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Abstract

The Sylvester-Kac matrix is a tridiagonal matrix with integer entries and integer eigenvalues that appears in a variety of applicative problems. We show that it belongs to a four dimensional linear space of tridiagonal matrices that can be simultaneously reduced to triangular form. We name this space after the matrix.

1 Introduction

By knowing an eigenvector a of a matrix M it is possible, at least in principle, to perform a step of the reduction by similarity of M to triangular form. For particular matrices the reduction step can be repeated almost without effort, since the reduction preserves the structure of M in such a way that it is easy to obtain an eigenvector of the reduced matrix from a. At the end of this progressive reduction we obtain a triangular matrix similar to M. In particular, this is possible for the Sylvester-Kac tridiagonal matrix [3, 4], also known as Clement matrix [2],

$$S_n = \begin{pmatrix} 0 & n-1 & & \\ 1 & 0 & \ddots & \\ & \ddots & \ddots & 1 \\ & & n-1 & 0 \end{pmatrix}, \tag{1}$$

that has integer eigenvalues and appears in a variety of applicative problems, see [4, 1] and the references therein. Vectors having all the entries equal to one, of the appropriate size, can be used as eigenvectors at every step of the reduction of S_n in triangular form.

In this paper we show that S_n belongs to a four dimensional linear space of tridiagonal matrices that can be progressively and simultaneously reduced to triangular form. Thus, in some sense, S_n is not as special as it seems.

After some preliminaries, we study in Section 3 some conditions under which the matrix obtained from a step of reduction from a banded matrix M is in turn banded with the same bandwidth of M and with, as much as possible, the same entries in the outermost diagonals. In Section 4 we restrict our attention to the case where M is tridiagonal. The property of progressive reducibility translates into a linear system that, in spite of having more equations that unknowns, turns out to be underdetermined, as we show in section 5. By solving the system we find a four dimensional linear space of progressively and simultaneously triangularizable tridiagonal matrices that seems appropriate to call the Sylvester-Kac matrix space.

2 Preliminaries

The Moore-Penrose pseudoinverse of a $n \times k$ full rank matrix A is given by

$$A^+ = \begin{cases} A^T(AA^T)^{-1} & \text{if } n \le k, \\ (A^TA)^{-1}A^T & \text{if } n \ge k. \end{cases}$$

Observe that A^+ is $k \times n$ and has full rank. Moreover $A^+ = A^{-1}$ in the case where $n = k$.

Let A and B be two matrices of dimensions $n \times k$ and $n \times (n-k)$ respectively, where $1 \le k \le n-1$. If A and B have full rank and $A^T B = O$, then the $n \times n$ matrix $V = \begin{pmatrix} A & B \end{pmatrix}$ is nonsingular and it is easy to verify that

$$V^{-1} = \begin{pmatrix} A^+ \\ B^+ \end{pmatrix}.$$

Let M be a $n \times n$ matrix such that $MA = A\Lambda$, where Λ is $k \times k$. Then

$$V^{-1}MV = \begin{pmatrix} A^+ \\ B^+ \end{pmatrix} M \begin{pmatrix} A & B \end{pmatrix} = \begin{pmatrix} \Lambda & A^+MB \\ O & B^+MB \end{pmatrix}. \tag{2}$$

In the following we assume $k = 1$, so that A is a vector that we denote with a, and Λ is a scalar that we denote with λ. With these notations $Ma = \lambda a$. We assume that the vector a has no zero entries. By virtue of this assumption we can set $a(1) = 1$.

For our purposes, we choose B as a unit lower bidiagonal matrix such that $B(i+1, i) = -a(i)/a(i+1)$ for $1 \le i \le (n-1)$. This implies $a^T B = 0$. The following lemma provides a convenient form for B^+. In the Lemma and throughout the paper we borrow colon and dot operation notations from MATLAB*.

Lemma 1 *We have*

$$B^+ = \text{tril}((1./a(1:(n-1)))a^T) + wa^T,$$

being w a suitable vector with $n-1$ components.

