at 0, then by Proposition 3.2 we get that $x^0 \in \partial C$ and $w^0 \in \partial\Omega^*$. Since all the cluster points of $\{z^k\}$ and $\{v^k\}$ are in Ω and C^*, respectively, so we have $\limsup x^0 v^k \leq 1$ and $\limsup w^0 z^k \leq 1$.

Assume by contradiction that $\limsup x^0 v^k < 1$, since $\{x^k\}$ and $\{z^k\}$ have the same set of cluster points, then there exists a subsequence $\{z^{k_i}\}$ such that $\limsup z^{k_i} v^{k_i} < 1$, a contradiction. ⊡

4 Conditions and Algorithms

In this section, we present conditions and algorithms which, given an approximated oracle Θ, (approximately) solve the problem (CRP). In this presentation, we first establish a hierarchy of abstract conditions ensuring convergence, and then for each we propose implementable procedures which realize the abstract conditions.

All these algorithms follow the generic cutting plane scheme sketched in the previous paragraph. More in details, a non decreasing sequence of feasible values $\{\gamma_k\}$ is produced, and for each γ_k the oracle Θ is called, thereby producing either a value l^k such that condition (16) holds, or points z^k and v^k satisfying conditions (15) and (17). By repeatedly calling the oracle, if necessary, we can construct a procedure which either proves that γ_k satisfies condition (16), or produces a better feasible value $\gamma_{k+1} < \gamma_k$. In the latter case, the algorithms produces points x^k and w^k such that

$$x^k \in \Omega,\ w^k \in C^* \ and\ x^k w^k = 1, \tag{19}$$

and $\gamma_{k+1} = dx^k + ew^k$.

Under suitable assumptions, the bounded sequence of points $\{(x^k, w^k)\}$ converges to an optimal solution.

Algorithm 1 Prototype Algorithm

---

0. Let (x^0, w^0) be the best available feasible solution, $\gamma_1 = dx^0 + ew^0$.
he (If no feasible solution is available, then set $\gamma_1 = +\infty$). $k = 1$.
1. If optimality condition (10) holds, then γ_k is the optimal value and stop;
2. Otherwise, select a feasible point (x^k, w^k) such that $dx^k + ew^k < \gamma_k$, set $\gamma_{k+1} = dx^k + ew^k$.
3. $k = k + 1$, goto 1.

An important feature for the convergence of Algorithm 1 is that $\{\gamma_k\}$ is a decreasing sequence and bounded below:

$$0 \le \gamma_\infty < \cdots < \gamma_k < \gamma_{k-1} < \cdots < \gamma_1,$$

where $\gamma_\infty = \lim_{k\to\infty} \gamma_k$. Therefore, $\{D(\gamma_k)\}$ is a "non-increasing sequence", i.e.,

$$D(\gamma_\infty) \subseteq \cdots \subseteq D(\gamma_{k+1}) \subseteq D(\gamma_k) \subseteq \cdots \subseteq D(\gamma_1).$$

Algorithm 1 is too general to deduce any meaningful property. At least two important points are still unsaid:

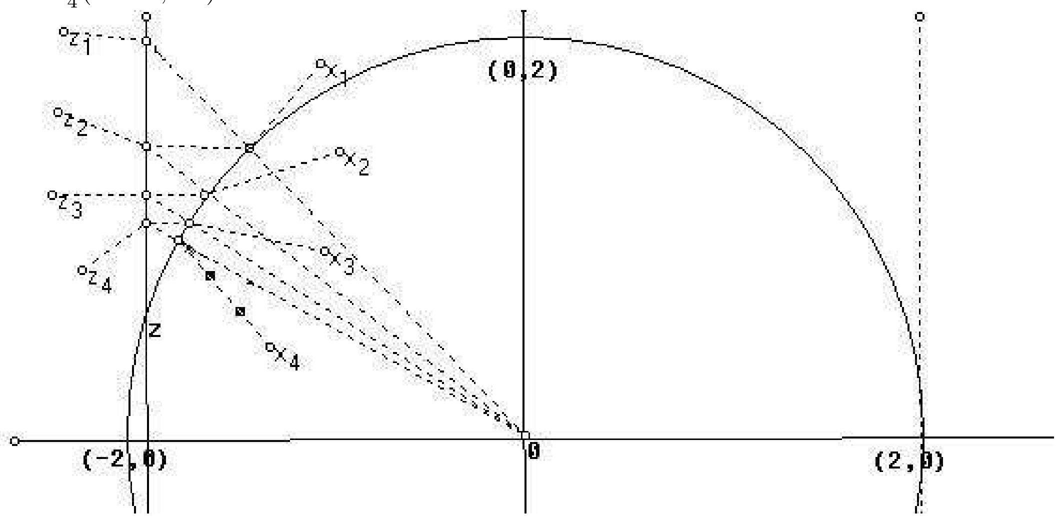
Question 4.1 How to check optimality condition (10)?

Question 4.2 How to select (x^k, w^k) once you know that condition (10) is not fulfilled?

Note that Question 4.1 and Question 4.2 are closely related to each other, i.e., if we can find a feasible point (x^k, w^k) such that $dx^k + ew^k < \gamma_k$ in Question 4.2, then Question 4.1 is answered at the same time. We start by answering Question 4.2. Assume that we have any constructive procedure that answers Question 4.1 by eventually producing a point $(z^k, v^k) \in D(\gamma_k)$ such that $v^k z^k > 1$. As it has been explained before, we can use this point to find a feasible point (x^k, w^k) such that $dx^k + ew^k < \gamma_k$. So the question is: does this method provide a convergent algorithm? the answer is no.

Example 4.1 Let $d = (0, 1)$ and $e = (0, 0)$; $\Omega = \{(x_1, x_2) \mid -1.8 \leq x_1 \leq 1.96, x_2 \geq 0\}$, $C = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 4\}$. Therefore, $C^* = \{(x_1, x_2) \mid x_1^2 + x_2^2 \leq 1/4\}$.

In this problem, we can find a sequence of points $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ converging to a non-optimal point (x, w) where $x = (-1.8, 0.87)$ and $w = \frac{1}{4}(-1.8, 0.87)$. However, the optimal point is $(\bar{x}, \bar{w})$ where $\bar{x} = (-1.96, 0.4)$ and $\bar{w} = \frac{1}{4}(-1.96, 0.4)$.



Example 4.1 shows that if there is no further restriction for the way to select $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$, Algorithm 1 may not converge to an optimal solution. We aim at providing general and weak assumptions under which convergence can be proved. We propose the following conditions:

$$\liminf v^k z^k \leq 1, \tag{20}$$

$$v^k z^k - 1 \geq \varepsilon \max\{vz - 1 \mid (z, v) \in D(\gamma_k)\}, \tag{21}$$

where $\varepsilon \in (0, 1)$.

Proposition 4.1 *If conditions* (20) *and* (21) *hold, then the sequence* $\{\gamma_k\}$ *converges to the optimal value.*

Proof: Since the sequence $\{\gamma_k\}$ is non-increasing and has a lower bound, then there exists a limit $\bar{\gamma}$ of $\{\gamma_k\}$. Let γ^* be the optimal value of problem (CRP), we get that γ^* is not greater than γ_k for all k, which implies that $\gamma^* \leq \bar{\gamma}$. Assume by contradiction that $\bar{\gamma}$ is not optimal, then we have $\bar{\gamma} > \gamma^*$.

Since $\bar{\gamma} \leq \gamma_k$ for all k, then condition (21) implies that

$$v^k z^k - 1 \geq \varepsilon \max\{vz - 1 \mid (z, v) \in D(\bar{\gamma})\}$$

for all k. It follows from condition (20) that

$$\max\{vz - 1 \mid (z, v) \in D(\bar{\gamma})\} \leq 0,$$

which means that $\bar{\gamma} \leq \gamma^*$, a contradiction. $\boxdot$

Note that condition (20) is difficult to check. In the following paragraphs, we aim to construct sequences of points $\{(z^k, v^k)\}$ satisfying condition (20). We introduce parameters λ_1^k, λ_2^k and vectors $y^k \in \mathbb{R}^n$, $u^k \in \mathbb{R}^n$ satisfying the following condition.

$$\begin{gathered}(z^k, v^k) - (\lambda_1^k y^k, \lambda_2^k u^k) = (x^k, w^k) \\ \text{where } \lambda_1^k \geq 0,\ \lambda_2^k \geq 0 \text{ and } \|y^k\| \in \{0, 1\},\ \|u^k\| \in \{0, 1\},\end{gathered} \tag{22}$$

Therefore, we have

$$v^k z^k - x^k w^k = v^k(z^k - x^k) + x^k(v^k - w^k) = \lambda_1^k v^k y^k + \lambda_2^k x^k u^k,$$

or

$$v^k z^k - x^k w^k = z^k(v^k - w^k) + w^k(z^k - x^k) = \lambda_1^k w^k y^k + \lambda_2^k z^k u^k.$$

Remark 4.1 Since condition (20) holds if and only if $\liminf(v^k z^k - x^k w^k) \leq 0$, which is either guaranteed by conditions

$$\liminf \lambda_1^k v^k y^k \leq 0 \tag{23}$$

and

$$\limsup \lambda_2^k x^k u^k \leq 0; \tag{24}$$

or by conditions

$$\limsup \lambda_1^k w^k y^k \leq 0 \tag{25}$$

and

$$\liminf \lambda_2^k z^k u^k \leq 0. \tag{26}$$

Remark 4.2 Condition (23) is equivalent to $\liminf v^k(z^k - x^k) \leq 0$ and condition (26) is equivalent to $\liminf z^k(v^k - w^k) \leq 0$. Since $x^k w^k = 1$, then conditions (24) and (25) are equivalent to $\limsup v^k x^k \leq 1$ and $\limsup z^k w^k \leq 1$, respectively.

It is easy to see that, if $\{v^k\}$ is bounded and $\liminf \lambda_1^k = 0$, then condition (23) holds; if $\{w^k\}$ is bounded and $\limsup \lambda_1^k = 0$, then condition (25) holds; if $\{x^k\}$ is bounded and $\limsup \lambda_2^k = 0$, then condition (24) holds; if $\{z^k\}$ is bounded and $\liminf \lambda_2^k = 0$, then condition (26) holds.

Lemma 4.1 *If $\lim \lambda_1^k = 0$ and $\lim \lambda_2^k = 0$, then $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ have the same set of cluster points.*

Proof: Let $(\bar{x}, \bar{w})$ be a cluster point of $\{(x^k, w^k)\}$ and $(x^{k_i}, w^{k_i}) \to (\bar{x}, \bar{w})$, then we have

$$\lim(z^{k_i}, v^{k_i}) = \lim(x^{k_i} + \lambda_1^{k_i} y^{k_i}, w^{k_i} + \lambda_2^{k_i} u^{k_i}) = (\bar{x}, \bar{w}).$$

In the same way, any cluster point of $\{(z^k, v^k)\}$ is also a cluster point of $\{(x^k, w^k)\}$. $\boxdot$

Lemma 4.2 *If $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ have the same non-empty set of cluster points, then condition* (20) *also holds.*

Proof: Let $(\bar{z}, \bar{v})$ be a cluster point of $\{(z^k, v^k)\}$, then $(\bar{z}, \bar{v})$ is also a cluster point of $\{(x^k, w^k)\}$. Since $w^k x^k = 1$ for all k, we get that $\bar{v}\bar{z} = 1$, which implies that $\lim_{k\to\infty} \inf v^k z^k \leq 1$. $\boxdot$

Lemma 4.1 and 4.2 state that, when the set of cluster points of $\{(x^k, w^k)\}$ or $\{(z^k, v^k)\}$ is non-empty, if $\lim \lambda_1^k = 0$ and $\lim \lambda_2^k = 0$, then condition (20) holds. In the following, we try to give the conditions under which $\lim \lambda_1^k = 0$ and $\lim \lambda_2^k = 0$. We assume that the following conditions hold.

$$dz^k + ev^k \leq dx^{k-1} + ew^{k-1}, \tag{27}$$

$$dx^k + ew^k \leq dz^k + ev^k. \tag{28}$$

Lemma 4.3 *If conditions* (27) *and* (28) *hold, then we have $d\lambda_1^k y^k + e\lambda_2^k u^k \to 0$.*

Proof: Conditions (27) and (28) imply $dx^k + ew^k \leq dx^{k-1} + ew^{k-1}$ for all k, i.e., $\{dx^k + ew^k\}$ is non-increasing. As it is well known, 0 is a lower bound of the sequence $\{dx^k + ew^k\}$, we get that the sequence $\{dx^k + ew^k\}$ is convergent, that is $dx^{k-1} + ew^{k-1} - (dx^k + ew^k) \to 0$, which further implies that $dz^k + ev^k - (dx^k + ew^k) \to 0$, i.e., $d\lambda_1^k y^k + e\lambda_2^k u^k \to 0$. $\boxdot$

In order that $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ have the same set of cluster points, we need to ask more properties on λ_1^k, λ_2^k, y^k and u^k. Thus the following ways are presented.

