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Hard life with weak binders

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Abstract

We introduce *weak binders*, a lightweight construct to deal with fresh names in nominal calculi. Weak binders do not define the scope of names as precisely as the standard ν-binders, yet they enjoy strong semantic properties. We provide them with a denotational semantics, an equational theory, and a trace inclusion preorder. Furthermore, we present a trace-preserving mapping between weak binders and ν-binders.

1 Introduction

Over the last few years *naming* has been envisaged as a suitable abstraction for capturing and handling a variety of computational concepts, like distributed objects, cryptographic keys, session identifiers. Also, the dynamicity issues usually arising in distributed computing (e.g., network reconfiguration, module versioning) may be usefully explained in terms of naming disciplines such as fresh name generation, binding and scoping rules. The π-calculus [9, 14] is probably the most illustrative example of nominal calculi, in which many of the concepts outlined above have been formally modelled and explained.

Nominal calculi manipulate names via explicit binders that define their scope. The standard example is the π-calculus restriction operator νn. A ν-binder also declares that a fresh name has to be created. A broad variety of formal theories [6, 7, 16, 13, 10, 11, 3] developed in the last few years shows the intrinsic difficulties of handling naming and freshness.

This paper aims at contributing to this line of research. Our motivating starting point is to understand what is the actual gain in using ν-binders to deal with fresh names. Indeed, the equational theory of ν-binders allows for freely moving them almost anywhere in a process (except escaping from a recursion). So, one might wonder whether ν-binders can be omitted in a process, and replaced by a more primitive construct, e.g. an atomic action to be interpreted as a *gensym()* that explicitly creates a fresh name.

We introduce a nominal calculus with *weak binders*, a construct for generating fresh names as an atomic action, without explicit ν-binders. Our calculus slightly extends Bergstra and Klop' Basic Process Algebras [2], by allowing parametrized atomic actions $\alpha(r)$, that abstract from dispatching the action α to the object r. Objects can be freshly created through the special action $new(n)$, our "weak binder".

We study under which conditions a weakly bound process can be treated coherently with a process with ν-binders. For instance, in the weakly bound process $p = new(n) \cdot \alpha(n) + new(m) \cdot \alpha(m)$ there is no confusion between the scopes of the "bound" names n and m, and so p is equivalent to the "strongly bound" process $P = \nu n.\nu m.(new(n) \cdot \alpha(n) + new(m) \cdot \alpha(m))$. We shall then say that p is *well-bound*, and that P is its *bindification*. This transformation makes precise the scopes of names in weakly bound processes, by inserting the ν-binders at the right points. This is not always possible, however, e.g. in the process $new(n) \cdot (\varepsilon + new(n)) \cdot \alpha(n)$ there is an inherent ambiguity, because we cannot tell whether the action α has to be done on the object created by the first or by the second *new*. When bindification is possible, we prove that the semantics of the weakly bound and of the bindified processes are trace equivalent.

A further contribution is the definition of a trace inclusion preorder $\precsim$ for weakly bound processes. We prove that when $p \precsim q$, the traces of p are included in those of q. We then compare this preorder with a trace inclusion preorder for strongly bound processes. Preorders of processes are a relevant and non-trivial aspect of subtyping/subeffecting for type and effect systems [1]. Also, such preorders can be exploited to study the compliance of contracts with implementations and subcontract relations in calculi for Web services [4, 5].

We envisage the impact of our approach as follows. Our main result is the formal definition of an agile methodology for handling the freshness of names without resorting to explicit binders. Indeed, the overall outcome of our semantical investigation consists in the full characterization of weak binders. We have proved that weak binders still enjoy interesting semantic properties, comparably to what can be obtained through ν-binders. We have successfully exploited the theory of weak binders to develop the static machinery (a type and effect system and a model checker) of a linguistic framework for resource usage control [1]. As a downside, we have found that weak binders, having a weaker structure than ν-binders, may make the life hard when going into the proofs.

The paper is organized as follows. We first introduce a calculus with explicit ν-binders, we give its operational and denotational semantics, and we show them fully abstract. We then remove ν-binders, and define a denotational semantics and an equational theory of weakly bound processes. Then, we define the bindify transformation, and we state its correctness: the bindification of a weakly bound process p is trace equivalent to p. After that, we compare the equational theories and the trace inclusion preorders of strongly bound and weakly bound processes. We conclude by reporting our experience about using weak binders, and by discussing some related work. The Appendixes contain the proofs of all our statements.

2 Strongly bound processes

We now introduce a process calculus with name binders, building upon Basic Process Algebras (BPAs, [2]). Our calculus shares with BPAs the primitives for sequential composition, for non-deterministic choice, and for recursion (though with a slightly different syntax). Quite differently from BPAs, our atomic actions (called *events*) have a parameter, which indicates the *resource* upon which the action is performed. Resources $r, r', \ldots \in \mathsf{Res}$ are system objects that can either be already available in the environment or be created at run-time. Resources can be accessed through a given finite set of *actions* $\alpha, \alpha', \mathit{new}, \ldots \in \mathsf{Act}$. The special action *new* represents the creation of a fresh resource: this means that for each dynamically created resource r, the event $\mathit{new}(r)$ must precede any other $\alpha(r)$. An *event* $\alpha(r) \in \mathsf{Ev}$ abstracts from accessing the resource r through the action α. We also have events the target of which is a *name* $n, n', \ldots \in \mathsf{Nam}$, to be bound by an outer ν. Since the name binders are explicit in this calculus, we call its processes *strongly bound*, whose abstract syntax is given in Def. 2.1.

Definition 2.1. Syntax of strongly bound processes

P, Q	$::=$	ε	empty process
		h	variable
		$\alpha(\rho)$	event ($\rho \in \mathsf{Res} \cup \mathsf{Nam}$)
		$\nu n.P$	resource binding
		$P \cdot Q$	sequential composition
		$P + Q$	choice
		$\mu h.P$	recursion

In a recursion $\mu h.P$, the free occurrences of h in P are bound by μ. In the construct $\nu n.\,P$, the ν acts as a binder for the free occurrences of the name n in P. The intended meaning is to keep track of the binding between n and a freshly created resource. A process is *closed* when it has no free names and variables.

The behaviour of a strongly bound process is described by the set of sequential traces (typically denoted by $\eta, \eta', \ldots \in \mathsf{Ev}^*$) of its events. As usual, ε denotes the empty trace, and $\varepsilon\eta = \eta = \eta\varepsilon$. The trace semantics $[\![P]\!]^{op}$ of a closed, strongly bound process P, is a function from finite set of resources to sets of traces (Def. 2.2). We first introduce an auxiliary labelled transition relation $P, \mathcal{R} \xrightarrow{a} P', \mathcal{R}'$ (where $a \in \mathsf{Ev} \cup \{\varepsilon\}$ and $\mathcal{R} \subset \mathsf{Res}$). The set $\mathcal{R}$ in configurations accumulates the resources created at run-time, so that no resource can be created twice, e.g.

$$\begin{aligned} (\nu n.\, new(n)) \cdot (\nu n.\, new(n)), \emptyset &\xrightarrow{\quad\varepsilon\quad} new(r_0) \cdot (\nu n.\, new(n)), \{r_0\} \\ &\xrightarrow{new(r_0)} \nu n.\, new(n), \{r_0\} \\ &\not\xrightarrow{\qquad\;} new(r_0), \{r_0\} \end{aligned}$$

The labelled transition relation is then exploited in the definition of $[\![P]\!]^{op}$, which contains two kinds of traces. First, we include in $[\![P]\!]^{op}$ all the traces for *terminating* executions, i.e. those leading to ε. Then, we add all the prefixes of *all* executions, and mark these truncated traces with a trailing ! symbol. Here, we just let ! be a distinguished event not in Ev. Including these $\eta!$ prefixes in $[\![P]\!]^{op}$ is useful, since they allow us to observe non-terminating computations.