Proof. We embed B in a $n \times n$ unit lower bidiagonal matrix $\widetilde{B}$. It is simple to observe that $\widetilde{B}^{-1} = \text{tril}((1./a)a^T)$. Since $(\widetilde{B}^{-1}(1:(n-1),:) - B^+)B = O$ it follows that the rows of the matrix $\widetilde{B}^{-1}(1:(n-1),:) - B^+$ are all proportional to a^T. ∎

*MATLAB is a registered trademark of The Mathworks, Inc.

Let us set $L = \mathrm{tril}((1./a(1:(n-1)))a^T)$, so that $B^+ = L + wa^T$. Since $a^T B = 0$ implies $a^T B^{+T} = 0$, by using (2) and Lemma 1 we obtain

$$\begin{pmatrix} a & B^{+T} \end{pmatrix}^{-1} M \begin{pmatrix} a & B^{+T} \end{pmatrix} = \begin{pmatrix} \lambda & a^+ M B^{+T} \\ O & B^T M B^{+T} \end{pmatrix} = \begin{pmatrix} \lambda & a^+ M B^{+T} \\ O & B^T M L^T \end{pmatrix}. \qquad (3)$$

Since we pursue a reduction process that starts with M we set $M = M^{(1)}$ and $M^{(2)} = B^T M L^T$. Obviously, we can compute $M^{(2)}$ from $M^{(1)}$ once we know the eigenvector a. We are interested in the case where $M^{(2)}$ bears a strong resemblance with $M^{(1)}$ in order that the reduction step can be repeated.

Example 1 Let us consider the Sylvester-Kac matrix (1). If $e = (1, \ldots, 1)^T$ then $S_n e = (n-1)e$. If we set $M^{(1)} = S_n$ and $a = e$ in (3) we find $M^{(2)} = S_{n-1} - I$, where I is the identity matrix. ■

In the preceding example we found that many entries of $M^{(1)}$ appear again in $M^{(2)}$ and that the two matrices have the same bandwidth. This is not accidental, as we are going to prove.

3 The case where M is banded

We are interested in the case where $M^{(1)}$ is banded. Let $b \geq 0$ be an integer, we say that $M^{(1)}$ has lower (upper) bandwidth b if $M^{(1)}(i,j) = 0$ if $i > j + b$ ($j > i + b$).

Theorem 1 *The matrices $M^{(2)}$ and $M^{(1)}$ have the same bandwidths. Moreover, the more external lower diagonal of $M^{(2)}$ coincides with the more external lower diagonal of the $(n-1) \times (n-1)$ leading principal submatrix of $M^{(1)}$.*

Proof. The claims about the lower bandwidth and the more external lower diagonals of $M^{(2)}$ and $M^{(1)}$ follow from the fact that L^T and B^T are unit upper triangular matrices. Now assume that $M^{(1)}$ has upper bandwidth b and let $p \geq b + 1$. Then

$$[M^{(1)} L^T(:,p)](1:p-b) = \frac{\lambda}{a(p)} a(1:p-b)^T$$

Hence the first $p - b - 1$ components of the vector $B^T M^{(1)} L^T(:,p)$ are equal to zero. ■

Now, let us consider the more external upper diagonal of $M^{(2)}$.

Theorem 2 *We have*

$$M^{(2)}(p-b,p) = M^{(1)}(p-b+1,p+1) \frac{a(p+1)a(p-b)}{a(p)a(p-b+1)},$$

for $p = (b+1), \ldots, (n-1)$.