4.1 The First Way

Given any point (z^k, v^k) satisfying condition (21), we propose the following two conditions to choose (y^k, u^k) and $(\lambda_1^k, \lambda_2^k)$.

$$\frac{\lambda_1^k}{\|z^k\|} = \frac{\lambda_2^k}{\|v^k\|} = \lambda^k, \tag{29}$$

$$y^k = \frac{z^k}{\|z^k\|};\ u^k = \frac{v^k}{\|v^k\|}. \tag{30}$$

Remark 4.3 If $z^k = 0$ or $v^k = 0$, then we have $v^k z^k = 0$, which implies that condition (21) doesn't hold. Therefore, $z^k \neq 0$, $u^k \neq 0$ follows by $v^k z^k \neq 0$ and thus (y^k, u^k) is well-defined for all k.

For $x^k = z^k(1 - \frac{\lambda_1^k}{\|z^k\|})$ and $w^k = v^k(1 - \frac{\lambda_2^k}{\|v^k\|})$, we have $y^k = \frac{x^k}{\|x^k\|} = \frac{z^k}{\|z^k\|}$ and $u^k = \frac{w^k}{\|w^k\|} = \frac{v^k}{\|v^k\|}$.

Remark 4.4 Condition (28) is equivalent to

$$d\lambda_1^k y^k + e\lambda_2^k u^k \geq 0$$

for all k, thus it is implied by conditions (29) and (30) since $d\lambda_1^k y^k + e\lambda_2^k u^k = \frac{\lambda^k}{1-\lambda^k}(dx^k + ew^k)$, where $dx^k + ew^k > 0$ for all k.

Lemma 4.4 *Suppose that the set of optimal solutions of problem* (CRP) *is non-empty. If conditions* (27), (29) *and* (30) *hold, then* $\lambda^k \to 0$.

Proof: Let $\hat{\gamma}$ be the optimal value of problem (CRP), Remark 2.1 states that $\hat{\gamma} > 0$. As it has been shown in Remark 4.4, condition (28) is implied by conditions (29) and (30). Then by Lemma 4.3 we have $d\lambda_1^k y^k + e\lambda_2^k u^k \to 0$. Therefore, $\lambda^k \to 0$ since $d\lambda_1^k y^k + e\lambda_2^k u^k = \frac{\lambda^k}{1-\lambda^k}(dx^k + ew^k)$ and $dx^k + ew^k \geq \hat{\gamma} > 0$ for all k. ⊡

Lemma 4.4 gives conditions under which $\lambda^k \to 0$. Thus we have obtained sufficient conditions guaranteeing the convergence of $\{(x^k, w^k)\}$.

Theorem 4.1 *Suppose that the set of optimal solutions of problem* (CRP) *is non-empty. If conditions* (21), (27), (29) *and* (30) *hold, then any cluster point of* $\{(x^k, w^k)\}$ *is globally optimal in problem* (CRP).

Proof: Let $(\bar{x}, \bar{w})$ be any cluster point of $\{(x^k, w^k)\}$, by taking subsequences if necessary, let $x^k \to \bar{x}$ and $w^k \to \bar{w}$.

Since conditions (27), (29) and (30) hold, then by Lemma 4.4 we have $\lambda^k \to 0$. This implies that $z^k \to \bar{x}$ and $v^k \to \bar{w}$ since $(z^k, v^k)(1 - \lambda^k) = (x^k, w^k)$ for all k. Therefore, we have $\liminf v^k z^k \leq \bar{x}\bar{w} = 1$ and thus by Proposition 4.1 we get that γ_k converges to the optimal value, i.e., $(\bar{x}, \bar{w})$ is an optimal point. ⊡

Therefore, conditions (21), (27), (29) and (30) ensure that a bounded sequence $\{(x^k, w^k)\}$ is convergent.

Although Algorithm 2 is convergent, it also presents other questions.

Algorithm 2 Algorithm Using the First Way

0. Let (x^0, w^0) be the best available feasible solution, $\gamma_1 = dx^0 + ew^0$.
(If no feasible solution is available, then set $\gamma_1 = +\infty$). $k = 1$.
1. If optimality condition (10) holds, then γ_k is the optimal value and stop;
2. Otherwise, select a point (z^k, v^k) and a feasible point (x^k, w^k) satisfying conditions (21), (27), (29) and (30). Set $\gamma_{k+1} = dx^k + ew^k$.
3. $k = k + 1$, goto 1.

Question 4.3 How to construct the sequence of points $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ satisfying conditions (21), (27), (29) and (30)?

Sub-procedure 4.2 provides one possible answer on Question 4.3. Give any feasible and non-optimal value γ, this sub-procedure ends in a finite number of steps and outputs the desired points (z', v') and (x', w'). The proof will be given later. In fact, this sub-procedure not only produces the desired points but also constructs sequences of convex sets. As it will be shown, these convex sets play an important role in finding these points.

Before giving Sub-procedure 4.2, we give Sub-procedure 4.1 with closed convex sets Ω and S.

Theorem 4.2 *[10, Proposition 17]*

Let Ω be a convex set in $\mathbb{R}^n$ such that $\Omega = \{x : \tilde{g}(x) \leq 0\}$, $\tilde{g}$ is a convex function. Assume that $0 \in int\,\Omega$ and let S_k, $k = 1, 2, ...$ be a sequence of polyhedrons satisfying

1) $z^k \in S_k \backslash \Omega$;

2) $S_{k+1} = S_k \cap \{x : \langle p^k, x - y^k\rangle + \alpha_k \leq 0\}$, where $y^k \in [0, z^k) \backslash int\,\Omega$, $0 \leq \alpha_k \leq \tilde{g}(y^k)$, $p^k \in \partial\tilde{g}(y^k)$ and $\alpha_k - \tilde{g}(y^k) \to 0$.

Then any cluster point $\bar{z}$ of the sequence $\{z^k\}$ belongs to $\partial\Omega$.

Theorem 4.2 gives an outer approximation method and proves that all the cluster points of of this method belong to Ω. Replying on the method provided by this theorem, we get Sub-procedure 4.1.

Subprocedure 4.1

1. Ω is a closed convex set such that $\Omega = \{g(x) \leq 0\}$ and $0 \in int\,\Omega$, $S \supseteq \Omega$ is a closed convex set.

2. Given a point $x \in S \backslash \Omega$, select a point $y \in (0, x) \cap \partial\Omega$ and a sub-gradient $p \in \partial g(y)$. Set $S = S \cap \{x : \langle p, x - y\rangle + g(y) \leq 0\}$

Remark 4.5 Sub-procedure 4.1 also works with fixed $w \in$ int Ω, it is not necessary to assume that $0 \in int\,\Omega$.

It is obvious that Sub-procedure 4.1 is a simplified form of the method provided by Theorem 4.2. Although the proof of Theorem 4.2 has shown that Sub-procedure 4.1 cut off x from Ω without cutting any point from Ω, we can still use a more simple way to explain it.

Remark 4.6 Let $H = \{z : \langle p, z - y\rangle + g(y) = 0\}$ and $H^+ = \{z : \langle p, z - y\rangle + g(y) \geq 0\}$. Note that for any point $z \in \Omega$, we have

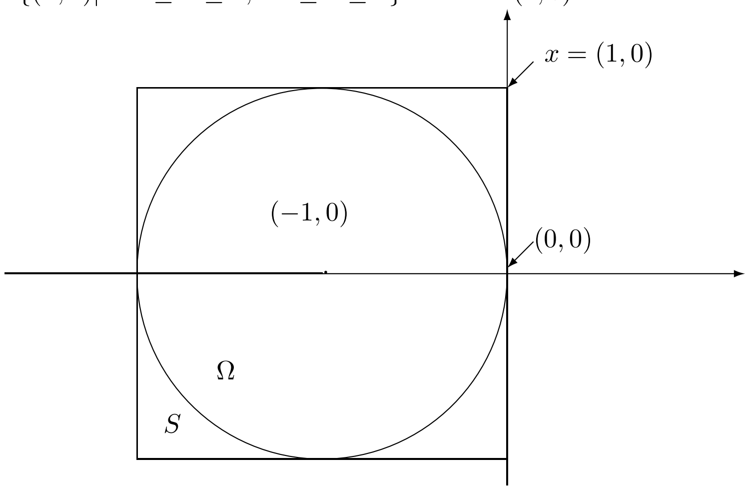
$$\langle p, z - y\rangle + g(y) \leq g(z) \leq 0 \tag{31}$$

Therefore, no point in Ω is cut away form S. In fact, it follows from the definition of the hyperplane H that it is a tangent hyperplane of Ω at y and $0 \in int\ H^-$ since $0 \in int\ \Omega$.

Consider the point $x \in S \backslash \Omega$ in Sub-procedure 4.1, it is easy to see that x and Ω are separated by H strictly: Assume by contradiction that $x \in H^-$, then $z \in int\ H^-$ for all $z \in (0, x)$ [28, Theorem 6.1], this contradicts the fact that $y \in (0, x)$ and $y \in H$.

Remark 4.6 shows that Sub-procedure 4.1 constructs a hyperplane separating Ω and x strictly. x is removed from S and Ω is still included in S. Note that the condition $0 \in int\ \Omega$ is required, otherwise Sub-procedure 4.1 may not be able to construct a hyperplane separating Ω and x strictly [Example 4.2].

Example 4.2 Let $\Omega \subseteq \mathbb{R}^2$ such that $\Omega = \{(u, v)|(u + 1)^2 + v^2 \leq 1\}$, $S = \{(u, v)| -1 \leq u \leq 1, -1 \leq v \leq 1\}$ and $x = (1, 0)$.



Apply Sub-procedure 4.1 to Ω and x, we get that $y = (0, 0)$ and the separating hyperplane is $\{(u, v)|u = 0\}$. It's easy to see that x still belongs to $S = S \cap \{(u, v)|u \leq 0\}$, i,e, Sub-procedure 4.1 can not remove x from S.$\boxdot$

In the following, we give Sub-procedure 4.2 to obtain $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ satisfying conditions (21), (27), (29) and (30). In this sub-procedure, we assume that $0 \in \text{int }\Omega$ and $0 \in \text{int } C^*$ since Sub-procedure 4.1 is used. The computational procedure is the following: Given a feasible value γ and two closed convex sets S and Q satisfying conditions (11) and (12), respectively. Denote $S^1 = S$

and $Q^1 = Q$, $i = 1$ and select a value l^i satisfying condition (16), or select v^i, z^i satisfying conditions (15) and (17). Moreover, take (y^i, u^i) satisfying condition (30). Take a small enough positive value ε', if $l^i \leq \varepsilon'$, then sub-procedure ends and outputs $l' = l^i$. Use Sub-procedure 4.1 with S^i and z^i to get S^{i+1} if $z^i \notin \Omega$; and with Q^i and v^i to get Q^{i+1} if $v^i \notin C^*$. Choose appropriate λ_1^i, λ_2^i such that (x^i, w^i) satisfying

$$x^i = z^i - \lambda_1^i y^i, w^i = v^i - \lambda_2^i u^i, x^i w^i = 1, \tag{32}$$

where $\lambda_1^i/\|z^i\| = \lambda_2^i/\|v^i\| = \lambda^i$. If $x^i \in \Omega$ and $w^i \in C^*$, then sub-procedure ends; otherwise, set $i = i + 1$ and iterate.