Definition 2.2. Trace semantics of strongly bound processes

$$\alpha(r),\, \mathcal{R} \xrightarrow{\alpha(r)} \varepsilon,\, \mathcal{R} \cup \{r\} \qquad \nu n.\, P,\, \mathcal{R} \xrightarrow{\varepsilon} P\{r/n\},\, \mathcal{R} \cup \{r\} \quad \text{if } r \notin \mathcal{R}$$

$$\varepsilon \cdot P,\, \mathcal{R} \xrightarrow{\varepsilon} P,\, \mathcal{R} \qquad P \cdot Q,\, \mathcal{R} \xrightarrow{a} P' \cdot Q,\, \mathcal{R}' \;\text{ if } P,\, \mathcal{R} \xrightarrow{a} P',\, \mathcal{R}'$$

$$P + Q,\, \mathcal{R} \xrightarrow{\varepsilon} P,\, \mathcal{R} \qquad P + Q,\, \mathcal{R} \xrightarrow{\varepsilon} Q,\, \mathcal{R} \qquad \mu h.\, P,\, \mathcal{R} \xrightarrow{\varepsilon} P\{\mu h.\, P/h\},\, \mathcal{R}$$

The trace semantics $[\![P]\!]^{op}(\mathcal{R})$ is then defined as

$$[\![P]\!]^{op}(\mathcal{R}) = \{\, \eta \mid P,\, \mathcal{R} \xrightarrow{\eta} \varepsilon,\, \mathcal{R}' \,\} \cup \{\, \eta! \mid P,\, \mathcal{R} \xrightarrow{\eta} P',\, \mathcal{R}' \,\}$$

Example 1. Consider the following strongly bound processes:

$$P_0 = \mu h.\, \alpha(r) \cdot h \qquad P_1 = \mu h.\, h \cdot \alpha(r) \qquad P_2 = \mu h.\, \nu n.\, (\varepsilon + \alpha(n) \cdot h)$$

Then, $[\![P_0]\!]^{op}(\emptyset) = \alpha(r)^*!$, i.e. P_0 generates traces with an arbitrary, finite number of $\alpha(r)$. Note that all the traces of P_0 are non-terminating (as indicated by the !) since there is no way to exit from the recursion. Instead, $[\![P_1]\!]^{op}(\emptyset) = \{!\}$, i.e. P_1 loops forever, without generating any events. The semantics of $[\![P_2]\!]^{op}(\emptyset)$ consists of all the traces of the form $\alpha(r_1) \cdots \alpha(r_k)$ or $\alpha(r_1) \cdots \alpha(r_k)!$, for all $k \geq 0$ and pairwise distinct resources r_i. □

The denotational semantics $[\![P]\!]^s_\theta$ of a strongly bound process P is given below (Def. 2.4) as a function Y in a cpo D_s, to be defined in a while. We first let D_0 be $\{\, X \subseteq \mathsf{Ev}^* \cup \mathsf{Ev}^*! \mid\, ! \in X \,\}$, that is the cpo of sets X of traces such that $! \in X$. Then we let D_h be the cpo $\mathcal{P}_{fin}(\mathsf{Res}) \rightharpoonup D_0$ (where $\rightharpoonup$ denotes partiality). Finally, D_s is the cpo $(\mathsf{Nam} \to \mathsf{Res}) \to D_h$ Intuitively, $[\![P]\!]^s_\theta(\chi)(\mathcal{R})$ contains all the possible traces of P. The first argument $\chi \in \mathsf{Nam} \to \mathsf{Res}$ records the bindings between names and resources. The second argument $\mathcal{R} \in \mathcal{P}_{fin}(\mathsf{Res})$ is a finite set of resources which indicates those already used, so to make them unavailable for future creations. As usual, the parameter θ binds the free variables of P (in our case, to values in D_h).

Before giving the semantics, it is convenient to introduce some auxiliary definitions that help in composing traces sequentially (see Def. 2.3 below).

The operator $\odot$ ensures that all the events after a $!$ are discarded. For instance, the process $P = (\mu h.\, h) \cdot \alpha(r)$ will never fire the event $\alpha(r)$, because of the infinite loop that precedes the event. The composition of the semantics of the first component $\mu h.\, h$ is $\{\,!\,\}$, while the semantics of $\alpha(r)$ is $\{\,!, \alpha(r), \alpha(r)\,!\,\}$. Combining the two semantics results in $\{\,!\,\} \odot \{\,!, \alpha(r), \alpha(r)\,!\,\} = \{\,!\,\}$.

The operator $\boxdot$ takes two semantics and combines their traces sequentially. While doing that, it records the resources created, so to avoid that a resource is generated twice. For instance, let $P = (\nu n.\, new(n)) \cdot (\nu n'.\, new(n'))$. The traces of the right-hand side $\nu n'.\, new(n')$ must not generate the same resources used in the left-hand side $\nu n.\, new(n)$, e.g. $new(r_0)new(r_0)$ is *not* a possible trace of P.

The definition of $\boxdot$ exploits the auxiliary operator R, that computes the set of resources occurring in a trace η. Also, $\downarrow \in \mathsf{R}(\eta)$ indicates that η is terminating, i.e. it does not contain any $!$s.

Definition 2.3. Let $X \in D_0$, and $x \in \mathsf{Ev} \cup \{\,!\,\}$. We define $x \odot X$ and $\eta \odot X$ as follows:

$$x \odot X = \begin{cases} \{\, x\,\eta \mid \eta \in X \,\} & \text{if } x \neq ! \\ \{x\} & \text{if } x = ! \end{cases} \qquad (a_1 \cdots a_n) \odot X = a_1 \odot \cdots \odot a_n \odot X$$

Given $Y_0, Y_1 \in D_s$, their composition $Y_0 \boxdot Y_1$ is:

$$Y_0 \boxdot Y_1 = \lambda \chi, \mathcal{R}.\ \bigcup \{\, \eta_0 \odot Y_1(\chi)(\mathcal{R} \cup \mathsf{R}(\eta_0)) \mid \eta_0 \in Y_0(\chi)(\mathcal{R}) \,\}$$

where $\mathsf{R}(\eta)$ is defined inductively as follows:

$$\mathsf{R}(\varepsilon) = \{\downarrow\} \qquad \mathsf{R}(\eta\, \alpha(r)) = \mathsf{R}(\eta) \cup \{r\} \text{ if } ! \notin \eta \qquad \mathsf{R}(\eta\,!\,\eta') = \mathsf{R}(\eta) \setminus \{\downarrow\}$$

Definition 2.4. Denotational semantics of strongly bound processes

$$\llbracket \varepsilon \rrbracket^s_\theta = \lambda\chi, \mathcal{R}.\, \{\,!, \varepsilon\} \qquad\qquad \llbracket h \rrbracket^s_\theta = \lambda\chi, \mathcal{R}.\, \theta(h)(\mathcal{R})$$

$$\llbracket \alpha(\rho) \rrbracket^s_\theta = \lambda\chi, \mathcal{R}. \begin{cases} \{\,!, \alpha(\rho), \alpha(\rho)\,!\} & \text{if } \rho = r \\ \{\,!, \alpha(\chi(n)), \alpha(\chi(n))\,!\} & \text{if } \rho = n \end{cases} \qquad \llbracket P \cdot Q \rrbracket^s_\theta = \llbracket P \rrbracket^s_\theta \boxdot \llbracket Q \rrbracket^s_\theta$$

$$\llbracket \nu n.\, P \rrbracket^s_\theta = \lambda\chi, \mathcal{R}.\, \bigcup_{r \notin \mathcal{R}} \llbracket P \rrbracket^s_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\}) \qquad \llbracket P + Q \rrbracket^s_\theta = \llbracket P \rrbracket^s_\theta \sqcup \llbracket Q \rrbracket^s_\theta$$

$$\llbracket \mu h. P \rrbracket^s_\theta = \lambda\chi, \mathcal{R}.\, \bigcup_{i \geq 0} \left(\lambda Z.\, \lambda\bar{\mathcal{R}}.\, \llbracket P \rrbracket^s_{\theta\{Z/h\}}(\chi)(\bar{\mathcal{R}})\right)^i (\lambda\mathcal{R}.\{\,!\})\ (\mathcal{R})$$

The semantics of an event $\alpha(r)$ comprises the possible "truncations" of $\{\alpha(r)\}$, i.e. $!, \alpha(r)\,!$ and $\alpha(r)$ (notice that $!$ is always included in the semantics of all P, coherently with the definition of the trace semantics). The semantics of $\alpha(n)$ is similar, but it looks in χ for the resource associated with n. The semantics of $\nu n.\, P$ joins the semantics of P, where the parameters $\mathcal{R}$ and χ are updated to record the binding of n with r, for *all* the resources r not yet used in $\mathcal{R}$. The semantics of $P \cdot Q$ combines the semantics of P and Q with the operator $\boxdot$. The semantics of $P + Q$ is the least upper bound of the semantics of P and Q. The semantics of a recursion $\mu h.\, P$ is the least upper bound of $f^i(\lambda\mathcal{R}.\{\,!\})$, where $f(Z) = \lambda\bar{\mathcal{R}}.\, \llbracket P \rrbracket^s_{\theta\{Z/h\}}(\chi)(\bar{\mathcal{R}})$. Since f is continuous and $\lambda\mathcal{R}.\{\,!\}$ is the bottom element of the cpo D_h, then $f^i(\lambda\mathcal{R}.\{\,!\})$ is an ω-chain, and its least upper bound is the least fixed point of f.