PROOF. Again, we assume that $M^{(1)}$ has upper bandwidth b and let $p \geq b+1$. Observe that

$$[M^{(1)}L^T(:,p)](p-b+1) = \frac{1}{a(p)}\big(\lambda a(p-b+1) - M^{(1)}(p-b+1,p+1)a(p+1)\big)$$

and this implies that

$$[B^T M^{(1)} L^T(:,p)](p-b) = M^{(1)}(p-b+1,p+1)\frac{a(p+1)a(p-b)}{a(p)a(p-b+1)}.$$

■

As a consequence, the more external upper diagonal of $M^{(2)}$ coincides with the more external upper diagonal of the $(n-1)\times(n-1)$ trailing principal submatrix of $M^{(1)}$ if

$$a(p+1)a(p-b) = a(p)a(p-b+1). \tag{4}$$

4 The case where M is tridiagonal

We now restrict our attention to the case where $M^{(1)}$ is tridiagonal, i.e. has upper and lower bandwidths $b=1$. By the previous two theorems we know that $M^{(2)}$ is tridiagonal we have a complete information about the external diagonals of $M^{(2)}$. Now, we turn our attention to the main diagonal of $M^{(2)}$.

Theorem 3 *If $M^{(1)}$ is tridiagonal then*

$$M^{(2)}(1,1) = M^{(1)}(1,1) - \frac{1}{a(2)}M^{(1)}(2,1). \tag{5}$$

and for $p = 2,\ldots,(n-1)$

$$M^{(2)}(p,p) = M^{(1)}(p,p) - \frac{a(p)}{a(p+1)}M^{(1)}(p+1,p) + \frac{a(p-1)}{a(p)}M^{(1)}(p,p-1). \tag{6}$$

PROOF. The equality (5) is immediate. To obtain the equality (6) we observe that, if $M^{(1)}$ is tridiagonal then for $p = 2:(n-1)$

$$\begin{array}{l}
[M^{(1)}L^T(:,p)](1:p-1) = \frac{\lambda}{a(p)}a(1:p-1)^T \\
[M^{(1)}L^T(:,p)](p) = \frac{1}{a(p)}(\lambda a(p) - M^{(1)}(p,p+1)a(p+1)) \\
[M^{(1)}L^T(:,p)](p+1) = M^{(1)}(p+1,p) \\
[ML^T(:,p)](j) = 0 \text{ for } j > p+1
\end{array}$$

This implies, for $p = 2,\ldots,(n-1)$

$$M^{(2)}(p,p) = \lambda - \frac{a(p+1)}{a(p)}M^{(1)}(p,p+1) - \frac{a(p)}{a(p+1)}M^{(1)}(p+1,p). \tag{7}$$

For $p = 2, \ldots, (n-1)$

$$\lambda = \frac{1}{a(p)} \sum_{i=-1}^{1} M^{(1)}(p, p+i) a(p+i),$$

and by substituting in (7) we obtain (6). ∎

If condition (4) holds our formulas become simpler as a consequence of the following result, whose simple proof is left to the reader.

Lemma 2 *Let v be a vector without zero entries. Then*

$$v(p+1)v(p-1) = v(p)^2, \tag{8}$$

for $p = 2, \ldots, n-1$ if and only if $v(i) = v(2)^{i-1}/v(1)^{i-2}$ for $i = 1, \ldots, n$.

Since our vector a has no zero entries we can use the previous Lemma. For shortness we set $\rho = a(2)$ so that, since $a(1) = 1$, $a(i) = \rho^{i-1}$. We find

$$M^{(2)}(1,1) = M^{(1)}(1,1) - \frac{1}{\rho} M^{(1)}(2,1). \tag{9}$$

and for $p = 2, \ldots, n-1$

$$M^{(2)}(p,p) = M^{(1)}(p,p) + \frac{1}{\rho}(M^{(1)}(p,p-1) - M^{(1)}(p+1,p)). \tag{10}$$

If $M^{(2)}$ has an eigenvector of the form $(1, \sigma, \ldots, \sigma^{n-2})^T$, then we can compute from $M^{(2)}$ a reduced matrix $M^{(3)}$ in the same way that led us from $M^{(1)}$ to $M^{(2)}$. We limit ourselves to the case where $\sigma = \rho$, since we are interested in finding matrices for which the reduction process is as close as possible to the one of the Sylvester-Kac matrix. With this assumption it is natural to look at ρ as to a known parameter. We remark that $\rho = 1$ for the Sylvester-Kac matrix.