Subprocedure 4.2 a) Let S and Q be the closed convex sets satisfying conditions (11) and (12).
Let $S^1 = S$ and $Q^1 = Q$. Set $i = 1$.
b) Use the oracle Θ to select l^i satisfying (16) or finds (z^i, v^i) satisfying (15) and (17). Take (y^i, u^i) as defined in condition (30).
If Θ finds $l^i \leq \varepsilon'$, then $l' = l^i$ and stop.
If $z^i \notin \Omega$, then use Sub-procedure 4.1 with S^i and z^i to get a convex set S^{i+1}; else, $S^{i+1} = S^i$.
If $v^i \notin C^*$, then use Sub-procedure 4.1 with Q^i and v^i to get a convex set Q^{i+1}; else, $Q^{i+1} = Q^i$.
Choose $\lambda^i = 1 - \frac{1}{\sqrt{v^i z^i}}$ such that $x^i w^i = 1$.
If $x^i \in \Omega$ and $w^i \in C^*$, then $x' = x^i$, $w' = w^i$, $z' = z^i$, $v' = v^i$, $l' = l^i$, $Q' = Q^{i+1}$ and $S' = S^{i+1}$, stop;
c) Otherwise, set $i = i + 1$, return to b);

Sub-procedure 4.2 generates sequences of points $\{(x^i, w^i)\}$, $\{(z^i, v^i)\}$ and sets $\{S^i\}$, $\{Q^i\}$. It is necessary and useful to explore their properties and relationships.

Remark 4.7 In Sub-procedure 4.2, Ω and C^* are included in S and Q, respectively. As it has been shown, Sub-procedure 4.1 constructs a hyperplane separating strictly z^i, v^i and Ω, C^*, respectively. By using this hyperplane, Sub-procedure 4.1 cuts off z^i, v^i from S^i, Q^i, respectively. Thus we get two decreasing sequences:

$$\Omega \subseteq \cdots \subseteq S^i \subseteq S^{i-1} \subseteq \cdots \subseteq S^1, \tag{33}$$

$$C^* \subseteq \cdots \subseteq Q^i \subseteq Q^{i-1} \subseteq \cdots \subseteq Q^1, \tag{34}$$

There two stopping criteria in Sub-procedure 4.2. The first is $l^i \leq \varepsilon'$ and the second is $(x^i, w^i) \in \Omega \times C^*$, let's discuss the difference of these two conditions.

Remark 4.8 If Sub-procedure 4.2 stops when $l^i \leq \varepsilon'$, then we have $\max\{vz - 1 \mid (z, v) \in D(\gamma)\} \leq \varepsilon'$, which implies that γ is an approximated optimal value. Thus we need not go to the next iteration.

If Sub-procedure 4.2 stops when $(x^i, w^i) \in \Omega \times C^*$, then we get a new feasible solution (x^i, w^i). Let $\gamma = dx^i + ew^i$ and perform Sub-procedure 4.2 again.

Proposition 4.2 *Suppose that Q and S are compact. If the feasible value γ is not optimal, then Sub-procedure* 4.2 *ends in a finite number of steps and it either reports $l' \leq \varepsilon'$ or reports (x', w'), (z', v') satisfying conditions* (21), (27), (29) *and* (30).

Proof: Assume by contradiction that there exists an infinite number of steps, which implies that $(x^i, w^i) \notin \Omega \times C^*$ for all i. Since $\{(z^i, v^i)\}$ is contained in $S \times Q$ and S, Q are bounded, we get that the sequence $\{(z^i, v^i)\}$ is also bounded and there exists a cluster point $(\bar{z}, \bar{v})$.

Theorem 4.2 guarantees that $(\bar{z}, \bar{v}) \in \Omega \times C^*$. Furthermore, $\bar{z}\bar{v} \geq 1 + \varepsilon'$ follows by $v^i z^i - 1 \geq l^i > \varepsilon'$. Since $x^i = (1 - \lambda^i) z^i$, $w^i = (1 - \lambda^i) v^i$ and $\lambda^i = 1 - \frac{1}{\sqrt{z^i v^i}} \in (0, 1)$ for all i, then there exists $\bar{\lambda} = 1 - \frac{1}{\sqrt{\bar{z}\bar{v}}}$ such that $(1 - \bar{\lambda})(\bar{z}, \bar{v})$ is a cluster point of $\{(x^i, w^i)\}$. It follows from $\bar{v}\bar{z} > 1$ that $\bar{\lambda} > 0$. The fact $0 \in \text{int } \Omega$ and $0 \in \text{int } C^*$ implies that $((1 - \bar{\lambda})\bar{z}, (1 - \bar{\lambda})\bar{v}) \in (\text{int } \Omega) \times (\text{int } C^*)$. Therefore, there exists $I > 0$ such that $(x^I, w^I) \in \Omega \times C^*$, a contradiction. This establishes the first assertion of proposition.

By taking $\gamma = \gamma_k$ and condition (**??**), we can get that condition (27) holds. Moreover, by selecting λ^i satisfying (29), (z^i, v^i) satisfying (15) and (y^i, u^i) satisfying (30), we get that conditions (21), (29) and (30) hold. $\boxdot$

Proposition 4.2 states that, when the feasible value γ is not optimal, Sub-procedure 4.2 ends in a finite number of steps. Let's show what happens when the feasible value γ is optimal.

Proposition 4.3 *Suppose that Q and S are compact. If the feasible value γ is optimal, then Sub-procedure* 4.2 *ends in a finite number of steps and reports $l' \leq \varepsilon'$.*

Proof: Assume by contradiction that there exists an infinite number of (z^i, v^i), then there exists a cluster point $(\bar{z}, \bar{v}) \in D(\gamma)$, which implies that $\max\{xy \mid (x, y) \in D(\gamma)\} \leq 1$ since γ is optimal. Therefore, there exists $K > 0$ such that $v^K z^K \leq 1 + \varepsilon'$ and so $l^k \leq v^k z^k - 1 \leq \varepsilon'$. $\boxdot$

4.2 The Second Way

In this subsection, we propose the following conditions. Let τ_1 and τ_2 be two positive values, we choose λ_1^k and λ_2^k satisfying the following conditions.

$$\text{either } \lambda_1^k = 0 \text{ or } dy^k \geq \tau_1, \tag{35}$$

$$\text{either } \lambda_2^k = 0 \text{ or } eu^k \geq \tau_2. \tag{36}$$

Lemma 4.5 *If conditions* (27), (35) *and* (36) *hold, then we have $\lambda_1^k \to 0$ and $\lambda_2^k \to 0$.*

Proof: Conditions (35) and (36) imply that $\lambda_1^k dy^k \geq 0$ and $\lambda_2^k eu^k \geq 0$, this means that $dz^k \geq dx^k$, $ev^k \geq ew^k$ for all k and so condition (28) holds. Lemma

4.3 states that when conditions (27) and (28) hold, we have $d\lambda_1^k y^k + e\lambda_2^k u^k \to 0$. Therefore, we have $d\lambda_1^k y^k \to 0$ and $e\lambda_2^k u^k \to 0$ since $\lambda_1^k dy^k \geq 0$ and $\lambda_2^k eu^k \geq 0$.

Assume by contradiction that $\lambda_1^k \nrightarrow 0$, then there exist $\theta > 0$, $I > 0$ and a subsequence $\{\lambda_1^{k_i}\}$ such that $\lambda_1^{k_i} \geq \theta$ for all $i \geq I$, which implies that $dy^{k_i} \geq \tau_1$ for all $i \geq I$. Therefore, $\lambda_1^{k_i} dy^{k_i} \geq \theta\tau_1$ for all $i \geq I$, a contradiction. In the same way, we can prove that $\lambda_2^k \to 0$. $\boxdot$

Lemma 4.5 implies that, conditions (27), (35) and (36) hold, then $\{(x^k, w^k)\}$ and $\{(z^k, v^k)\}$ have the same set of cluster points.

Theorem 4.3 *If conditions* (21)*,* (27)*,* (35) *and* (36) *hold, then any cluster point of* $\{(x^k, w^k)\}$ *is globally optimal in problem* (CRP)*.*

Proof: Since conditions (27), (35) and (36) hold, then by Lemma 4.5 we get that $\{(x^k, w^k)\}$ and $\{(z^k, v^k)\}$ have the same set of cluster points, and thus condition (20) holds. Proposition 4.1 states that, when conditions (20) and (21) hold, $\{\gamma_k\}$ converges to the optimal value. $\boxdot$

Therefore, conditions (21), (27), (35) and (36) ensure that a bounded sequence $\{(x^k, w^k)\}$ is convergent. The following question is presented.

Question 4.4 How to construct the sequence of points $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ satisfying conditions (21), (27), (35) and (36)?

Remark 4.9 When the optimal solution is non-empty, Remark 2.1 states that the optimal value γ^* is positive.

Given any bounded sequence $\{(x^k, w^k)\}$, if we choose (y^k, u^k) satisfying condition (30), let $M^k = \max\{\|x^k\|, \|w^k\|\}$ and $M = \sup_k\{M^k\}$. Since $\{(x^k, w^k)\}$ is bounded, we get that M is a finite number. Take positive values τ_1 and τ_2 such that $(\tau_1 + \tau_2)M \leq \gamma^*$, we have

$$dx^k + ew^k \geq \gamma^* \geq \tau_1\|x^k\| + \tau_2\|w^k\|,$$

which implies that λ_1^k and λ_2^k can not be both 0.

Sub-procedure 4.3 provides one possible answer on Question 4.4. Give any feasible and non-optimal value γ, this sub-procedure ends in a finite number of steps and outputs the desired points (z', v') and (x', w'). The proof will be given later. In fact, this sub-procedure not only produces the desired points but also constructs sequences of convex sets. As it will be shown, these convex sets play an important role in finding these points.

In the following, we give Sub-procedure 4.3 to obtain $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ satisfying conditions (21), (27), (30), (35) and (36). In this sub-procedure, we assume that $0 \in \text{int } \Omega$ and $0 \in \text{int } C^*$ since Sub-procedure 4.1 is used. The computational procedure is the following:

Subprocedure 4.3 a) Let S and Q be the closed convex sets satisfying conditions (11) and (12).
Let $S^1 = S$ and $Q^1 = Q$. Set $i = 1$.
b) Use the oracle Θ to select l^i satisfying (16) or finds (z^i, v^i) satisfying (15) and (17).
If Θ finds $l^i \le \varepsilon'$, then $l' = l^i$ and stop.
c) Set (y^i, u^i) according to (30).
If $z^i \notin \Omega$, then use Sub-procedure 4.1 with S^i and z^i to get a convex set S^{i+1}; else, $S^{i+1} = S^i$.
If $v^i \notin C^*$, then use Sub-procedure 4.1 with Q^i and v^i to get a convex set Q^{i+1}; else, $Q^{i+1} = Q^i$.
If $\max\{dy^i, eu^i\} \le 0$, goto e).
Else, set $\tau^i = \frac{1}{4}\max\{dy^i, eu^i\}$.
If $dy^i < eu^i$, then set $\bar{v}^i = v^i$ and goto $c1$). Otherwise, set $\bar{z}^i = z^i$ and goto $c2$).
c1) If $dy^i \ge \tau^i$, then set $\bar{z}^i = z^i$ and $\lambda_1^i = (1 - \frac{1}{\sqrt{\bar{v}^i \bar{z}^i}})\|\bar{z}^i\|$, $\lambda_2^i = (1 - \frac{1}{\sqrt{\bar{v}^i \bar{z}^i}})\|\bar{v}^i\|$.

Else, choose $\bar{z}^i$ satisfying condition

$$\bar{z}^i = \begin{cases} z^i & \text{if } z^i \in \Omega \\ (0, z^i) \cap \partial\Omega & \text{else,} \end{cases} \tag{37}$$

Set $\lambda_1^i = 0$ and $\lambda_2^i = (1 - \frac{1}{\bar{z}^i \bar{v}^i})\|\bar{v}^i\|$.
goto d).

c2) If $eu^i \ge \tau^i$, then set $\bar{v}^i = v^i$ and $\lambda_1^i = (1 - \frac{1}{\sqrt{\bar{v}^i \bar{z}^i}})\|\bar{z}^i\|$, $\lambda_2^i = (1 - \frac{1}{\sqrt{\bar{v}^i \bar{z}^i}})\|\bar{v}^i\|$.