The following theorem states that the denotational semantics of strongly bound processes is fully abstract with respect to their operational semantics.

Theorem 2.5 (Full abstraction)**.** Let $\mathcal{R}$ be a finite sets of resources, and let $\emptyset$ be the empty mapping. Then, for all closed strongly bound processes P:

$$\llbracket P \rrbracket^{op}(\mathcal{R}) \ = \ \llbracket P \rrbracket^s_\emptyset(\emptyset)(\mathcal{R})$$

3 Weakly bound processes

In strongly bound processes, the ν-binders precisely define the scope of names. However, classical equational theories [8] for these processes usually allow binders to be floated out, towards the top-level, e.g. in $P_0 + \nu n.\, P_1 = \nu n.\, P_0 + P_1$, under the usual conditions. Indeed, the binder can always be brought outside a context, provided that 1) no recursion boundary is crossed, i.e. in $\mu h.\, \nu n.P$ the binder cannot be moved outside, and 2) no name in the context is captured. Because of this, it is often convenient to define a *normal form* for processes, where all the binders are placed at their top-most position, i.e. at the top-level or just under a recursion. These are standard and well-known facts about process algebras.

One might wonder what information is actually carried by the presence of the ν-binders. From an operational point of view, we can see them as the points where resources are created. In our setting, this information is also carried by the *new* events. Therefore, it is interesting to study whether, under this assumption, we can neglect placing binders in our processes, and let the *new* events to define, at least in some loose way, the scope of names.

To this purpose, we now introduce *weakly bound* processes, which have no ν-binders (Def. 3.1). For instance, let $p = new(n) \cdot \alpha(n) + new(m) \cdot \alpha'(m)$. Here, the event $new(n)$ binds the name n, while $new(m)$ binds m. We shall later on define a semantics of weakly bound processes such that p is equivalent to the strongly bound process $(\nu n.new(n) \cdot \alpha(n)) + (\nu m.new(m) \cdot \alpha'(m))$, as the intuition suggests.

While weakly bound processes may make our reasoning more agile, we must not neglect that, unlike in the strongly bound case, weakly bound processes are possible where names have no clear scope. E.g., in $new(n) \cdot (new(n)+\varepsilon)\cdot\alpha(n)$ it is not clear what binds the last occurrence of n. Roughly, these troublesome processes are those that can be derived from a strongly bound process by neglecting to α-convert some name while enlarging the scope of a ν-binder, yielding to unwanted name captures. We shall return to this point in Sect. 4.

Definition 3.1. Syntax of weakly bound processes

p, q	$::=$	ε	empty process
		h	variable
		$\alpha(\rho)$	event ($\rho \in \mathsf{Res} \cup \mathsf{Nam}$)
		$new(n)$	resource creation
		$p \cdot q$	sequential composition
		$p + q$	choice
		$\mu h.p$	recursion

Free names in weakly bound processes have to be dealt with quite peculiarly, because of the absence of ν-binders. Consider e.g. $p = p' \cdot \alpha(n)$. To tell whether n is free in p we have to inspect p'. For example if $p' = new(n)$, we shall consider n as non-free; instead, if $p' = \varepsilon$, the name n is obviously free. Given p', we define which names are *bound* by p', so to extend the scope of the names of p' when it occurs at the left of another process, as in $p' \cdot p''$. Non-determinism complicates matters: it might happen than a process p' binds a name to a resource only in some, but not all, of its execution, e.g. $p' = new(n) + \varepsilon$. So, we define two sets of names, the *must-bound* names $bn^{\Box}(p)$ and the *may-bound* names $bn^{\diamond}(p)$, for the names that are bound in *every* execution of p, and the names that are bound in *some* execution of p, respectively (see Def. 3.2). So, if $p' = \mathit{new}(n) + \varepsilon$, we have $bn^{\Box}(p') = \emptyset$ and $bn^{\diamond}(p') = \{n\}$. Note that the sets $bn^{\Box}(p')$ and $bn^{\diamond}(p')$ can be seen as static approximations for the actual run-time bindings created by the process p'. Of course, $bn^{\Box}(p) \subseteq bn^{\diamond}(p)$. Note that no "weak" binding can escape a recursion, as real ν-binders cannot cross recursive contexts. So, in $(\mu h.\, \mathit{new}(n) \cdot h + \varepsilon) \cdot \alpha(n)$ the last n is free, and is unrelated to the $\mathit{new}(n)$ event under the μh. Therefore, the bound names (both *must* and *may*) of a recursion are empty.

Definition 3.2. Must-bound names $\mathbf{bn^{\Box}(p)}$, may-bound names $\mathbf{bn^{\diamond}(p)}$

$$bn^{\Box}(\varepsilon) = bn^{\Box}(h) = \emptyset \qquad bn^{\Box}(\alpha(\rho)) = \begin{cases} \{n\} & \text{if } \alpha = \mathit{new} \text{ and } \rho = n \\ \emptyset & \text{otherwise} \end{cases}$$

$$bn^{\Box}(p \cdot q) = bn^{\Box}(p) \cup bn^{\Box}(q) \qquad bn^{\Box}(p + q) = bn^{\Box}(p) \cap bn^{\Box}(q) \qquad bn^{\Box}(\mu h.\, p) = \emptyset$$

$$bn^{\diamond}(p) = \begin{cases} bn^{\Box}(p) & \text{if } p = \varepsilon,\ h,\ \alpha(\rho),\ \mu h.\, p' \\ bn^{\diamond}(p') \cup bn^{\diamond}(p'') & \text{if } p = p' + p'' \text{ or } p = p' \cdot p'' \end{cases}$$

We can now define the free names $\mathit{fn}(p)$ of a weakly bound process p. This is mostly standard, except that must-bound names are checked to single out captured names. The choice of using must-bound names instead of may-bound names is done so that, e.g. in $p = (\mathit{new}(n)+\varepsilon) \cdot \alpha(n)$ we consider n as free. This has the nice property that, whenever $\mathit{fn}(p) = \emptyset$, in no execution of p we will attempt to fire an event $\alpha(n)$ without a proper binding for n.

Definition 3.3. Free names $\mathbf{fn(p)}$

$$\mathit{fn}(h) = \emptyset \qquad \mathit{fn}(\alpha(\rho)) = \begin{cases} \{n\} & \text{if } \rho = n \text{ and } \alpha \neq \mathit{new} \\ \emptyset & \text{otherwise} \end{cases}$$

$$\mathit{fn}(\mu h.\, p) = \mathit{fn}(p) \qquad \mathit{fn}(p + q) = \mathit{fn}(p) \cup \mathit{fn}(q)$$

$$\mathit{fn}(\varepsilon) = \emptyset \qquad \mathit{fn}(p \cdot q) = \mathit{fn}(p) \cup (\mathit{fn}(q) \setminus bn^{\Box}(p))$$

We now define a denotational semantics of weakly bound processes. Unlike in the case of strongly bound processes, where the result of the semantics was a set of event traces, here we also need to keep track of the bindings generated by the *new* events. We shall then use sets of pairs (η, χ) instead of sets of traces η. Note that this difference – the extra χ – between the semantic domains for the strongly/weakly bound processes is exactly the same difference between the classic domains for programming languages with static/dynamic scoping.

As we did with strongly bound processes in Def. 2.3, we introduce the auxiliary operators $\odot$ and $\boxdot$ to handle sequential composition.

The operator $\odot$ merges two pairs (η, χ), so ensuring that all the events after a ! are discarded, as well as the bindings created after the !. For example, $(\eta!, \chi) \odot (\eta', \chi') = (\eta!, \chi)$, discarding both η' and χ'. Here we also use two cpos, D_1 and D_w, to play the role of D_0 and D_s used for strongly bound processes. We let D_1 be the cpo of sets X of pairs (η', χ') such that there exists a pair in X with $\eta' = !$. Formally, D_1 is the cpo $\{\, X \subseteq (\mathsf{Ev}^* \cup \mathsf{Ev}^*\,!) \times (\mathsf{Nam} \to \mathsf{Res}) \mid \exists \chi'.\,(\,!, \chi') \in X \,\}$.