Theorem 4 *Let the matrix $M^{(1)}$ have an eigenvector of the form $(1, \rho, \ldots, \rho^{n-1})^T$ associated with an eigenvalue λ_1. Then the matrix $M^{(2)}$ has an eigenvector of the form $(1, \rho, \ldots, \rho^{n-2})^T$ associated with an eigenvalue λ_2 if and only if*

$$\lambda_2 - \lambda_1 = \rho(M^{(1)}(p+1,p+2) - M^{(1)}(p,p+1)) + \frac{1}{\rho}(M^{(1)}(p,p-1) - M^{(1)}(p+1,p)). \tag{11}$$

for $p = 1, \ldots, n-1$ where $M^{(1)}(1,0) = M^{(1)}(n,n+1) = 0$.

Proof. The matrix $M^{(1)}$ has an eigenvector of the form $(1, \rho, \ldots, \rho^{n-1})^T$ associated with an eigenvalue λ_1 if and only if, for $p = 1, \ldots, n$

$$\lambda_1 = \sum_{i=-1}^{1} M^{(1)}(p, p+i)\rho^i, \tag{12}$$

where $M^{(1)}(1,0) = M^{(1)}(n,n+1) = 0$. Analogously the matrix $M^{(2)}$ has an eigenvector $(1,\rho,\ldots,\rho^{n-2})^T$ associated with an eigenvalue λ_2 if and only if, for $p = 1,\ldots,n-1$

$$\lambda_2 = \sum_{i=-1}^{1} M^{(2)}(p,p+i)\rho^i, \tag{13}$$

where $M^{(2)}(1,0) = M^{(2)}(n-1,n) = 0$. By the virtue of (10) and of the theorems 1 and 2, the equalities (12) and (13) are equivalent to the equalities (12) and (11). ■

5 The matrix space

By virtue of Theorem 4 the possibility to complete the reduction process is equivalent to the solvability of a linear homogeneous system of $\sum_{k=1}^{n-1} k = n(n-1)/2$ equations in $3n-2$ unknowns, i.e., the $2n-2$ off diagonal entries of $M^{(1)}$, the n eigenvalues λ_i, $i = 1,\ldots,n$. We discuss a possible elimination algorithm in the case where $n = 5$. The results that we obtain can be readily generalized. It turns out that, for every n, the system has a four dimensional space of solutions. We discuss a possible general parametrization of this space and the choice of a suitable basis.

For $n = 5$, equations (11) hold for all the four reduction steps, and give rise to an homogeneous system of 10 linear equations in 13 unknowns. The matrix of the system is the following

$$\left(\begin{array}{cccc|cccc|ccccc}
-\rho & \rho & 0 & 0 & -1/\rho & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & -\rho & \rho & 0 & 1/\rho & -1/\rho & 0 & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & -\rho & \rho & 0 & 1/\rho & -1/\rho & 0 & 1 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -\rho & 0 & 0 & 1/\rho & -1/\rho & 1 & -1 & 0 & 0 & 0 \\
\hline
0 & -\rho & \rho & 0 & -1/\rho & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & -\rho & \rho & 1/\rho & -1/\rho & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\
0 & 0 & 0 & -\rho & 0 & 1/\rho & -1/\rho & 0 & 0 & 1 & -1 & 0 & 0 \\
\hline
0 & 0 & -\rho & \rho & -1/\rho & 0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
0 & 0 & 0 & -\rho & 1/\rho & -1/\rho & 0 & 0 & 0 & 0 & 1 & -1 & 0 \\
\hline
0 & 0 & 0 & -\rho & -1/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1
\end{array}\right)$$

The unknowns, ordered by column, are the entries $M^{(1)}(i,i+1)$, $i = 1,\ldots,4$ that we shorten with α_i, the entries $M^{(1)}(i+1,i)$, $i = 1,\ldots,4$ that we shorten with β_i, and the eigenvalues λ_i, $i = 1,\ldots,5$. We immediately observe that the solutions of the system in the case where $\rho \neq 1$ can be obtained by the solutions in the case where $\rho = 1$ just by multiplying the α_i by $1/\rho$ and the β_i by ρ. Hence, from now on we assume $\rho = 1$.