Else, choose $\bar{v}^i$ satisfying condition

$$\bar{v}^i = \begin{cases} v^i & \text{if } v^i \in C^* \\ (0, v^i) \cap \partial C^* & \text{else,} \end{cases} \tag{38}$$

Set $\lambda_2^i = 0$ and $\lambda_1^i = (1 - \frac{1}{\bar{v}^i \bar{z}^i})\|\bar{z}^i\|$.
goto d).

d) If $\bar{v}^i \bar{z}^i - 1 < \frac{\varepsilon l^i}{\bar{\varepsilon}}$, goto e).
Else if $x^i \in \Omega$ and $w^i \in C^*$, then $x' = x^i$, $w' = w^i$, $\tau' = \tau^i$, $z' = \bar{z}^i$, $v' = \bar{v}^i$, $Q' = Q^{i+1}$ and $S' = S^{i+1}$, stop;
Else, goto e).
e) Set $i = i + 1$, return to b);

Sub-procedure 4.3 generates sequences of points $\{(x^i, w^i)\}$, $\{(z^i, v^i)\}$ and sets $\{S^i\}$, $\{Q^i\}$. It is necessary and useful to explore their properties and relations.

Remark 4.10 Let (z^i, v^i) be generated by Sub-procedure 4.3, assume that $dy^i < eu^i$. When $dy^i \ge \tau^i$, we have $x^i = \frac{\bar{z}^i}{\sqrt{\bar{v}^i \bar{z}^i}}$ and $w^i = \frac{\bar{v}^i}{\sqrt{\bar{v}^i \bar{z}^i}}$. When $dy^i < \tau^i$, we get that $x^i = \bar{z}^i$ and $w^i = \frac{\bar{v}^i}{\bar{v}^i \bar{z}^i}$. It follows from the fact $\bar{v}^i \bar{z}^i \ge 1$ that $x^i \in (0, \bar{z}^i]$ and $w^i \in (0, \bar{v}^i]$ for all i. For $\bar{z}^i \in (0, z^i]$ and $\bar{v}^i \in (0, v^i]$, we get that $x^i \in (0, z^i]$ and $w^i \in (0, v^i]$ for all i.

Lemma 4.6 *If $z^i \to \bar{z}$ and $x^i \in (0, z^i]$ for all i, then all cluster points of $\{x^i\}$ are in $[0, \bar{z}]$.*

Proof: For $z^i \to \bar{z}$ and $x^i \in (0, z^i]$, we have $\{z^i\}$ is bounded and thus $\{x^i\}$ is also bounded. Take λ^i such that $x^i = \lambda^i z^i$ for all i, we have $\lambda^i \in (0, 1]$ for all i. Thus all the cluster points of $\{\lambda^i\}$ are in $[0, 1]$, which implies that all the cluster points of $\{\lambda^i z^i\}$ are in $[0, \bar{z}]$. $\boxdot$

Lemma 4.7 *Suppose that D is a closed convex set and $0 \in \text{int } D$. If $z^i \to \bar{z} \in D$ and*

$$x^i = \begin{cases} z^i & \text{if } z^i \in D \\ (0, z^i) \cap \partial D & \text{else,} \end{cases}$$

then $x^i \to \bar{z}$.

Proof: Since $x^i \in (0, z^i]$ for all i and $z^i \to \bar{z}$, then by Lemma 4.6 we get that all the cluster points of x^i are in $[0, \bar{z}]$.

Assume by contradiction that there exists a cluster point $\bar{x}$ of $\{x^i\}$ such that $\bar{x} \neq \bar{z}$. For $0 \in \text{int } D$ and $\bar{z} \in D$ we have $\bar{x} \in \text{int } D$. Let $x^{i_k} \to \bar{x}$, there exists $K > 0$ such that $x^{i_k} \in \text{int } D$ for all $k \geq K$, which implies that $x^{i_k} = z^{i_k}$ for all $k \geq K$. Therefore, $z^{i_k} \to \bar{x} \neq \bar{z}$, a contradiction. $\boxdot$

Proposition 4.4 *Suppose that Q and S are compact. If the feasible value γ is not optimal, then Sub-procedure* 4.5 *ends in a finite number of steps and it either reports $l' \leq \varepsilon'$ or reports (x', w'), (z', v') satisfying conditions* (21)*,* (30)*,* (35) *and* (36)*.*

Proof: Assume by contradiction that there exists an infinite number of (z^i, v^i), which implies that $(x^i, w^i) \notin \Omega \times C^*$ for all i. Since $\{(z^i, v^i)\}$ is contained in $S \times Q$ and S, Q are bounded, we get that the sequence $\{(z^i, v^i)\}$ is also bounded. Let $(\bar{z}, \bar{v})$ be a cluster point of $\{(z^i, v^i)\}$.

Since Sub-procedure 4.3 never stops, we get that $v^i z^i \geq 1 + \varepsilon'$ for all i and $\bar{z}\bar{v} \geq 1 + \varepsilon'$. Then $(\bar{z}, \bar{v})$ is not a cluster point of $\{(x^i, w^i)\}$ follows by $x^i w^i = 1$ for all i. Theorem 4.2 guarantees that $(\bar{z}, \bar{v}) \in \Omega \times C^*$, which means that $(\bar{z}, \bar{v})$ is a feasible point and thus $d\bar{z} + e\bar{v} > 0$. As it has been shown in Remark 4.10, $x^i \in (0, z^i]$ and $w^i \in (0, v^i]$ for all i, thus Lemma 4.6 guarantees that there exists a cluster point $(\bar{x}, \bar{w})$ of $\{(x^i, w^i)\}$ such that $\bar{x} \in [0, \bar{z}]$ and $\bar{w} \in [0, \bar{v}]$.

By taking subsequences if necessary, let $x^i \to \bar{x}$ and $w^i \to \bar{w}$, $z^i \to \bar{z}$ and $v^i \to \bar{v}$. In Sub-procedure 4.3, $\bar{z}^i$ either satisfies condition (37) or equals z^i for all i, and $\bar{v}^i$ either satisfies condition (38) or equals v^i for all i. When there exists a subsequence $\{\bar{z}^{i_k}\}$ satisfying condition (37), Lemma 4.7 guarantees that $\bar{z}^{i_k} \to \bar{z}$ and thus we have $\bar{z}^i \to \bar{z}$. In the same way, we get that $\bar{v}^i \to \bar{v}$, which implies that $\lim v^i z^i = \lim \bar{v}^i \bar{z}^i$.

Since $v^i z^i \geq l^i$ for all i, there are only a finite number of $\{(\bar{z}^i, \bar{v}^i)\}$ such that $\bar{v}^i \bar{z}^i < \frac{\varepsilon l^i}{\bar{\varepsilon}}$. By $d\bar{z} + e\bar{v} > 0$ we get that there are only a finite number of (z^i, v^i) such that $dz^i \leq 0$ and $ev^i \leq 0$, which implies that there exists $I > 0$ such that $\max\{dy^i, eu^i\} = \max\{\frac{dz^i}{\|z^i\|}, \frac{ev^i}{\|v^i\|}\} > 0$ for all $i \geq I$.

Without loss of generality, let $dy^i < eu^i$ for all i. If there exists a subsequence $\{y^{i_k}\}$ such that $dy^{i_k} < \tau^{i_k}$ for all k, then by definition of τ^i we have $eu^{i_k} \geq \tau^{i_k}$. Furthermore, since $\bar{z}^{i_k} = x^{i_k} \in \Omega$ for all k then we have $\bar{z} = \bar{x}$. In this case, we have $\bar{v} \neq \bar{w}$, which implies that $\bar{w} \in \text{int } C^*$ since $\bar{w} \in (0, \bar{v})$, $0 \in \text{int } C^*$ and $\bar{v} \in C^*$. Therefore, there exists $I^1 > 0$ such that $w^i \in C^*$ for all $i \geq I^1$ and thus exists a point $(x^{i_K}, w^{i_K}) \in \Omega \times C^*$, a contradiction.

Let's consider the case that there are only finite number of y^i satisfying $dy^i < \tau^i$, which implies that there exists $I^2 > 0$ such that $\min\{dy^i, eu^i\} \geq \tau^i$ and thus $\lambda_1^i = (1 - \frac{1}{\sqrt{\bar{v}^i \bar{z}^i}})\|\bar{z}^i\|$, $\lambda_2^i = (1 - \frac{1}{\sqrt{\bar{v}^i \bar{z}^i}})\|\bar{v}^i\|$ for all $i \geq I^2$. Since $\bar{v}^i \bar{z}^i \geq 1 + \frac{\varepsilon\varepsilon'}{\bar{\varepsilon}}$ for all i, we get that $\lambda_1^i \nrightarrow 0$ and $\lambda_2^i \nrightarrow 0$. Thus we have $\bar{x} \neq \bar{z}$ and $\bar{w} \neq \bar{v}$. Therefore, $\bar{x} \in \text{int } \Omega$ and $\bar{w} \in \text{int } C^*$, there exists a point $(x^{I_3}, w^{I_3}) \in \Omega \times C^*$, a contradiction.

By definition we get that y^i and u^i satisfy condition (30). When $dy^i < \tau^i$ we get that $eu^i \geq \tau^i$ and thus $\lambda_1^i = 0$, $\lambda_2^i \neq 0$; otherwise, we have $\lambda_2^i = 0$. Therefore, λ_1^i and λ_2^i satisfy conditions (35) and (36) for all i. Moreover, $v'z' \geq \frac{\varepsilon}{\bar{\varepsilon}} l'$ implies that condition (21) holds. ⊡

Proposition 4.4 states that, when the feasible value γ is not optimal, Sub-procedure 4.3 ends in a finite number of steps. In fact, when the feasible value γ is optimal, we can use the same way used in the proof of Proposition 4.3 to prove that Sub-procedure 4.3 still ends in a finite number of steps and reports $l' \leq 1 + \varepsilon'$.

In the following parts, we give another sub-procedure that can also generate the desired points in a finite number of steps.

$$y^i = \begin{cases} \frac{z^i}{\|z^i\|} & \text{If } dz^i \geq 0 \\ -\frac{z^i}{\|z^i\|} & \text{Else} \end{cases} \tag{39}$$

$$u^i = \begin{cases} \frac{v^i}{\|v^i\|} & \text{If } ev^i \geq 0 \\ -\frac{v^i}{\|v^i\|} & \text{Else} \end{cases} \tag{40}$$

Subprocedure 4.4 a) Let S and Q be the closed convex sets satisfying conditions (11) and (12).
Let $S^1 = S$ and $Q^1 = Q$. Set $i = 1$.
b) Use the oracle Θ to select l^i satisfying (16) or finds (z^i, v^i) satisfying (15) and (17).
If Θ finds $l^i \leq \varepsilon'$, then $l' = l^i$ and stop.
c) Set (y^i, u^i) according to (39) and (40).
If $z^i \notin \Omega$, then use Sub-procedure 4.1 with S^i and z^i to get a convex set S^{i+1}; else, $S^{i+1} = S^i$.
If $v^i \notin C^*$, then use Sub-procedure 4.1 with Q^i and v^i to get a convex set Q^{i+1}.; else, $Q^{i+1} = Q^i$.
If $\max\{d\frac{z^i}{\|z^i\|}, e\frac{v^i}{\|v^i\|}\} \leq 0$, goto e).
Else, set $\tau^i = \max\{\frac{dz^i}{4\|z^i\|}, \frac{ev^i}{4\|v^i\|}\}$.

If $d\frac{z^i}{\|z^i\|} < e\frac{v^i}{\|v^i\|}$, then set $\bar{v}^i = v^i$ and goto c1). Otherwise, set $\bar{z}^i = z^i$ and goto c2).

c1) If $dz^i \geq \tau^i\|z^i\|$, then set $\bar{z}^i = z^i$, $\lambda_1^i = (1 - \frac{1}{\sqrt{\bar{v}^i\bar{z}^i}})\|\bar{z}^i\|$ and $\lambda_2^i = (1 - \frac{1}{\sqrt{\bar{v}^i\bar{z}^i}})\|\bar{v}^i\|$.

Else, choose $\bar{z}^i$ satisfying condition (37).

If $dy^i < \tau^i$, then set $\lambda_1^i = 0$ and $\lambda_2^i = (1 - \frac{1}{\bar{z}^i\bar{v}^i})\|\bar{v}^i\|$.