Definition 3.4. Let $a \in \mathsf{Ev} \cup \{\,!\}$, $X \in D_1$, $(\eta, \chi), (\eta', \chi') \in X$. We define $\odot$ as follows:

$$(a, \chi) \odot (\eta', \chi') = \begin{cases} (a, \chi) & \text{if } a = ! \\ (a\eta', \chi') & \text{otherwise} \end{cases}$$

$$(\eta, \chi) \odot (\eta', \chi') = (a_1, \chi) \odot \cdots \odot (a_k, \chi) \odot (\eta', \chi') \ \text{ if } \eta = a_1 \cdots a_k$$

$$(\eta, \chi) \odot X = \{\, (\eta, \chi) \odot (\bar{\eta}, \bar{\chi}) \mid (\bar{\eta}, \bar{\chi}) \in X \,\}$$

The operator $\boxdot$ takes two semantics Y_0 and Y_1 and combines their traces sequentially. In $Y_0 \boxdot Y_1$ the bindings (i.e. the χ) generated by Y_0 are passed to Y_1, so that e.g. $new(n) \cdot \alpha(n)$ works as expected.

Definition 3.5. Let $D_w = (\mathsf{Nam} \to \mathsf{Res}) \to \mathcal{P}_{fin}(\mathsf{Res}) \rightharpoonup D_1$ be the cpo of functions from functions from names to resources, to the finite subsets of Res to D_1 (where $\rightharpoonup$ denotes partiality). Given $Y_0, Y_1 \in D_w$, their composition $Y_0 \boxdot Y_1$ is:

$$Y_0 \boxdot Y_1 = \lambda \chi, \mathcal{R}.\ \bigcup \{\, (\eta_0, \chi_0) \odot Y_1(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0)) \mid (\eta_0, \chi_0) \in Y_0(\chi)(\mathcal{R}) \,\}$$

The denotational semantics $[\![p]\!]^w_\theta$ of a weakly bound process p is defined as a function $Y \in D_w$, where we assume that $Y(\chi)(\mathcal{R})$ is defined only if $\mathcal{R} \supseteq ran(\chi)$. The parameter θ is a mapping from the free variables h of p to D_h.

Definition 3.6. Denotational semantics of weakly bound processes

Below, we let $set_\chi I = \{\, (\eta, \chi) \mid \eta \in I \,\}$.

$$[\![\varepsilon]\!]^w_\theta = \lambda\chi, \mathcal{R}.\, set_\chi\{\,!, \varepsilon\} \qquad [\![h]\!]^w_\theta = \lambda\chi, \mathcal{R}.\, set_\chi \theta(h)(\mathcal{R})$$

$$[\![\alpha(\rho)]\!]^w_\theta = \lambda\chi, \mathcal{R}.\, \begin{cases} set_\chi\{\,!, \alpha(\rho), \alpha(\rho)\,!\} & \text{if } \rho = r \\ set_\chi\{\,!, \alpha(\chi(n)), \alpha(\chi(n))\,!\} & \text{if } \rho = n \in dom(\chi) \\ \{(\,!, \chi)\} \cup \bigcup_{r \notin \mathcal{R}} set_{\chi\{r/n\}}\{\alpha(r), \alpha(r)\,!\} & \text{if } \begin{array}{l} \rho = n \notin dom(\chi) \\ \text{and } \alpha = new \end{array} \end{cases}$$

$$[\![p \cdot q]\!]^w_\theta = [\![p]\!]^w_\theta \boxdot [\![q]\!]^w_\theta \qquad [\![p + q]\!]^w_\theta = [\![p]\!]^w_\theta \sqcup [\![q]\!]^w_\theta$$

$$[\![\mu h.p]\!]^w_\theta = \lambda\chi, \mathcal{R}.\, set_\chi \bigcup_{i \geq 0} \big(\lambda Z.\lambda\bar{\mathcal{R}}.\, fst([\![p]\!]^w_{\theta\{Z/h\}}(\chi|_{dom(\chi)\setminus bn^\diamond(p)})(\bar{\mathcal{R}}))\big)^i (\lambda\mathcal{R}.\{\,!\})\ (\mathcal{R})$$

The semantics above is similar to the one for strongly bound processes, so we just comment on the differences. First, each trace η has now been bundled with its generated bindings χ. Related to this, now the $new(n)$ event creates the actual binding, which augments the χ at hand. Note that we assume the operators $\cup$ and $\sqcup$ to be undefined when one of the arguments is undefined. This must hold also for $\odot$ and $\boxdot$, so making e.g. the semantics of $(new(n) + \varepsilon) \cdot \alpha(n)$ undefined when $n \notin dom(\chi)$, since in one branch $\alpha(n)$ is evaluated without a proper binding for n.

The semantics of recursion variables h is peculiar. First, note that we chose the semantics parameter θ so that $\theta(h)$ is an element of D_h and not of D_w. This is because, when recursion is involved, the bindings of names must not be propagated: this is strictly related to the fact that ν-binders cannot cross a recursive context in strongly bound processes. For example, in the strongly bound process $\mu h.\, \nu n.\, P \cdot h \cdot P'$ there is no way for the resource bound to n to be "passed" to the inner "call" to h; similarly, if the inner "call" generates a binding, it cannot be "returned" so to interfere with P'. Of course, this would change if we allowed a more complex form of recursion where h can take a resource as an argument. Returning to the semantics of h, since $\theta(h) \in D_h$ needs no χ, then it suffices to pass it an $\mathcal{R}$, and then augment the returned set of traces with χ. This is accomplished by the set_χ function.

The semantics of the recursion $\mu h.\, p$ is quite similar to the one for strongly bound processes. For the reasons explained above, we compute a fixed point over D_h and not D_w. This means that we have to adapt the semantics of p, which is in D_w, to a function in D_h. More concretely, we just need to provide χ to $[\![p]\!]^w$ and ignore the χ returned by it. The latter is done by a trivial left projection, the *fst* in the actual formula. The χ we pass, instead, is the top-level χ – the one provided to the whole recursive process – after the bindings which affect $bn^\diamond(p)$ have been filtered out. This filtering is needed to prevent from name confusion e.g. in $new(n) \cdot (\mu h.\, new(n) \cdot p)$, where the outer n is unrelated to the inner one. Aside from this, the fixed point is computed exactly as for the strongly bound processes, exploiting the continuity of $f(Z) = \lambda\bar{\mathcal{R}}.\, fst([\![p]\!]^w_{\theta\{Z/h\}}(\chi')(\bar{\mathcal{R}}))$.

4 Bindifying weakly bound processes

To make precise the scope of names in weakly bound processes, we shall translate them into strongly bound processes, through the transformation *bindify* (Def. 4.2). This transformation will insert the ν-binders at the right points, provided that the introduced scopes of names do not interfere dangerously. We shall call *well-bound* those weakly bound processes that can be safely translated into strongly bound ones. To help the intuition, we shall first give some examples.

Example 2. Consider the weakly bound processes:

$$p_1 = new(n) \cdot new(n) \cdot \alpha(n) \qquad p_2 = \alpha(n) \cdot new(n) \qquad p_3 = new(n) + \alpha(n)$$
$$p_4 = (\varepsilon + new(n)) \cdot \alpha(n) \qquad p_5 = new(n) \cdot (\mu h.\, (\varepsilon + new(n) \cdot h)) \cdot \alpha(n)$$

The processes p_1, p_2, p_3, p_4 are not well-bound. If p_1 were such, its bindification would either be $\nu n.\, new(n) \cdot (\nu n.\, new(n)) \cdot \alpha(n)$ – where α is performed on the resource generated by the outer ν-binder – or $\nu n.\, new(n) \cdot (\nu n.\, new(n) \cdot \alpha(n))$ – where α acts on the resource of the inner binder. Because of this possible ambiguity, we treat p_1 as not well-bound. The process p_2 is not well-bound, too, because it would produce an ill-formed trace $\alpha(r)new(r)$ where the event $\alpha(r)$ is fired *before* the event $new(r)$ that signals the creation of r. Similarly, the process p_3 is not well-bound, because its bindification would give rise to the ill-formed trace $\alpha(r)$. The process p_4 is not well-bound, because choosing the branch ε would lead to a similar situation. Observe that the denotation of p_1 contains the non-sense trace $new(r)new(r)\alpha(r)$, while the semantics of p_2, p_3 and p_4 are undefined, because $\boxdot$ and $\sqcup$ are strict. Finally, the process p_5 is well-bound, because the μ-binder clearly separates the scope of the outer $new(n)$ from that of the inner one. $\square$

The following definition formalizes when a process is well-bound. The empty process, variables and events are well-bound. A recursion is well-bound when its body is such. A choice $p + q$ is well-bound when both p and q are well-bound. Additionally, we require that the may-bound names of p are disjoint from the free names of q, and *viceversa* (e.g. $new(n) + \alpha(n)$ is not well-bound). A sequence $p \cdot q$ is well-bound when both p and q are well-bound, and furthermore (i) the may-bound names of q are disjoint from the names of p (e.g. $\alpha(n) \cdot new(n)$ and $new(n) \cdot new(n)$ are not well-bound), and (ii) the free names of q are either must-bound in p, or they are not may-bound in p (e.g. $(\varepsilon + new(n)) \cdot \alpha(n)$ is not well-bound).