It is possible to perform a two steps elimination strategy in order to reduce the matrix of the system to block triangular form. In particular, in the first step we subtracted the ninth equation from the tenth, the sixth and the seventh from the eighth and ninth respectively, the second to the fourth from the fifth to the seventh respectively. In the second step we subtracted the eighth equation from the tenth, the fifth an the sixth equation from the eighth and ninth respectively. By doing so these last two equations become identical and one of the two can be eliminated. This is a key feature of the elimination strategy and leads actually to the reduction of the number of equations from 10 to 9. The reduced matrix is as follows.

$$\left(\begin{array}{cccc|cccc|ccccc}
-1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & 0\\
0 & -1 & 1 & 0 & 1 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 0\\
0 & 0 & -1 & 1 & 0 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 1 & -1 & 0 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & -2 & 1 & 0 & 0 & -1 & 2 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & -2 & 1 & 0 & -1 & 2 & -1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 1 & -2 & 1 & -1 & 2 & -1 & 0 & 0\\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 3 & -1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -3 & 3 & -1
\end{array}\right)$$

After the reduction, the dimension of the space of solutions is almost self evident.

Theorem 5 *The space of solutions has dimension four.*

Proof. After assigning free values to $\beta_1,\ \lambda_1,\ \lambda_2,\ \lambda_3$, one can solve the last two equations for λ_4 and λ_5, equations $5, 6, 7$ for $\beta_i,\ i = 2, \ldots, 4$, and equations $4, 3, 2, 1$ for $\alpha_i,\ i = 4, 3, 2, 1$. ■

We define Sylvester-Kac space the four dimensional matrix space determined by the solutions.

Analogous results hold for any n. The homogeneous system of $n(n-1)/2$ equations can be reduced to $3n-6$ equations. The unknowns $\beta_1,\ \lambda_1,\ \lambda_2,\ \lambda_3$ can be freely chosen and it is possible to verify that the other unknowns are given by the following formulas:

$$\lambda_i = \tfrac{1}{2}(i-2)(i-3)\lambda_1 - (i-1)(i-3)\lambda_2 + \tfrac{1}{2}(i-1)(i-2)\lambda_3, \quad i = 4, \ldots, n,$$

$$\beta_i = i\beta_1 + \tfrac{1}{2}i(i-1)(\lambda_1 - 2\lambda_2 + \lambda_3),, \qquad i = 2, \ldots, n-1,$$

$$\begin{aligned}\alpha_i = (n-i)(-\beta_1 - \tfrac{1}{2}(n+i-5)\lambda_1 + (n+i-4)\lambda_2 \\ -\tfrac{1}{2}(n+i-3)\lambda_3), \qquad i = 1, \ldots, n-1.\end{aligned}$$

Let us denote with $S_n(\beta_1, \lambda_1, \lambda_2, \lambda_3)$ the generic matrix of the Sylvester-Kac space. By means of the previous formulas we can immediately write down a basis. In the case where $n = 5$ we obtain

$$S_5(1,0,0,0) = \begin{pmatrix} 4 & -4 & 0 & 0 & 0\\ 1 & 2 & -3 & 0 & 0\\ 0 & 2 & 0 & -2 & 0\\ 0 & 0 & 3 & -2 & -1\\ 0 & 0 & 0 & 4 & -4 \end{pmatrix},$$

$$S_5(0,1,0,0)=\begin{pmatrix} 3 & -2 & 0 & 0 & 0\\ 0 & 4 & -3 & 0 & 0\\ 0 & 1 & 3 & -3 & 0\\ 0 & 0 & 3 & 0 & -2\\ 0 & 0 & 0 & 6 & -5 \end{pmatrix},$$

$$S_5(0,0,1,0)=\begin{pmatrix} -8 & 8 & 0 & 0 & 0\\ 0 & -9 & 9 & 0 & 0\\ 0 & -2 & -6 & 8 & 0\\ 0 & 0 & -6 & 1 & 5\\ 0 & 0 & 0 & -12 & 12 \end{pmatrix},$$

$$S_5(0,0,0,1)=\begin{pmatrix} 6 & -6 & 0 & 0 & 0\\ 0 & 6 & -6 & 0 & 0\\ 0 & 1 & 4 & -5 & 0\\ 0 & 0 & 3 & 0 & -3\\ 0 & 0 & 0 & 6 & -6 \end{pmatrix}.$$