Else, choose $x^i \in \partial\Omega$ such that $\bar{z}^i \in (0, x^i]$ and $\lambda_2^i = (1 - \frac{1}{x^i\bar{v}^i})\|\bar{v}^i\|$.

Goto d).

c2) If $ev^i \geq \tau^i\|v^i\|$, then set $\bar{v}^i = v^i$, $\lambda_1^i = (1 - \frac{1}{\sqrt{\bar{v}^i\bar{z}^i}})\|\bar{z}^i\|$ and $\lambda_2^i = (1 - \frac{1}{\sqrt{\bar{v}^i\bar{z}^i}})\|\bar{v}^i\|$.

Else, choose $\bar{v}^i$ satisfying condition (38).

If $eu^i < \tau^i$, then set $\lambda_2^i = 0$ and $\lambda_1^i = (1 - \frac{1}{\bar{z}^i\bar{v}^i})\|\bar{z}^i\|$.

Else, choose $w^i \in \partial C^*$ such that $\bar{v}^i \in (0, w^i]$ and $\lambda_1^i = (1 - \frac{1}{w^i\bar{z}^i})\|\bar{z}^i\|$.

Goto d).

d) If $\bar{v}^i\bar{z}^i - 1 < \frac{\varepsilon l^i}{\bar{\varepsilon}}$, goto *e*).

Else if $x^i \in \Omega$ and $w^i \in C^*$, then $x' = x^i$, $w' = w^i$, $\tau' = \tau^i$, $v' = \bar{v}^i$, $z' = \bar{z}^i$, $Q' = Q^{i+1}$ and $S' = S^{i+1}$, stop;

Else, goto e).

e) Set $i = i + 1$, return to b);

Remark 4.11 In Sub-procedure 4.4, we always have $x^i = \mu_1^i z^i$ and $w^i = \mu_2^i v^i$ where $\mu_1^i > 0$ and $\mu_2^i > 0$. This implies that, for any τ' and (x', w') produced by Sub-procedure 4.4, we have $\tau' = \max\{\frac{dx'}{4\|x'\|}, \frac{ew'}{4\|w'\|}\}$.

Let the bounded sequence $\{(x^k, w^k)\}$ and $\{\tau^k\}$ be generated by Sub-procedure 4.4. Assume by contradiction that $\tau^k \to 0$, then we have $\lim\max\{\frac{dx^k}{4\|x^k\|}, \frac{ew^k}{4\|w^k\|}\} = 0$. Let $(\bar{x}, \bar{w})$ be any cluster point of $\{(x^k, w^k)\}$, we get that $d\bar{x} + e\bar{w} \leq 0$, which contradicts condition (6).

Proposition 4.5 *Suppose that Q and S are compact. If the feasible value γ is not optimal, then Sub-procedure* 4.4 *ends in a finite number of steps and it either reports $l' \leq \varepsilon'$ or reports (x', w'), (z', v') satisfying conditions* (21)*,* (27)*,*(35) *and* (36)*.*

Proof: Assume by contradiction that there exists an infinite number of (z^i, v^i), which implies that $(x^i, w^i) \notin \Omega \times C^*$ for all i. Since $\{(z^i, v^i)\}$ is contained in $S \times Q$ and S, Q are bounded, we get that the sequence $\{(z^i, v^i)\}$ is also bounded. Let $(\bar{z}, \bar{v})$ be a cluster point of $\{(z^i, v^i)\}$. Since Sub-procedure 4.4 never stops, we get that $v^i z^i \geq 1 + \varepsilon'$ for all i and $\bar{z}\bar{v} \geq 1 + \varepsilon'$. Then $(\bar{z}, \bar{v})$ is not a cluster point of $\{(x^i, w^i)\}$ follows by $x^i w^i = 1$ for all i. Theorem 4.2 guarantees that $(\bar{z}, \bar{v}) \in \Omega \times C^*$, which means that $(\bar{z}, \bar{v})$ is a feasible point and thus $d\bar{z} + e\bar{v} > 0$.

By taking subsequences if necessary, let $x^i \to \bar{x}$ and $w^i \to \bar{w}$, $z^i \to \bar{z}$ and $v^i \to \bar{v}$. In Sub-procedure 4.4, $\bar{z}^i$ either satisfies condition (37) or equals z^i for all i, and $\bar{v}^i$ either satisfies condition (38) or equals v^i for all i. When there exists a subsequence $\{\bar{z}^{i_k}\}$ satisfying condition (37), Lemma 4.7 guarantees that

$\bar{z}^{i_k} \to \bar{z}$ and thus we have $\bar{z}^i \to \bar{z}$. In the same way, we get that $\bar{v}^i \to \bar{v}$, which implies that $\lim v^i z^i = \lim \bar{v}^i \bar{z}^i$. Since $v^i z^i \geq l^i$ for all i, there are only a finite number of $\{(\bar{z}^i, \bar{v}^i)\}$ such that $\bar{v}^i \bar{z}^i < \frac{\varepsilon l^i}{\bar{\varepsilon}}$. By $d\bar{z} + e\bar{v} > 0$ we get that there are only a finite number of (z^i, v^i) such that $dz^i \leq 0$ and $ev^i \leq 0$, which implies that there exists $I > 0$ such that $\max\{\frac{dz^i}{\|z^i\|}, \frac{ev^i}{\|v^i\|}\} > 0$ for all $i \geq I$.

Without loss of generality, let $\frac{dz^i}{\|z^i\|} < \frac{ev^i}{\|v^i\|}$ for all i. If there exists a subsequence $\{z^{i_k}\}$ such that $dz^{i_k} < \tau^{i_k}\|z^{i_k}\|$ for all k, then by definition of τ^{i_k} we have $ev^{i_k} \geq \tau^{i_k}\|v^{i_k}\|$, which implies that $u^{i_k} = \frac{v^{i_k}}{\|v^{i_k}\|}$ and thus $w^{i_k} \in (0, v^{i_k}]$ for all k. Moreover, by the selection of $\{x^{i_k}\}$ we have $\bar{z}^{i_k} \in (0, x^{i_k}] \subseteq \Omega$ and so $\bar{z} \in (0, \bar{x}]$. Thus we have $\bar{v} \neq \bar{w}$ and hence $\bar{w} \in$ int C^* since $0 \in$ int C^* and $\bar{v} \in C^*$. Therefore, there exists $I > 0$ such that $w^i \in C^*$ for all $i \geq I$ and thus exists a point $(x^{i_K}, w^{i_K}) \in \Omega \times C^*$, a contradiction.

Consider the case that there are only finite number of z^i satisfying $dz^i < \tau^i\|z^i\|$, which implies that there exists $I^1 > 0$ such that $\min\{\frac{dz^i}{\|z^i\|}, \frac{ev^i}{\|v^i\|}\} \geq \tau^i$ and thus $\lambda_1^i = (1 - \frac{1}{\sqrt{\bar{v}^i \bar{z}^i}})\|\bar{z}^i\|$, $\lambda_2^i = (1 - \frac{1}{\sqrt{\bar{v}^i \bar{z}^i}})\|\bar{v}^i\|$ for all $i \geq I^1$. Since $\bar{v}^i \bar{z}^i \geq 1 + \frac{\varepsilon\varepsilon'}{\bar{\varepsilon}}$ for all i, we get that $\lambda_1^i \not\to 0$ and $\lambda_2^i \not\to 0$. Thus we have $\bar{x} \in$ int Ω and $\bar{w} \in$ int C^* follows from $y^i = \frac{z^i}{\|z^i\|}$ and $u^i = \frac{v^i}{\|v^i\|}$, hence there exists a point $(x^{I_2}, w^{I_2}) \in \Omega \times C^*$, a contradiction.

Since y^i and u^i satisfying conditions (39) and (40) for all i, by the selection of λ_1^i and λ_2^i, we get that conditions (35) and (36) hold. Moreover, $v'z' \geq \frac{\varepsilon}{\bar{\varepsilon}} l'$ implies that condition (21) holds. By setting $\gamma = \gamma_k$ and condition (15), we get that condition (27) holds. $\boxdot$

Proposition 4.5 states that, when the feasible value γ is not optimal, Sub-procedure 4.4 ends in a finite number of steps. In fact, when the feasible value γ is optimal, we can use the same way used in the proof of Proposition 4.3 to prove that Sub-procedure 4.4 still ends in a finite number of steps and reports $l' \leq 1 + \varepsilon'$.

Corollary 4.1 *Suppose that condition* (42) *holds. If* $\tau > \max\{\|d\|, \|e\|\}$*, then conditions* (35) *and* (36) *hold.*

Proof: By conditions (42) and $\tau > \max\{\|d\|, \|e\|\}$ we get that $dy^k \geq \tau - \|e\| > 0$ and $eu^k \geq \tau - \|d\| > 0$. Let $\tau_1 = \tau - \|e\|$ and $\tau_2 = \tau - \|d\|$, then conditions (35) and (36) hold. $\boxdot$

Corollary 4.2 *Suppose that* $\tau > \max\{\|d\|, \|e\|\}$. *If conditions* (21), (27) *and* (42) *hold, then any cluster point of* $\{(x^k, w^k)\}$ *is globally optimal in problem* (CRP).

Proof: Corollary 4.1 shows that, when condition (42) holds and $\tau > \max\{\|d\|, \|e\|\}$, conditions (35) and (36) hold. $\boxdot$

Remark 4.12 When $d = 0$ or $e = 0$, condition $\tau > \max\{\|d\|, \|e\|\}$ and (42) can not both hold: Let $e = 0$, we have $dy \geq \tau > \|d\|$, which contradicts the assumption that $\|y\| \in \{0, 1\}$. Therefore, the set of conditions in Corollary 4.2 can not be applied in many cases.

4.3 The Third Way

Let τ be a positive value, in the second way, we propose the following conditions:

$$\lambda_1^k = \lambda_2^k, \tag{41}$$

$$dy^k + eu^k \geq \tau. \tag{42}$$

Remark 4.13 Condition (28) is equivalent to

$$d\lambda_1^k y^k + e\lambda_2^k u^k \geq 0$$

for all k, thus it is implied by conditions (41) and (42) since $\lambda_1^k = \lambda_2^k \geq 0$.

Lemma 4.8 *If conditions* (27)*,* (41) *and* (42) *hold, then* $\lambda_1^k = \lambda_2^k \to 0$.

Proof: As it has been shown in Remark 4.13, condition (28) is implied by conditions (41) and (42). Then by Lemma 4.3 we have $d\lambda_1^k y^k + e\lambda_2^k u^k \to 0$ since conditions (27) and (28) hold. Therefore, $\lambda_1^k = \lambda_2^k \to 0$ since $dy^k + eu^k \geq \tau$ for all k and $\lambda_1^k = \lambda_2^k$. $\boxdot$

Remark 4.14 When d or e equals 0, we don't require that $\lambda_1^k \to 0$ or $\lambda_2^k \to 0$, then condition (41) is not required in Lemma 4.8, i.e., if $e = 0$, then condition (42) and (27) are sufficient to get $d\lambda_1^k y^k \to 0$ and thus $\lambda_1^k \to 0$.

Lemma 4.8 gives conditions under which $\lambda_1^k \to 0$ and $\lambda_2^k \to 0$. Thus we have obtained sufficient conditions guaranteeing the convergence of $\{(x^k, w^k)\}$.

Theorem 4.4 *If conditions* (21)*,* (27)*,* (41) *and* (42) *hold, then any cluster point of* $\{(x^k, w^k)\}$ *is globally optimal in problem* (CRP).

Proof: Let $(\bar{x}, \bar{w})$ be any cluster point of $\{(x^k, w^k)\}$. By taking subsequences if necessary, let $x^k \to \bar{x}$ and $w^k \to \bar{w}$.

Since conditions (27), (41) and (42) hold, then by Lemma 4.8 we have $\lambda_1^k = \lambda_2^k \to 0$, which implies that $z^k \to \bar{x}$ and $v^k \to \bar{w}$, which implies that condition (20) holds. Therefore, Proposition 4.1 states that γ_k converges to the optimal value, and thus $(\bar{x}, \bar{w})$ is an optimal point. $\boxdot$

Therefore, conditions (21), (27), (41) and (42) ensure that a bounded sequence $\{(x^k, w^k)\}$ is convergent.