Definition 4.1. Well-bound processes

A weakly bound process p is *well-bound* when $wb(p)$, defined inductively as follows:

$$wb(\varepsilon) = wb(h) = wb(\alpha(\rho)) = true \qquad wb(\mu h.\, p) \text{ if } wb(p)$$

$$wb(p + q) \text{ if } wb(p), wb(q), bn^{\diamond}(p) \cap fn(q) = bn^{\diamond}(q) \cap fn(p) = \emptyset$$

$$wb(p \cdot q) \text{ if } wb(p), wb(q), bn^{\diamond}(q) \cap (bn^{\diamond}(p) \cup fn(p)) = (bn^{\diamond}(p) \setminus bn^{\Box}(p)) \cap fn(q) = \emptyset$$

We now introduce the *bindify* transformation, which is defined on well-bound processes only. The may-bound names are lifted to the leftmost position of the bindified process, and they are placed within the scope of a ν-binder. In the case of a recursion $\mu h.\, p$, the may-bound names of p are lifted to the leftmost position *within* the recursion, i.e. they do not escape the scope of the μh.

Definition 4.2. Bindification

If $wb(p)$, the bindification $bindify(p)$ of p is a strongly bound process, defined as follows:

$$bindify(p) = \nu\, bn^{\diamond}(p).\, \beta(p)$$

where the auxiliary operator β is defined inductively as follows:

$$\beta(\varepsilon) = \varepsilon \qquad \beta(\alpha(\rho)) = \alpha(\rho) \qquad \beta(p + q) = \beta(p) + \beta(q)$$

$$\beta(h) = h \qquad \beta(\mu h.\, p) = \mu h.\, bindify(p) \qquad \beta(p \cdot q) = \beta(p) \cdot \beta(q)$$

Example 3. Recall from Sect. 1 the process $p = new(n) \cdot \alpha(n) + new(m) \cdot \alpha(m)$. It is easy to check that p is well-bound, and that its may-bound names are:

$$bn^{\diamond}(p) = bn^{\diamond}(new(n) \cdot \alpha(n)) \cup bn^{\diamond}(new(m) \cdot \alpha(m)) = \{n, m\}$$

Then the bindification of p is the strongly bound process:

$$bindify(p) \;=\; \nu n.\nu m.(new(n) \cdot \alpha(n) + new(m) \cdot \alpha(m))$$

Example 4. Recall the process $p_5 = new(n) \cdot (\mu h.\, (\varepsilon + new(n) \cdot h)) \cdot \alpha(n)$ from Ex. 2. It is easy to check that p_5 is well-bound. Its may-bound names are computed as follows:

$$bn^\diamond(p_5) = bn^\diamond(new(n)) \cup bn^\diamond(\mu h.\,(\varepsilon + new(n) \cdot h)) \cup bn^\diamond(\alpha(n)) = \{n\} \cup \emptyset = \{n\}$$

The bindification of p_5 is then computed as follows:

$$\begin{aligned} bindify(p_5) &= \nu n.\, \beta\big(new(n) \cdot (\mu h.\,(\varepsilon + new(n) \cdot h)) \cdot \alpha(n)\big) \\ &= \nu n.\, \big(\beta(new(n)) \cdot \mu h.\, bindify(\varepsilon + new(n) \cdot h) \cdot \beta(\alpha(n))\big) \\ &= \nu n.\, new(n) \cdot (\mu h.\, \nu n.\, \beta(\varepsilon + new(n) \cdot h)) \cdot \alpha(n) \\ &= \nu n.\, new(n) \cdot (\mu h.\, \nu n.\, (\varepsilon + new(n) \cdot h)) \cdot \alpha(n) \end{aligned}$$

We now state the correctness of bindification (Theorem 4.3). The "strong" semantics of $bindify(p)$ contains exactly the traces of the "weak" semantics of p.

Theorem 4.3. For all closed, weakly bound processes p such that $wb(p)$, $[\![p]\!]^w_\emptyset(\emptyset)(\emptyset)$ is defined, and:

$$[\![bindify(p)]\!]^s_\emptyset(\emptyset)(\emptyset) \quad = \quad fst([\![p]\!]^w_\emptyset(\emptyset)(\emptyset))$$

5 Equational theories and trace inclusion

In this section we provide strongly bound and weakly bound processes with an equational theory and a trace inclusion preorder. We shall state their correctness, i.e. the equational theory preserves the set of traces, while the preorder preserves their inclusion. Finally, we shall highlight some differences between the two calculi.

We first give in Def. 5.1 an equational theory of strongly bound processes.

Definition 5.1. An equational theory of strongly bound processes

The relation $=$ over strongly bound processes is the least congruence including α-conversion of names and variables such that:

$$P + P = P \qquad (P + P') + P'' = P + (P' + P'') \qquad P + P' = P' + P$$

$$(P \cdot P') \cdot P'' = P \cdot (P' \cdot P'') \qquad \varepsilon \cdot P = P = P \cdot \varepsilon$$

$$(P + P') \cdot P'' = P \cdot P'' + P' \cdot P'' \qquad P \cdot (P' + P'') = P \cdot P' + P \cdot P''$$

$$\mu h.\mu h'.P = \mu h'.\mu h.P \qquad \mu h.P = P\{\mu h.\,P/h\} \qquad \nu n.\varepsilon = \varepsilon$$

$$\nu n.\nu n'.P = \nu n'.\nu n.P \qquad \nu n.(P + P') = (\nu n.P) + P' \ \text{ if } n \notin fn(P')$$

$$\nu n.(P \cdot P') = P \cdot (\nu n.P') \ \text{ if } n \notin fn(P) \quad \nu n.(P \cdot P') = (\nu n.P) \cdot P' \ \text{ if } n \notin fn(P')$$

The operation $+$ is associative, commutative and idempotent; $\cdot$ is associative, has identity ε, and distributes over $+$. The binders μ and ν allow for α-conversion of bound names and variables, and can be rearranged. A μh can be introduced/eliminated when h does not occur free. A νn can be extruded when it does not bind a free occurrence of n. A $\mu h. P$ can be folded/unfolded as usual.

As expected, the equational theory above is not complete, e.g. $[\![P]\!]^s = [\![P']\!]^s$ does not imply $P = P'$. E.g., $\mu h. \alpha(r) \cdot h$ cannot be equated to $\mu h. \alpha(r) \cdot \alpha(r) \cdot h$, yet they have the same traces $\alpha(r)^* !$. However, the equational theory is sound w.r.t. our semantics, as established by the first item Theorem 5.3 below.

We then define a preorder $P \preceq Q$ betweeen strongly bound processes. The preorder $\preceq$ includes equivalence, and it is closed under contexts. Also, a strongly bound process P can be arbitrarily "weakened" to $P + Q$.

Definition 5.2. A trace inclusion preorder of strongly bound processes

The relation $\preceq$ over strongly bound processes is the least precongruence such that:

$$P \preceq Q \quad \text{if } P = Q \qquad\qquad P \preceq P + Q$$

The following theorem states that the equational theory $=$ and the preorder $\preceq$ agree with the semantics of strongly bound processes.

Theorem 5.3. For all closed, strongly bound processes P and Q:

- if $P = Q$, then $[\![P]\!]^s_\emptyset = [\![Q]\!]^s_\emptyset$.
- if $P \preceq Q$ then $[\![P]\!]^s_\emptyset(\chi)(\mathcal{R}) \subseteq [\![Q]\!]^s_\emptyset(\chi)(\mathcal{R})$, for all $\mathcal{R}$ and χ.

We now consider how to express an equational theory and a trace inclusion preorder for weak binders, in the same spirit of Def. 5.1 and Def. 5.2. In spite of their weaker structure, weakly bound processes still share many semantic-preserving equational properties with strongly bound processes, as summarized in Def. 5.4. Notably, the equations involving $+$ and $\cdot$ are identical with respect to Def. 5.1. The recursions μh can be rearranged, as before. Of course, here we do not have ν-binders, so the α-conversion of bound names can not be done, in general. As an important exception, we know that bound names inside a recursion can not escape, so their scope is completely known. In this case, we allow for α-conversion. Note that unfolding recursions is not allowed, otherwise we would have $\mu h.new(n) \cdot h \approx new(n) \cdot (\mu h.new(n) \cdot h) \approx new(n) \cdot new(n) \cdot (\mu h.new(n) \cdot h)$, so causing name confusion — indeed, the first two processes are well-bound, while the last one is not. As with strong binders, the equational theory below is not complete, yet it is sound w.r.t. the $[\![-]\!]^w$ semantics, as established by the first item of Theorem 5.6.