Obviously, S_5 in (1) is a matrix of the space, in particular $S_5 = S_5(1,4,2,0)$, and more generally $S_n = S_n(1, n-1, n-3, n-5)$.

It is clear that the Sylvester-Kac matrix space contains the scalar matrices as a linear subspace. It is natural to ask if the space contains nontrivial symmetric matrices.

Theorem 6 *The Sylvester-Kac matrix space contains a two dimensional subspace made up by symmetric matrices.*

Proof. We are looking for the solutions for which $\alpha_i = \beta_i$ for $i = 1, \ldots, n-1$. Since the scalar matrices are part of the subspace, there is no loss of generality in setting $\lambda_1 = 0$. The resulting system has $3n-6$ equations and $2n-2$ unknowns. We assign a free value to λ_2 and we solve the nonsingular $(n-1)\times(n-1)$ linear system

$$\begin{pmatrix} -2 & 1 & & \\ 1 & -2 & \ddots & \\ & \ddots & \ddots & 1\\ & & 1 & -2 \end{pmatrix} \begin{pmatrix} \alpha_1\\ \vdots\\ \vdots\\ \alpha_{n-1} \end{pmatrix} = -\lambda_2 \begin{pmatrix} 1\\ \vdots\\ \vdots\\ 1 \end{pmatrix}. \tag{14}$$

The next $n-2$ equations are satisfied by choosing $\lambda_3 = 3\lambda_2$. From the last $n-3$ equations we obtain the remaining λ_i for $i = 4, \ldots, n$. ∎

For $n = 5$ a basis containing a nontrivial symmetric matrix is formed by $S_5(1,0,0,0)$, $S_5(0,1,1,1) = I_5$, S_5 and

$$S_5(2,3,2,0)=\begin{pmatrix} 1 & 2 & 0 & 0 & 0\\ 2 & -2 & 3 & 0 & 0\\ 0 & 3 & -3 & 3 & 0\\ 0 & 0 & 3 & -2 & 2\\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}.$$

Observe that $S_5(1,0,0,0)$ is anticentrosymmetric, S_5 (the Sylvester-Kac matrix) is centrosymmetric, $S_5(2,3,2,0)$ is symmetric and centrosymmetric (this additional property is not accidental, since the solution of system (14) is such that $\alpha_i = \alpha_{n-i}$ for $i = 1, \ldots, \lfloor n/2 \rfloor$). For generic n the same properties are enjoyed by the basis formed by $S_n(1,0,0,0)$, $S_n(0,1,1,1) = I_n$, S_n and $S_n(\frac{n-1}{2},3,2,0)$. It follows that the Sylvester-Kac space can be expressed as direct sum of a three dimensional space of centrosymmetric matrices and a one dimensional space of anticentrosymmetric matrices.

References

[1] T. Boros, P. Rózsa, *An explicit formula for singular values of the Sylvester-Kac matrix*, Linear Algebra Appl. 421(2007), 407–416.

[2] A. Edelman, E. Kostlan, *The road from Kac's matrix to Kac's random polynomials*, in: J. G. Lewis (Ed.), Proceedings of the Fifth SIAM on Applied Linear Algebra, Philadelphia (1994), 503–507.

[3] O. Holtz, *Evaluation of Sylvester type determinants using orthogonal polynomials*, Proceedings of the Fourth ISSAC Congress, World Scientific (2005), 395–405.

[4] O. Taussky, J. Todd, *Another look to a matrix of Mark Kac*, Linear Algebra Appl. 150(1991), 341–360.