Question 4.5 How to construct the sequence of points $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ satisfying conditions (21), (27), (41) and (42)?

Sub-procedure 4.5 provides one possible answer on Question 4.5. Give any feasible and non-optimal value γ, this sub-procedure ends in a finite number of steps and outputs the desired points (z', v') and (x', w'). The proof will be given later. In fact, this sub-procedure not only produces the desired points but also constructs sequences of convex sets. As it will be shown, these convex sets play an important role in finding these points.

In the following, we give Sub-procedure 4.5 to obtain $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ satisfying conditions (21), (27), (41) and (42). In this sub-procedure, we assume that $0 \in \text{int } \Omega$ and $0 \in \text{int } C^*$ since Sub-procedure 4.1 is used. The computational procedure is the following: Given a feasible value γ and two closed convex sets S and Q satisfying conditions (11) and (12), respectively. Denote $S^1 = S$ and $Q^1 = Q$, $i = 1$.Use the oracle Θ to select l^i satisfying (16) or finds (z^i, v^i) satisfying (15) and (17). Moreover, take (y^i, u^i) satisfying condition

$$y^i = \begin{cases} \frac{z^i}{\|z^i\|} & \text{if } dz^i \geq 0 \\ 0 & \text{else} \end{cases} \tag{43}$$

$$u^i = \begin{cases} \frac{v^i}{\|v^i\|} & \text{if } ev^i \geq 0 \\ 0 & \text{else} \end{cases} \tag{44}$$

use Sub-procedure 4.1 with S^i and z^i to get S^{i+1} if $z^i \notin \Omega$; and with Q^i and v^i to get Q^{i+1} if $v^i \notin C^*$. Choose appropriate λ_1^i, λ_2^i such that (x^i, w^i) satisfying condition (32). If $x^i \in \Omega$ and $w^i \in C^*$, then sub-procedure ends; otherwise, set $i = i + 1$ and iterate.

Subprocedure 4.5 a) Let S and Q be the closed convex sets satisfying conditions (11) and (12).
Let $S^1 = S$ and $Q^1 = Q$. Set $i = 1$.
b) Use the oracle Θ to select l^i satisfying (16) or finds (z^i, v^i) satisfying (15) and (17).
If Θ finds $l^i \leq \varepsilon'$, then $l' = l^i$ and stop.
c) Take y^i and u^i according to (43), (44), respectively.
If $z^i \notin \Omega$, then use Sub-procedure 4.1 with S^i and z^i to get a convex set S^{i+1}; else, $S^{i+1} = S^i$.
If $v^i \notin C^*$, then use Sub-procedure 4.1 with Q^i and v^i to get a convex set Q^{i+1}.; else, $Q^{i+1} = Q^i$.
If $\max\{dz^i, ev^i\} < 0$, goto e).
If $dz^i < 0$, then set $\bar{v}^i = v^i$.
    Choose $\bar{z}^i$ satisfying condition (37).
    Set $\lambda_1^i = \lambda_2^i = (1 - \frac{1}{\bar{z}^i \bar{v}^i})\|\bar{v}^i\|$.
Else if $ev^i < 0$, then set $\bar{z}^i = z^i$.
    Choose $\bar{v}^i$ satisfying condition (38).
    Set $\lambda_1^i = \lambda_2^i = (1 - \frac{1}{\bar{z}^i \bar{v}^i})\|\bar{z}^i\|$.
Else, $\bar{z}^i = z^i$, $\bar{v}^i = v^i$, take $\lambda_1^i = \lambda_2^i$ such that $x^i w^i = 1$.
If $\bar{v}^i \bar{z}^i - 1 < \frac{\varepsilon l^i}{\bar{\varepsilon}}$, goto *e*); else, goto *d*).
d) If $x^i \in \Omega$ and $w^i \in C^*$, then $x' = x^i$, $w' = w^i$, $v' = \bar{v}^i$, $z' = \bar{z}^i$, $Q' = Q^{i+1}$

and $S' = S^{i+1}$, stop;
Else, goto e).
e) Set $i = i + 1$, return to b);

Sub-procedure 4.5 generates sequences of points $\{(x^i, w^i)\}$, $\{(z^i, v^i)\}$ and sets $\{S^i\}$, $\{Q^i\}$. It is necessary and useful to explore their properties and relationships.

In Sub-procedure 4.5, we use the same way as Sub-procedure 4.2 to cut off v^i, z^i from C^* and Ω, respectively. Then we get that $\{S^i\}$ and $\{Q^i\}$ satisfy conditions (33) and (34), respectively.

Lemma 4.9 *Suppose that* $\{(x^k, w^k)\}$ *is bounded. If we choose* (y^k, u^k) *satisfying conditions* (43) *and* (44)*, then condition* (42) *holds.*

Proof: Since $\{(x^k, w^k)\}$ is bounded, then the set of optimal solutions of problem (CRP) is non-empty. By Remark 2.1 we get that the optimal value $\gamma^* > 0$. This implies that $dx^k + ew^k \geq \gamma^* > 0$ for all k.

Assume by contradiction that there exists no $\tau > 0$ such that $dy^k + eu^k \geq \tau$ for all k. Let $M = \max\{\|x^k\|, \|w^k\|\}$ and $\tau^* = \frac{\gamma^*}{M}$, there exists $K > 0$ such that $dy^K + eu^K < \tau^*$, that is

$$\max\{0, dx^K/\|x^K\|\} + \max\{0, ew^K/\|w^K\|\} < \tau^*,$$

thus we have

$$dx^K + ew^K \leq \max\{0, dx^K\} + \max\{0, ew^K\} < \gamma^*,$$

a contradiction. $\boxdot$

Proposition 4.6 *Suppose that* Q *and* S *are compact. If the feasible value* γ *is not optimal, then Sub-procedure* 4.5 *ends in a finite number of steps and it either reports* $l' \leq \varepsilon'$ *or reports* (x', w'), (z', v') *satisfying conditions* (21), (27), (41) *and* (42).

Proof: Assume by contradiction that there exists an infinite number of (z^i, v^i), which implies that $(x^i, w^i) \notin \Omega \times C^*$ for all i. Since $\{(z^i, v^i)\}$ is contained in $S \times Q$ and S, Q are bounded, we get that the sequence $\{(z^i, v^i)\}$ is also bounded. Let $(\bar{z}, \bar{v})$ be a cluster point of $\{(z^i, v^i)\}$.

Since Sub-procedure 4.5 never stops, we get that $v^i z^i \geq 1 + \varepsilon'$ for all i and $\bar{z}\bar{v} \geq 1 + \varepsilon'$. Then $(\bar{z}, \bar{v})$ is not a cluster point of $\{(x^i, w^i)\}$ follows by $x^i w^i = 1$ for all i, that is $\lambda_1^i = \lambda_2^i \nrightarrow 0$. Theorem 4.2 guarantees that $(\bar{z}, \bar{v}) \in \Omega \times C^*$, which means that $(\bar{z}, \bar{v})$ is a feasible point and thus $d\bar{z} + e\bar{v} > 0$. Since $x^i \in (0, z^i]$ and $w^i \in (0, v^i]$ for all i, Lemma 4.6 guarantees that there exists a cluster point $(\bar{x}, \bar{w})$ of $\{(x^i, w^i)\}$ such that $\bar{x} \in [0, \bar{z}]$ and $\bar{w} \in [0, \bar{v}]$.

By taking subsequences if necessary, let $x^i \to \bar{x}$ and $w^i \to \bar{w}$, $z^i \to \bar{z}$ and $v^i \to \bar{v}$. In Sub-procedure 4.5, $\bar{z}^i$ either satisfies condition (37) or equals z^i for all i, and $\bar{v}^i$ either satisfies condition (38) or equals v^i for all i. When there exists a subsequence $\{\bar{z}^{i_k}\}$ satisfying condition (37), Lemma 4.7 guarantees that $\bar{z}^{i_k} \to \bar{z}$ and thus we have $\bar{z}^i \to \bar{z}$. In the same way, we get that $\bar{v}^i \to \bar{v}$, which implies that $\lim v^i z^i = \lim \bar{v}^i \bar{z}^i$. Since $v^i z^i \geq l^i$ for all i, there are only a finite number of $\{(\bar{z}^i, \bar{v}^i)\}$ such that $\bar{v}^i \bar{z}^i < \frac{\varepsilon l^i}{\bar{\varepsilon}}$. By $d\bar{z} + e\bar{v} > 0$ we get that there are only a finite number of $\{(z^i, v^i)\}$ such that $\max\{dz^i, ev^i\} < 0$.

If there exists infinite number of i such that $dz^i \geq 0$ and $ev^i \geq 0$, which implies that $y^i = z^i/\|z^i\|$ and $u^i = v^i/\|v^i\|$. Therefore, $\|\lambda_1^i y^i\| \nrightarrow 0$ and $\|\lambda_2^i u^i\| \nrightarrow 0$ and thus we have $\bar{x} \neq \bar{z}$ and $\bar{w} \neq \bar{v}$. Then we have $\bar{x} \in \text{int } \Omega$ and $\bar{w} \in \text{int } C^*$ follows by $0 \in \text{int } \Omega \cap \text{int } C^*$ and $(\bar{z}, \bar{v}) \in \Omega \times C^*$, hence there exists $I > 0$ such that $(x^I, w^I) \in \Omega \times C^*$, a contradiction.

Without loss of generality, suppose that there exists I^1 such that $dz^i < 0$ for all $i \geq I^1$, which implies that $x^i = \bar{z}^i \in \Omega$ and thus $\bar{x} = \bar{z}$. Therefore, we have $\bar{w} \neq \bar{v}$ since $\bar{x}\bar{w} = 1$ and $\bar{z}\bar{v} \geq 1 + \varepsilon'$, which implies that $\bar{w} \in \text{int } C^*$ follows by $0 \in \text{int } C^*$ and $\bar{v} \in C^*$. Therefore, there exists $I^2 > 0$ such that $(x^{I^2}, w^{I^2}) \in \Omega \times C^*$, a contradiction.

When we set $\gamma = \gamma_k$, we get that condition (27) holds. Moreover, $v'z' - 1 \geq \frac{\varepsilon l'}{\bar{\varepsilon}}$ implies that condition (21) holds. We always choose the same step $\lambda_1^i = \lambda_2^i$, thus condition (41) holds. Lemma 4.9 guarantees that condition (42) holds. ⊡

Proposition 4.6 states that, when the feasible value γ is not optimal, Sub-procedure 4.5 ends in a finite number of steps. When the feasible value γ is optimal, we can use the same way used in the proof of Proposition 4.3 to prove that Sub-procedure 4.5 will end in a finite number of steps.

4.4 The Fourth Way

We still need to consider the case that either $\lambda_1^k \nrightarrow 0$ or $\lambda_2^k \nrightarrow 0$. In this section, we assume that the following conditions hold.

$$dy^k \geq 0 \text{ for all } k, \tag{45}$$

$$eu^k \geq 0 \text{ for all } k. \tag{46}$$

Remark 4.15 When conditions (45) and (46) hold, we have

$$dz^k + ev^k - (dx^k + ew^k) = \lambda_1^k dy^k + \lambda_2^k eu^k \geq 0,$$

i.e., condition (28) holds. When conditions (27), (45) and (46) hold, Lemma 4.3 states that $d\lambda_1^k y^k + e\lambda_2^k u^k \to 0$, then we get that $d\lambda_1^k y^k \to 0$ and $e\lambda_2^k u^k \to 0$ since $dy^k \geq 0$ and $eu^k \geq 0$.

Lemma 4.10 *If conditions* (27), (42) (45) *and* (46) *hold, then either* $\liminf \lambda_1^k = 0$ *or* $\liminf \lambda_2^k = 0$.