Definition 5.4. An equational theory of weakly bound processes

The relation $\approx$ over weakly bound processes is the least congruence including α-conversion of variables such that:

$$p + p \approx p \qquad (p + p') + p'' \approx p + (p' + p'') \qquad p + p' \approx p' + p$$

$$\varepsilon \cdot p \approx p \approx p \cdot \varepsilon \qquad (p \cdot p') \cdot p'' \approx p \cdot (p' \cdot p'')$$

$$(p + p') \cdot p'' \approx p \cdot p'' + p' \cdot p'' \qquad p \cdot (p' + p'') \approx p \cdot p' + p \cdot p''$$

$$\mu h.\mu h'.p \approx \mu h'.\mu h.p \qquad \mu h.p \approx \mu h.(p\{m/n\}) \text{ if } n \in bn^{\diamond}(p) \text{ and } m \notin p$$

Example 5. The equational theories shown above offer an opportunity to compare strong ν-binders with weak *new* binders. Consider the following equation: $new(n) \cdot p + new(n) \cdot q \approx new(n) \cdot (p + q)$. This is a trivial fact, since it directly follows from the distributive law. Its equivalent for strongly bound processes, $(\nu n.\, P) + (\nu n.\, Q) = \nu n.\, (P + Q)$, appears instead to be non trivial. Indeed, although Def. 5.1 comprises all the classic equations for ν-binders, the mentioned equation can not be derived from them, since we can not identify the two binders. Yet, in most process algebras, we expect the equation to be sound w.r.t. any reasonable process equivalence relation. So, in this case weak binders offer a simpler view.

We shall now introduce a preorder $p \precsim_{\mathcal{N}} q$ on weakly bound processes. Here, we use a set of names $\mathcal{N}$ as an index to the preorder relation. This index is needed to avoid name confusion, as we shall see below. When $p \precsim_{\mathcal{N}} q$ holds, then the semantics of p is included in that of q (second item of Theorem. 5.6).

Definition 5.5. A trace inclusion preorder of weakly bound processes

The relation $\precsim_{\mathcal{N}}$ over weakly bound processes is the least preorder such that:

$$p \precsim_{\emptyset} q \quad \text{if } p \approx q \qquad p \precsim_{\emptyset} p + q \qquad p \precsim_{\mathcal{N} \cup \mathcal{N}'} p'' \quad \text{if } p \precsim_{\mathcal{N}} p' \text{ and } p' \precsim_{\mathcal{N}'} p''$$

$$\mathcal{C}(p) \precsim_{\mathcal{N}} \mathcal{C}(q) \quad \text{if } p \precsim_{\mathcal{N}} q \text{ and } \mathcal{N} \cap (bn^{\diamond}(\mathcal{C}) \cup fn(\mathcal{C})) = \emptyset$$

$$p\sigma\{\mu h.\, p/h\} \precsim_{ran(\sigma)} \mu h.\, p \quad \text{if } ran(\sigma) \cap fn(p) = \emptyset$$

where $\mathcal{C} = p \cdot \bullet \mid \bullet \cdot p \mid p + \bullet \mid \bullet + p$ is a *context*, $\sigma : \mathsf{Nam} \to \mathsf{Nam}$ is an injective function with $dom(\sigma) = bn^{\diamond}(p)$, and $p\sigma\{\mu h.\, p/h\}$ is capture-avoiding.

The preorder $\precsim_{\mathcal{N}}$ includes $\approx$-equivalence (Def. 5.4). A process p can be arbitrarily "weakened" to $p + q$. The relation is closed under contexts, provided that the names in $\mathcal{N}$ are disjoint from those in the context. Note that, because of this side condition, $\precsim_{\mathcal{N}}$ is not a precongruence, unlike $\preceq$ for strongly bound processes. Folding/unfolding is possible, but in a weaker form than in Def. 5.1. To avoid name confusion and preserve well-boundedness, the unfolded names must be fresh. For instance, if $p = \mu h.\, new(n) \cdot \alpha(n) \cdot h$, then we shall have $new(n') \cdot \alpha(n') \cdot p \precsim_{\{n'\}} p$. The name n' in $\precsim_{\{n'\}}$ is needed to avoid name clashes. For instance, it prevents from using the previous unfolding in the context $\mathcal{C} = \bullet \cdot \alpha'(n')$, since the extruded $new(n')$ would bind the name n' in $\alpha'(n')$, as checked by the context rule above. The side condition on the rule for folding/unfolding is needed to ensure that all the processes smaller (w.r.t. $\precsim$) than a well-bound process are well-bound (Theorem 5.7). Omitting the disjointness condition between $fn(p)$ and the range of the substitution σ would lead to situations like $\alpha(n') \cdot new(n') \precsim_{\{n'\}} \mu h.\, \alpha(n') \cdot new(n)$, where the right-hand side is well-bound, while the left-hand side is not. Substitutions of names must be coherent with bindification, i.e. they must not affect names that would be put under a ν-binder by $\beta(-)$, e.g. $(new(n) \cdot \mu h.\, new(n))\{m/n\} = new(m) \cdot \mu h.\, new(n)$. Similarly, substitutions can trigger α-conversions to avoid name captures, e.g. $(\mu h.\, new(m) \cdot \alpha(n))\{m/n\} = \mu h.\, new(m') \cdot \alpha(m)$.

We now formally state that our syntactic preorder agrees with the semantics of weakly bound processes, as it yields trace inclusion. Note that trace inclusion requires the two semantics to be defined. Otherwise we have $new(n) \cdot \mu h.\, \alpha(n) \precsim_{\emptyset} new(n) \cdot \mu h.\, (new(n) + \varepsilon) \cdot \alpha(n)$: when the branch ε is chosen in the right-hand side, we find $\chi'(n) = \chi|_{dom(\chi) \setminus \{n\}}(n)$, so $\alpha(n)$ cannot be evaluated, and the whole semantics is undefined (while the semantics of the left-hand side is always defined). Note however that is q is well-bound, then also p is such (Theorem 5.7), and so by Theorem 4.3 both the semantics are defined.

Theorem 5.6. For all closed, weakly bound processes p and q:

- if $p \approx q$, then $[\![p]\!]^w_\emptyset = [\![q]\!]^w_\emptyset$.
- if $p \precsim_{\mathcal{N}} q$ and, then $fst([\![p]\!]^w_\emptyset(\chi)(\mathcal{R})) \subseteq fst([\![q]\!]^w_\emptyset(\chi)(\mathcal{R}))$, for all $\mathcal{R}$ and χ such that $dom(\chi) \cap \mathcal{N} = \emptyset$ and both the semantics are defined.

The projection *fst* in the statement above is necessary. Consider e.g. $p = new(n) \precsim_{\{n\}} \mu h.\, new(m) = q$. Here, the semantics of p and q agree on the η components, i.e. the truncations of $new(r)$ with $r \notin \mathcal{R}$, but p will augment χ with the new binding $\{r/n\}$, unlike q which does not affect χ.

The next theorem guarantees that *bindify* is well-defined, i.e. it maps $\approx$-equivalent weakly bound processes $=$-equivalent strongly bound processes. Moreover, processes smaller (w.r.t. $\precsim_{\mathcal{N}}$) than well-bound processes are well-bound.

Theorem 5.7. For all weakly bound processes p and q:

- if $p \approx q$, then $wb(p)$ if and only if $wb(q)$.
- if $p \approx q$ and $wb(p)$, then $bindify(p) = bindify(q)$.
- if $p \precsim_{\mathcal{N}} q$ and $wb(q)$, then $wb(p)$.

6 Conclusions

We have investigated weak binders – a construct for fresh name generation – as an alternative for ν-binders in nominal calculi. Weak binders allow for a looser reasoning, while still admitting a trace-preserving translation into strong binders. However, this comes at a cost: often, useful properties, e.g. trace inclusion (Th. 5.6), require more side conditions to be checked for ensuring sanity. Also, α-conversion of names can only be applied inside μ-binders. Compositionality is reduced, since e.g. $wb(p)$ and $wb(q)$ do not automatically imply $wb(p \cdot q)$ which – if needed – must be established by exploiting further assumptions. In our experiments with weak binders, we also found they sometimes lead to more intricate proofs, since particular care must be exercised with corner cases. For instance, handling recursion in an operational semantics for weakly bound processes seems to be quite complex. Indeed, naïve unfolding causes name confusion, so one has to resort to either renaming all bound names so that they are indeed globally fresh, or to record the “call frames” (entering/leaving the body of a recursion) in a stack. Since we need to keep track of this, run-time configurations become more complex, and we found our operational semantics (not presented in this paper) to be too inconvenient to be used in proofs. Even when using the denotational semantics (Def. 3.6), we felt that writing inductive statements for weak binders required more trial-and-error steps, w.r.t. strong binders. However, in some occasions weak binders may become a more agile tool. For instance, they can exploited to implement a type and effect inference algorithm for a calculus with side effects and explicit name binders (like [1]), on top of an existing algorithm for a calculus without binders. Each time a ν-binder is encountered, a fresh name is generated, similarly to fresh type variables in Hindley-Milner type inference. After solving the obtained type and effect constraints through unification, the resulting effect is bindified. Of course, this is not always possible, e.g. when the effect is not well-bound. Possible counter-measures consist in suitably extending let-polymorphism to ν-binders.