Proof: Assume by contradiction that $\liminf \lambda_1^k \neq 0$ and $\liminf \lambda_1^k \neq 0$, then there exists $\sigma > 0$ and $K > 0$ such that $\lambda_1^k \geq \sigma/2$ and $\lambda_2^k \geq \sigma/2$ for all $k \geq K$. Therefore, $\lambda_1^k dy^k + \lambda_2^k eu^k \geq \frac{\sigma}{2}\tau$ for all $k \geq K$. However, conditions (45), (46) imply that condition (28) holds, and Lemma 4.3 states that $\lambda_1^k dy^k + \lambda_2^k eu^k \to 0$ when conditions (27) and (28) hold, a contradiction. $\boxdot$

The following conditions are assumed.

$$v^k x^k \leq 1, \tag{47}$$

$$z^k w^k \leq 1. \tag{48}$$

In order to get the convergence of $\{(x^k, w^k)\}$, we also propose the following conditions:

$$dy^k \geq \tau_1, \tag{49}$$

$$eu^k \geq \tau_2. \tag{50}$$

Theorem 4.5 *Suppose that $\{v^k\}$ ($\{z^k\}$) is bounded. If there exists a subsequence satisfying conditions* (21)*,* (27)*,* (46)*,* (47) *and* (49) *(*(21)*,* (27)*,* (45)*,* (48) *and* (50)*), then any cluster point of $\{(x^k, w^k)\}$ is globally optimal in problem* (CRP)*.*

Proof: Let $(\bar{x}, \bar{w})$ be any cluster point of $\{(x^k, w^k)\}$. Let's consider the case that there exists a subsequence $\{(x^{k_i}, w^{k_i})\}$ satisfying conditions (21), (27), (46), (47) and (49).

When condition (49) holds, condition (45) also holds. Remark 4.15 shows that when conditions (27), (45) and (46) hold, we have $d\lambda_1^{k_i} y^{k_i} \to 0$ and thus by condition (49) we have $\lambda_1^{k_i} \to 0$. Then we get that condition (23) holds since $\{v^k\}$ is bounded.

Since $v^{k_i} x^{k_i} \leq 1$ for all i, we get that $\lambda_2^{k_i} u^{k_i} x^{k_i} \leq 0$ for all i. Thus condition (24) holds. Therefore, condition (20) holds. Proposition 4.1 states that $\{\gamma_k\}$ converges to the optimal value when conditions (20) and (21) hold, thus $(\bar{x}, \bar{w})$ is optimal. $\boxdot$

Corollary 4.3 *Suppose that $\{(z^k, v^k)\}$ is bounded. If $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ satisfy conditions* (21)*,* (27)*,* (46)*,* (47) *and* (49) *or conditions* (21)*,* (27)*,* (45)*,* (48) *and* (50) *at each step k, then any cluster point of $\{(x^k, w^k)\}$ is globally optimal in problem* (CRP)*.*

Proof: It suffices to show that there exists either subsequence satisfying conditions (21), (27), (46), (47) and (49), or a subsequence satisfying conditions (21), (27), (45), (48) and (50). $\boxdot$

When condition (42) holds and $\tau > \|e\|$, we get that condition (49) also holds with $\tau_1 = \tau - \|e\|$. Thus we have the following corollary.

Corollary 4.4 *Suppose that $\{v^k\}$ is bounded, and $\tau > \|e\|$. If there exists a subsequence satisfying conditions* (21)*,* (27)*,* (42)*,* (46) *and* (47)*, then any cluster point of $\{(x^k, w^k)\}$ is globally optimal in problem* (CRP)*.*

Proof: Let $\tau_1 = \tau - \|e\|$, since condition (42) holds and $eu^k \leq \|e\|$ for all k, we have $dy^k \geq \tau_1$ for all k, i.e., condition (49) holds. Thus by Theorem 4.5 we get that any cluster point of $\{(x^k, w^k)\}$ is globally optimal in problem (CRP). ⊡

Remark 4.16 Suppose that $\{z^k\}$ is bounded, and $\tau > \|d\|$. If there exists a subsequence satisfying conditions (21), (27), (42), (45) and (48), then any cluster point of $\{(x^k, w^k)\}$ is globally optimal in problem (CRP).

Remark 4.17 Let $\{\sigma_1^k\}$ and $\{\sigma_2^k\}$ be two positive sequences such that $\sigma_1^k \to 0$ and $\sigma_2^k \to 0$. In Theorem 4.5, conditions (47) and (48) can be relaxed to the following conditions, respectively.

$$v^k x^k \leq 1 + \sigma_1^k, \tag{51}$$

$$z^k w^k \leq 1 + \sigma_2^k. \tag{52}$$

Theorem 4.6 *Suppose that* $\{v^k\}$ *(*$\{z^k\}$*) is bounded. If there exists a subsequence satisfying conditions* (21), (27), (46), (49) *and* (51) *((*21), (27), (45), (50) *and* (52)*), then any cluster point of* $\{(x^k, w^k)\}$ *is globally optimal in problem* (CRP)*.*

Proof: Let's consider the case that there exists a subsequence $\{(x^{k_i}, w^{k_i})\}$ satisfying conditions (21), (27), (46), (49) and (51). Let $(\bar{x}, \bar{w})$ be a cluster point of $\{(x^k, w^k)\}$.

As it has been in the proof of Theorem 4.5, when $\{v^k\}$ is bounded, conditions (27), (45) and (49) guarantee that condition (23) holds. Since $v^{k_i} x^{k_i} \leq 1 + \sigma_1^{k_i}$ for all i, we get that $\lambda_2^{k_i} u^{k_i} x^{k_i} \leq \sigma_1^{k_i}$ for all i. Thus condition (24) holds. Therefore, condition (20) holds. Proposition 4.1 states that $\{\gamma_k\}$ converges to the optimal value when conditions (20) and (21) hold, thus $(\bar{x}, \bar{w})$ is optimal. ⊡

Remark 4.18 In Problem (CDC), the following conditions ensure that a bounded sequence $\{x^k\}$ is convergent:

$$(0, z^k) \cap \Omega \cap \partial C \neq \emptyset, \tag{53}$$

$$x^k \in (0, z^k) \cap \Omega \cap \partial C, \tag{54}$$

$$dz^k \leq dx^{k-1}, \tag{55}$$

$$v^k z^k - 1 \geq \varepsilon \max\{vz - 1 \mid (z, v) \in D(\gamma_k)\}, \tag{56}$$

$$v^k x^i \leq 1 \text{ for all } i < k. \tag{57}$$

When $e = 0$ and $\{v^k\}$ is bounded, these conditions are stronger than the set of conditions in Theorem 4.6. It's easy to show that, when $e = 0$, condition (56) collapses to condition (21), and (55) collapses to condition (27). Moreover, $e = 0$ implies that $eu^k \geq 0$ for all k, i.e., (46) holds. When Ω is bounded, we get that the optimal value $\gamma^* > 0$ and $\{x^k\}$ is bounded. Let $M = \max\{\|x^k\|\}$, then by condition (54) we have $dy^k = \frac{dx^k}{\|x^k\|} \geq \frac{\gamma^*}{M}$, i.e., condition (49) holds with $\tau_1 = \frac{\gamma^*}{M}$. Condition (57) and $\{v^k\}$ is bounded guarantee that $\limsup v^k x^k \leq 1$, which implies that there exists a positive sequence $\sigma_1^k \to 0$ such that $v^k x^k \leq 1 + \sigma_1^k$ for all k, i.e., condition (51) holds.

Let's give an algorithm finding point satisfying conditions conditions (21), (27), (46), (47) and (49); or conditions (21), (27), (45), (48) and (50).

Question 4.6 How to construct the sequence of points $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ satisfying conditions (21), (27), (46), (47) and (49), or conditions ((21), (27), (45), (48) and (50))?

Sub-procedure 4.6 provides one possible answer on Question 4.6. Give any feasible and non-optimal value γ, this sub-procedure ends in a finite number of steps and outputs the desired points (z', v') and (x', w'). The proof will be given later. In fact, this sub-procedure not only produces the desired points but also constructs sequences of convex sets. As it will be shown, these convex sets play an important role in finding these points.

In the following, we give Sub-procedure 4.6 to obtain $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ satisfying conditions (21), (27), (46), (47) and (49), or conditions ((21), (27), (45), (48) and (50)). In this sub-procedure, we assume that $0 \in \text{int } \Omega$ and $0 \in \text{int } C^*$ since Sub-procedure 4.1 is used. The computational procedure is the following:

Subprocedure 4.6 a) Let S and Q be the closed convex sets satisfying conditions (11) and (12).
Let $S^1 = S$ and $Q^1 = Q$. Set $i = 1$.
b) Use the oracle Θ to select l^i satisfying (16) or finds (z^i, v^i) satisfying (15) and (17).
If Θ finds $l^i \leq \varepsilon'$, then $l' = l^i$ and stop.
c) Set y^i,u^i according to (43), (44), respectively.
If $z^i \notin \Omega$, then use Sub-procedure 4.1 with S^i and z^i to get a convex set S^{i+1}; else, $S^{i+1} = S^i$.
If $v^i \notin C^*$, then use Sub-procedure 4.1 with Q^i and v^i to get a convex set Q^{i+1}; else, $Q^{i+1} = Q^i$.
If $\max\{\frac{dz^i}{\|z^i\|}, \frac{ev^i}{\|v^i\|}\} < 0$, goto e).
If $\frac{ev^i}{\|v^i\|} \geq \frac{dz^i}{\|z^i\|}$, then set $\lambda_1^i = 0$, $\bar{v}^i = v^i$.
    Choose $\bar{z}^i$ satisfying condition (37).
    Set $\lambda_2^i = (1 - \frac{1}{\bar{z}^i \bar{v}^i})\|\bar{v}^i\|$.
Else, set $\lambda_2^i = 0$, $\bar{z}^i = z^i$.
    Choose $\bar{v}^i$ satisfying condition (38).
    Set $\lambda_1^i = (1 - \frac{1}{\bar{z}^i \bar{v}^i})\|\bar{z}^i\|$.
If $\bar{v}^i \bar{z}^i - 1 < \frac{\varepsilon l^i}{\bar{\varepsilon}}$, goto e); else, goto d).
d)If $x^i \in \Omega$ and $w^i \in C^*$, then set $x' = x^i$, $w' = w^i$, $z' = \bar{z}^i$, $v' = \bar{v}^i$, $Q' = Q^{i+1}$ and $S' = S^{i+1}$, stop;

Else, goto e).
e) Set $i = i + 1$, return to b);

Sub-procedure 4.6 generates sequences of points $\{(x^i, w^i)\}$, $\{(z^i, v^i)\}$ and sets $\{S^i\}$, $\{Q^i\}$. It is necessary and useful to explore their properties and relations.

Remark 4.19 When $\frac{ev^i}{\|v^i\|} \geq \frac{dz^i}{\|z^i\|}$, we have $x^i = \bar{z}^i$, $u^i = \frac{v^i}{\|v^i\|}$ and $\lambda_2^i = (1 - \frac{1}{\bar{z}^i \bar{v}^i})\|\bar{v}^i\|$. Therefore, $w^i = \bar{v}^i(1 - \frac{\lambda_2^i}{\|\bar{v}^i\|}) = \frac{\bar{v}^i}{\bar{z}^i \bar{v}^i}$ and thus we have $x^i w^i = 1$. In the same way, when $\frac{ev^i}{\|v^i\|} < \frac{dz^i}{\|z^i\|}$, we can also get that $x^i w^i = 1$.

Proposition 4.7 *Suppose that Q and S are compact. If the feasible value γ is not optimal, then Sub-procedure* 4.6 *ends in a finite number of steps and it either reports $l' \leq \varepsilon'$ or reports (x', w'), (z', v') satisfying either conditions* (21), (27), (46), (47) *and* (49), *or conditions* (21), (27), (45), (48) *and* (50).

Proof: Assume by contradiction that there exists an infinite number of (z^i, v^i), which implies that $(x^i, w^i) \notin \Omega \times C^*$ for all i. Since $\{(z^i, v^i)\}$ is contained in $S \times Q$ and S, Q are bounded, we get that the sequence $\{(z^i, v^i)\}$ is also bounded. Let $(\bar{z}, \bar{v})$ be a cluster point of $\{(z^i, v^i)\}$.