Related work

A number of formal techniques have been developed to handle binding and freshness of names. The permutation model of sets introduced by Fraenkel-Mostowski has led to an elegant and powerful mathematical theory of naming [6]. The key observation of this approach is that α-conversion, binding and freshness can be defined through name permutations (or swappings). For instance, the freshness axiom for a name of a computational entity (i.e. an object, a process, a context, *etc.*) is expressed by saying that the fresh name does not belong to the support of the computational entity. Notably, in the permutation model the support of computational entities is *finite*. This mathematical theory has been used to model early and late semantics of the π-calculus [7]. Also, it has driven the design of a functional language, FreshML [16], which includes primitive mechanisms for handling fresh bindable names. In FreshML freshness is managed by resorting to a *gensym()* primitive to dynamically generate names, and a primitive for permuting names. Our notion of weakly bound processes exploits the *gensym()* primitive without resorting to an explicit primitive for handling name permutations. Indeed, the bindify trasformation singles out the names in the finite support of a weakly bound process. A monadic denotational semantics for FreshML has been used to handle freshness through a continuation monad on FM-sets [15]. This semantics allows for translating the usual domain-theoretic results in the context of FM-sets, and to use them to prove freshness-related properties.

The $\lambda\nu$-calculus presented in [12] extends the pure λ-calculus with names. In contrast to λ-bound variables, nothing can be substituted for a name, yet names can be tested for equality. Reduction in $\lambda\nu$ is confluent, and it allows for deterministic evaluation. Furthermore, all the observational equivalences that hold in the pure λ-calculus still hold in $\lambda\nu$. This has the practical consequence that all the equational techniques for transforming and verifying pure functional programs are also applicable to programs with local names. Nominal techniques have been implicitly used for reasoning about the semantics of functional languages with local state in [13], to prove when two functional programs are equivalent in every evaluation context.

Binding and freshness of names have been a main concern in process calculi. History-Dependent automata [10, 11] provide an automata-based model where states are equipped with name permutations to manage freshness and garbage collections of names. They automatically manage the creation and deallocation of names, while allowing for a compact representation of the system behaviour, by collapsing the states that only differ for the renaming of local names. The π-calculus is extended in [3] with an operational model where names are localized to their owners; each sequential process has its logical space on names and a local manager generates fresh names whenever necessary.

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A Proofs: strongly bound processes

The main result of this appendix is Theorem 2.5, that shows the operational and the denotational semantics of strongly bound processes are fully abstract. A number of intermediate definitions and lemmata are necessary, the proofs of which are contained herewith.

Remark A.1. To simplify the proof of full abstraction, hereafter we shall extend strongly bound processes with the process $!$, that models a non-terminated computation. The labelled transition system in Def. 2.2 is enriched with the following rule, that allows for observing the finite prefixes of non-terminating computations:

$$P, \mathcal{R} \xrightarrow{!} !, \mathcal{R}$$

The trace semantics $[\![P]\!]^{op}(\mathcal{R})$ is then defined as follows:

$$[\![P]\!]^{op}(\mathcal{R}) \;=\; \{\, \eta \mid P, \mathcal{R} \xrightarrow{\eta} \varepsilon, \mathcal{R}' \,\} \cup \{\, \eta\,! \mid P, \mathcal{R} \xrightarrow{\eta\,!} Q, \mathcal{R}' \text{ and } ! \notin \eta \,\}$$

Lemma A.2. For all strongly bound processes P and for all $\mathcal{R}$, if $\eta \in [\![P]\!]^{op}(\mathcal{R})$ and $! \in \eta$, then $\eta = \eta'\,!$ for some η' such that $! \notin \eta'$.

Proof. A straightforward inductive argument. □

We now define the function $\mathsf{R}(P)$, that computes the set of resources mentioned in P and reachable in some computations of P. To do that, $\mathsf{R}(P)$ performs a sort of reachability analysis, e.g. $\mathsf{R}(\alpha(r) \cdot (\mu h.\, h) \cdot \alpha(r'))$ contains r but not r', since the non-terminaing loop $\mu h.\, h$ makes $\alpha(r')$ unreachable. Having $\downarrow \in \mathsf{R}(P)$, means that P allows for some terminating computations. The function $\mathsf{T}(P)$ defined below exploits this fact to characterize the processes that may terminate.

Definition A.3. For all strongly bound processes P and for all functions Θ from variables h to $\mathsf{Res} \cup \{\downarrow\}$, we define $\mathsf{R}_\Theta(P)$ inductively as follows:

$$\mathsf{R}_\Theta(\varepsilon) = \{\downarrow\} \qquad \mathsf{R}_\Theta(!) = \emptyset \qquad \mathsf{R}_\Theta(h) = \Theta(h) \qquad \mathsf{R}_\Theta(\nu n.\, P) = \mathsf{R}_\Theta(P)$$

$$\mathsf{R}_\Theta(\alpha(\rho)) = \begin{cases} \{r, \downarrow\} & \text{if } \rho = r \\ \{\downarrow\} & \text{otherwise} \end{cases} \qquad \mathsf{R}_\Theta(P \cdot Q) = \begin{cases} (\mathsf{R}_\Theta(P) \setminus \{\downarrow\}) \cup \mathsf{R}_\Theta(Q) & \text{if } \downarrow \in \mathsf{R}_\Theta(P) \\ \mathsf{R}_\Theta(P) & \text{otherwise} \end{cases}$$

$$\mathsf{R}_\Theta(P + Q) = \mathsf{R}_\Theta(P) \cup \mathsf{R}_\Theta(Q) \qquad \mathsf{R}_\Theta(\mu h.\, P) = \mathsf{R}_{\Theta\{\{\downarrow\} \cap \mathsf{R}_{\Theta\{\emptyset/h\}}(P)/h\}}(P)$$

Also, we define $\mathsf{T}_\Theta(P)$ as follows:

$$\mathsf{T}_\Theta(P) \;=\; \{\downarrow\} \cap \mathsf{R}_\Theta(P)$$

Example 6. Let $P = \mu h.\, \nu n.\, h \cdot \alpha(r) + \alpha(n)$. Then:

$$\mathsf{R}_{\emptyset}(P) = \mathsf{R}_{\{\mathsf{T}_{\{\emptyset/h\}}(\nu n.\, h \cdot \alpha + \alpha(n))/h\}}(\nu n.\, h \cdot \alpha(r) + \alpha(n))$$

since $\mathsf{T}_{\{\emptyset/h\}}(\nu n.\, h \cdot \alpha(r) + \alpha(n)) = \mathsf{T}_{\{\emptyset/h\}}(h \cdot \alpha(r)) \cup \mathsf{T}_{\{\emptyset/h\}}(\alpha(n)) = \{\downarrow\}$, then:

$$\begin{aligned} &= \mathsf{R}_{\{\{\downarrow\}/h\}}(\nu n.\, h \cdot \alpha(r) + \alpha(n)) \\ &= \mathsf{R}_{\{\{\downarrow\}/h\}}(h \cdot \alpha(r)) \cup \mathsf{R}_{\{\{\downarrow\}/h\}}(\alpha(n)) \end{aligned}$$

and, since $\downarrow \in \mathsf{R}_{\{\{\downarrow\}/h\}}(h) = \{\downarrow\}$:

$$\begin{aligned} &= \mathsf{R}_{\{\{\downarrow\}/h\}}(h) \cup \mathsf{R}_{\{\{\downarrow\}/h\}}(\alpha(r)) \cup \{\downarrow\} \\ &= \{\downarrow, r\} \end{aligned}$$

Lemma A.4. For all P and Θ, we have that $\mathsf{T}_\Theta(P)$ equals to:

$$\begin{array}{ll} \{\downarrow\} & \text{if } P = \varepsilon \text{ or } P = \alpha(\rho) \\ \{\downarrow\} \cap \Theta(h) & \text{if } P = h \\ \emptyset & \text{if } P = \,! \\ \mathsf{T}_\Theta(Q) & \text{if } P = \nu n.\, Q \\ \mathsf{T}_\Theta(P_0) \cap \mathsf{T}_\Theta(P_1) & \text{if } P = P_0 \cdot P_1 \\ \mathsf{T}_\Theta(P_0) \cup \mathsf{T}_\Theta(P_1) & \text{if } P = P_0 + P_1 \\ \mathsf{T}_{\Theta\{\emptyset/h\}}(Q) & \text{if } P = \mu h.\, Q \end{array}$$