Since Sub-procedure 4.6 never stops, we get that $v^i z^i \geq 1 + \varepsilon'$ for all i and $\bar{z}\bar{v} \geq 1 + \varepsilon'$. Then $(\bar{z}, \bar{v})$ is not a cluster point of $\{(x^i, w^i)\}$ follows by $x^i w^i = 1$ for all i. Theorem 4.2 guarantees that $(\bar{z}, \bar{v}) \in \Omega \times C^*$, which means that $(\bar{z}, \bar{v})$ is a feasible point and thus $d\bar{z} + e\bar{v} > 0$. Since $x^i \in (0, z^i]$ and $w^i \in (0, v^i]$ for all i, Lemma 4.6 guarantees that there exists a cluster point $(\bar{x}, \bar{w})$ of $\{(x^i, w^i)\}$ such that $\bar{x} \in [0, \bar{z}]$ and $\bar{w} \in [0, \bar{v}]$.

By taking subsequences if necessary, let $x^i \to \bar{x}$ and $w^i \to \bar{w}$, $z^i \to \bar{z}$ and $v^i \to \bar{v}$. In Sub-procedure 4.6, $\bar{z}^i$ either satisfies condition (37) or equals z^i for all i, and $\bar{v}^i$ either satisfies condition (38) or equals v^i for all i. When there exists a subsequence $\{\bar{z}^{i_k}\}$ satisfying condition (37), Lemma 4.7 guarantees that $\bar{z}^{i_k} \to \bar{z}$ and thus we have $\bar{z}^i \to \bar{z}$. In the same way, we get that $\bar{v}^i \to \bar{v}$, which implies that $\lim v^i z^i = \lim \bar{v}^i \bar{z}^i$. Since $v^i z^i \geq l^i$ for all i, there are only a finite number of $\{(\bar{z}^i, \bar{v}^i)\}$ such that $\bar{v}^i \bar{z}^i < \frac{\varepsilon l^i}{\bar{\varepsilon}}$. By $d\bar{z} + e\bar{v} > 0$ we get that there are only a finite number of $\{(z^i, v^i)\}$ such that $\max\{dz^i, ev^i\} < 0$.

If there exists a subsequence satisfying $\frac{dz^{i_k}}{\|z^{i_k}\|} \leq \frac{ev^{i_k}}{\|v^{i_k}\|}$ for all k, then $\bar{z}^{i_k} = x^{i_k}$ and so $\bar{z} = \bar{x}$. In this case, we also have $u^{i_k} = \frac{v^{i_k}}{\|v^{i_k}\|}$ and $\bar{v} \neq \bar{w}$, which implies that $\bar{w} \in \text{int } C^*$ since $\bar{v} \in C^*$. Therefore, there exists $I > 0$ such that $w^i \in C^*$ for all $i \geq I$ and thus exists a point $(x^{i_K}, w^{i_K}) \in \Omega \times C^*$, a contradiction. In the same way, we get that there exists a point $(x^{i_K}, w^{i_K}) \in \Omega \times C^*$ when there exists a subsequence satisfying $\frac{dz^{i_k}}{\|z^{i_k}\|} > \frac{ev^{i_k}}{\|v^{i_k}\|}$ for all k.

It is obvious that y^i and u^i satisfy conditions (45) and (46), Lemma 4.9 states that condition (42) also holds and there exists $\tau > 0$ such that $dy' + eu' \geq \tau$. Therefore, by setting $\tau_1 < \frac{\tau}{2}$ and $\tau_2 < \frac{\tau}{2}$, we get that either condition (49) or (50) holds. If $\frac{dz'}{\|z'\|} \leq \frac{ev'}{\|v'\|}$, then condition (49) holds and thus $\lambda_2' = 0$, which implies that condition (47) holds. Otherwise, (50) and (48) hold. Moreover,

$v'z' - 1 \geq \frac{\varepsilon l'}{\bar{\varepsilon}}$ implies that condition (21) holds. By setting $\gamma = \gamma_k$ and condition (15), we get that $dz' + ev' \leq \gamma_k$ and so condition (27) holds. ⊡

Proposition 4.7 states that, when the feasible value γ is not optimal, Sub-procedure 4.6 ends in a finite number of steps. When the feasible value γ is optimal, we can use the same way used in the proof of Proposition 4.3 to prove that Sub-procedure 4.6 ends in a finite number of steps and reports $l' \leq 1 + \varepsilon'$.

By using Sub-procedure 4.6, we want to get a new algorithm that is able to generate sequences of points satisfying either conditions (21), (27), (46), (47) and (49), or conditions (21), (27), (45), (48) and (50). Generally, the procedure of this new algorithm is the following. Assume that a set of points (x^{k-1}, w^{k-1}) is known, set $\gamma_k = dx^{k-1} + ew^{k-1}$. Then use Sub-procedure 4.6 with γ_k to get points (z^k, v^k), (x^k, w^k) and iterate.

Remark 4.20 When $\{(z^k, v^k)\}$ and $\{(x^k, w^k)\}$ are bounded, if condition (57) holds, we have $\lim v^k x^k \leq 1$ and thus condition (51) holds. In the same way, we get that when condition

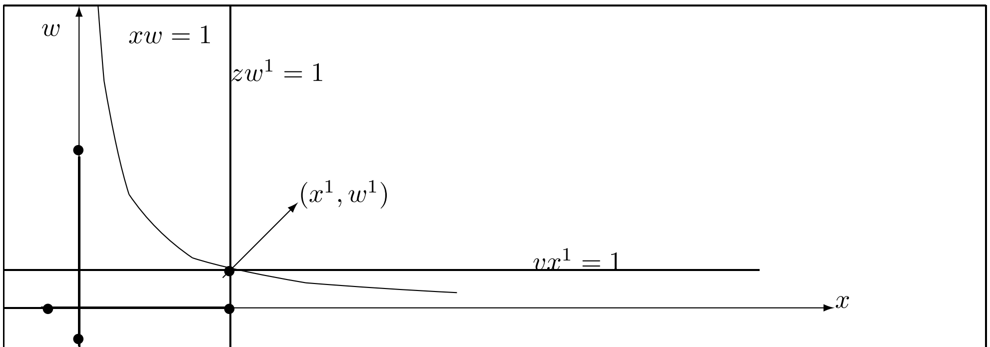
$$z^k w^i \leq 1 \text{ for all } i < k. \tag{58}$$

holds, then condition (52) holds.

However, usually we can not assume that both conditions (57) and (58) hold.

Remark 4.21 Conditions (57) and (58) make restrictions on the selection of (z^k, v^k). It should be noted that these two conditions may be inconsistent with other conditions, i.e., conditions (27), (28), (42), etc. In Example 4.3, there may exist no point satisfying conditions (28), (57) and (58) at the same time.

Example 4.3 Let $\Omega = [-\frac{1}{2}, 2]$ and $C^* = [-\frac{1}{2}, 2]$, thus $\Omega^* = [-2, \frac{1}{2}]$ and $C = [-2, \frac{1}{2}]$. $d = 1$ and $e = 1$. Take $(x^1, w^1) = (2, \frac{1}{2})$, then we get that $z^k \leq 2$ and $v^k \leq \frac{1}{2}$ for all $k \geq 2$. Therefore, all points (z^k, v^k) satisfying $z^k v^k > 1$ has $dz^k + ev^k < 0$, which contradicts condition (28).



The thick lines denote Ω and C^*, respectively

Example 4.3 shows that, although conditions (21), (27), (46), (57) and (49) ensure that a bounded sequence $\{(x^k, w^k)\}$ is convergent, we may not get such a sequence.

References

[1] A.S. Strekalovsky, I. Tsevendorj, "Testing the $\mathbb{R}$-strategy for a reverse convex problem", *J. Global Optim.* **13** (1998), 61–74.

[2] G. Bigi, A. Frangioni, Q.H. Zhang, "Outer approximation algorithms for canonical *DC* problems", *Technical Report, TR-07-06, Dipartimento di Informatica, Universita di Pisa*.

[3] H. Tuy, "Global minimization of a difference of two convex functions", *Math. Programming Studies* **30** (1987), 150–182.

[4] H. Tuy, "Canonical DC programming problem: outer approximation methods revisited", *Oper. Res. Lett.* **18** (1995), 99–106.

[5] H. Tuy, B.T. Tam, "Polyhedral annexation vs outer approximation for the decomposition of monotonic quasiconcave minimization problems", *Acta Math. Vietnam.* **20** (1995), 99–114.

[6] H. Tuy, "On nonconvex optimization problems with separated nonconvex variables", *J. Global Optim.* **2** (1992), 133–144.

[7] H. Tuy, F.A. Al-Khayyal, "Global optimization of a nonconvex single facility location problem by sequential unconstrained convex minimization", *J. Global Optim.* **2** (1992), 61–71.

[8] H. Tuy, *Convex Analysis and Global Optimization*, Kluwer Academic Publishers, 1998.

[9] H. Tuy, "A general deterministic approach to global optimization via d.c. programming", in J.B. Hiriart-Urruty (ed.) *FERMAT Days 85: Mathematics for Optimization*, North-Holland, Amsterdam (1986), 273–303.

[10] H. Tuy, "D.C. optimization: theory, methods and algorithms", in R. Horst, P.M. Pardalos (eds.), *Handbook of global optimization*, Kluwer Academic Publishers, Dordrecht (1995), 149–216.

[11] H. Tuy, "Convex programs with an additional reverse convex constraint", *J. Optim. Theory Appl.* **52** (1997), 463-486.

[12] H. Tuy, "On global optimality conditions and cutting plane algorithms", *J. Optim. Theory Appl.* **118** (2003), 201–216.

[13] H. Tuy, "An implicit space covering method with applications to fixed point and global optimization problems", *Acta Math. Vietnam.* **12** (1987), 93–102.

[14] J.B. Hiriart-Urruty, "Generalized differentiability, duality and optimization for problems dealing with difference of convex functions", in M. Beckmann, W. Krelle (eds.), *Convexity and Duality in Optimization*, Lecture notes in Economics and Mathematical Systems, 256, Springer, Berlin (1985), 37–70.

[15] J.B. Hiriart-Urruty, C. Lemaréchal, *Convex Analysis and Minimization Algorithms II*, Springer-Verlag, 1993.

[16] J. Fulop, "A finite cutting plane method for solving linear programs with an additional reverse constraint", *European J. Oper. Res.* **44** (1990), 395–409.

[17] L.T. Hoai An, P.D. Tao, "A continuous approach for globally solving linearly constrained quadratic zero-one programming problems", *Optimization* **50** (2001), 93–120.

[18] M. Borchardt, O. Engel, "A counterexample to a global optimization algorithm", *J. Global Optim.* **5** (1994), 371–372.

[19] M.D. Nghia, N.D. Hieu, "A method for solving reverse convex programming problems", *Acta Math. Vietnam.* **11** (1986), 241–252.

[20] N.V. Thoai, "A modified version of Tuy's method for solving d.c. programming problems", *Optimization* **19** (1988), 665–674.

[21] J.P. Penot, "What is quasiconvex analysis?", *Optimization* **47** (2000), 35–110.

[22] P.T. Thach, "Convex programs with several additional reverse convex constraints", *Acta Math. Vietnam.* **10** (1985), 35–57.

[23] P.T. Thach, "D.c sets, d.c. functions and nonlinear equations", *Math. Program.* **58** (1993), 415–428.

[24] D.T. Pham, S. El Bernoussi, "Numerical methods for solving a class of global nonconvex optimization problems", *International Series of Numerical Mathematics* **87** (1989), 97–132.

[25] R. Horst, P.M. Pardalos, *Handbook of global optimization*, Kluwer Academic Publishers, Dordrecht (1995).

[26] R. Horst, H. Tuy, *Global optimization*, Springer, Berlin, 1990.

[27] R. Horst, T.Q. Phong, N.V. Thoai, "On solving general reverse programming problems by a sequence of linear programs and line searches", *Ann. Oper. Res.* **25** (1990), 1–18.

[28] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.

[29] S. Ben Saad, S.E. Jacobsen, "A level set algorithm for a class of reverse convex programs", *Ann. Oper. Res.* **25** (1990), 19–42.

[30] S. Ben Saad, S.E. Jacobsen, "A new cutting plane algorithm for a class of reverse convex 0-1 integer programs", *Recent advances in global optimization*, Princeton University Press, Princeton, NJ. (1992), 152–164.

[31] T.Q. Phong, P.D. Tao, L.T. Hoai An, "A method for solving *D.C.* programming problems; application to fuel mixture nonconvex optimization problem", *J. Global Optim.* **6** (1995), 87-105.