Proof. Straightforward structural induction. □

Lemma A.5. For all P, h, $\mathcal{R}$, $\mathcal{R}'$, Θ, and for all $\chi : \mathsf{Nam} \to \mathsf{Res}$:

(5a) $\quad \mathcal{R} \subseteq \mathcal{R}' \implies \mathsf{T}_{\Theta\{\mathcal{R}/h\}}(P) \subseteq \mathsf{T}_{\Theta\{\mathcal{R}'/h\}}(P)$

(5b) $\quad \mathsf{T}_\Theta(P) = \mathsf{T}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(P)$

(5c) $\quad \mathsf{T}_{\Theta\{\mathsf{T}_{\Theta\{\emptyset/h\}}(P)/h\}}(P) = \mathsf{T}_{\Theta\{\emptyset/h\}}(P)$

(5d) $\quad \mathcal{R} \subseteq \mathcal{R}' \implies \mathsf{R}_{\Theta\{\mathcal{R}/h\}}(P) \subseteq \mathsf{R}_{\Theta\{\mathcal{R}'/h\}}(P)$

(5e) $\quad \mathsf{R}_\Theta(P) \subseteq \mathsf{R}_\Theta(P\chi) \subseteq \mathsf{R}_\Theta(P) \cup \mathit{ran}(\chi)$

(5f) $\quad \mathsf{R}_\Theta(P) \subseteq \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(P) \cup \Theta(h)$

(5g) $\quad \mathsf{R}_\Theta(\mu h.\, P) = \mathsf{R}_{\Theta\{\mathsf{R}_\Theta(\mu h.\, P)/h\}}(P)$

Proof. The item (5a) is implied by (5d). The item (5b) is straightforward by induction on the size of P. To prove (5c) it suffices to consider two cases. If $\mathsf{T}_{\Theta\{\emptyset/h\}}(P) = \emptyset$, then $\mathsf{T}_{\Theta\{\mathsf{T}_{\Theta\{\emptyset/h\}}(P)/h\}}(P) = \mathsf{T}_{\Theta\{\emptyset/h\}}(P)$, which is the thesis. Otherwise $\mathsf{T}_{\Theta\{\emptyset/h\}}(P) = \{\downarrow\}$, and so $\mathsf{T}_{\Theta\{\mathsf{T}_{\Theta\{\emptyset/h\}}(P)/h\}}(P) = \mathsf{T}_{\Theta\{\{\downarrow\}/h\}}(P) \supseteq \mathsf{T}_{\Theta\{\emptyset/h\}}(P)$ (the last inclusion is by (5a), since $\{\downarrow\} \supseteq \emptyset$). The thesis follows by noting that $\mathsf{T}_{\Theta\{\emptyset/h\}}(P)$ can only take one of two values, i.e. either $\emptyset$ or $\{\downarrow\}$. The item (5d) can be easily proved by induction on the size of P.

We now prove the item (5e) by induction on the size of P.

- If $P = \varepsilon$, $P = !$, $P = h$, $P = \alpha(r)$, or $P = \alpha(n)$ with $n \notin dom(\chi)$, then the thesis is implied by $P = P\chi$.
- If $P = \alpha(n)$ and $n \in dom(\chi)$ then the thesis follows from $\mathsf{R}_\Theta(P) = \{\downarrow\}$, $\mathsf{R}_\Theta(P\chi) = \{\chi(n), \downarrow\}$, and $ran(\chi) = \{\chi(n)\}$.
- If $P = P_0 \cdot P_1$, by the induction hypothesis we obtain:

$$\begin{array}{l} \mathsf{R}_\Theta(P_0) \subseteq \mathsf{R}_\Theta(P_0\chi) \subseteq \mathsf{R}_\Theta(P_0) \cup ran(\chi) \\ \mathsf{R}_\Theta(P_1) \subseteq \mathsf{R}_\Theta(P_1\chi) \subseteq \mathsf{R}_\Theta(P_1) \cup ran(\chi) \end{array}$$

  Note that, since $\chi : \mathsf{Nam} \to \mathsf{Res}$ and $\downarrow \notin \mathsf{Res}$, then $\mathsf{T}_\Theta(P_0) = \mathsf{T}_\Theta(P_0\chi)$.

  - If $\mathsf{T}_\Theta(P_0) = \{\downarrow\}$, then we obtain:

$$\begin{aligned} \mathsf{R}_\Theta(P) &= (\mathsf{R}_\Theta(P_0) \setminus \{\downarrow\}) \cup \mathsf{R}_\Theta(P_1) \\ &\subseteq (\mathsf{R}_\Theta(P_0\chi) \setminus \{\downarrow\}) \cup \mathsf{R}_\Theta(P_1\chi) \\ &= \mathsf{R}_\Theta(P\chi) \\ &\subseteq ((\mathsf{R}_\Theta(P_0) \cup ran(\chi)) \setminus \{\downarrow\}) \cup \mathsf{R}_\Theta(P_1) \cup ran(\chi) \\ &= (\mathsf{R}_\Theta(P_0) \cup \setminus\{\downarrow\}) \cup \mathsf{R}_\Theta(P_1) \cup ran(\chi) \\ &= \mathsf{R}_\Theta(P) \cup ran(\chi) \end{aligned}$$

  - If instead $\mathsf{T}_\Theta(P_0) = \emptyset$, then the thesis follows from: $\mathsf{R}_\Theta(P) = \mathsf{R}_\Theta(P_0)$ and $\mathsf{R}_\Theta(P\chi) = \mathsf{R}_\Theta(P_0\chi)$

- If $P = P_0 + P_1$, the induction hypothesis suffices.
- If $P = \nu n.\, P'$, then $\mathsf{R}_\Theta(P) = \mathsf{R}_\Theta(P')$ and $\mathsf{R}_\Theta(P\chi) = \mathsf{R}_\Theta(P'\chi')$ where $\chi' = \chi\mid_{dom(\chi)\setminus\{n\}}$. By the induction hypothesis we obtain

$$\mathsf{R}_\Theta(P') \subseteq \mathsf{R}_\Theta(P'\chi') \subseteq \mathsf{R}_\Theta(P') \cup ran(\chi')$$

  Since $ran(\chi') \subseteq ran(\chi)$, the thesis holds.
- If $P = \mu h.\, P'$, then $\mathsf{R}_\Theta(P) = \mathsf{R}_{\Theta\{\mathsf{T}_{\Theta\{\emptyset/h\}}(P')/h\}}(P')$ and $\mathsf{R}_\Theta(P\chi) = \mathsf{R}_{\Theta\{\mathsf{T}_{\Theta\{\emptyset/h\}}(P'\chi)/h\}}(P'\chi)$. Since $\mathsf{T}_{\Theta\{\emptyset/h\}}(P') = \mathsf{T}_{\Theta\{\emptyset/h\}}(P'\chi)$, the induction hypothesis then suffices.

For (5f), we proceed by induction on the size of P.

- if $P = \varepsilon$, $P = \alpha(\rho)$ or $P = !$, trivial.
- if $P = h'$, there are two subcases.
  - If $h' = h$, then:

$$\mathsf{R}_\Theta(h) = \Theta(h) \subseteq \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(h) \cup \Theta(h)$$

- If $h' \neq h$, then:

$$\mathsf{R}_\Theta(h') = \Theta(h') = \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(h') \subseteq \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(h') \cup \Theta(h)$$

- if $P = \nu n.\, P'$, then by the induction hypothesis:

$$\begin{aligned}\mathsf{R}_\Theta(\nu n.\, P') &= \mathsf{R}_\Theta(P') \\ &\subseteq \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(P') \cup \Theta(h) \\ &= \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(\nu n.\, P') \cup \Theta(h)\end{aligned}$$

- if $P = P_0 \cdot P_1$, there are two subcases.
  - If $\mathsf{T}_\Theta(P_0) = \{\downarrow\}$, then:

    $$\mathsf{R}_\Theta(P_0 \cdot P_1) = \mathsf{R}_\Theta(P_0) \cup \mathsf{R}_\Theta(P_1)$$

    and by the induction hypothesis:

    $$\subseteq \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(P_0) \cup \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(P_1) \cup \Theta(h)$$

    by (5b), $\mathsf{T}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(P_0) = \mathsf{T}_\Theta(P_0) = \{\downarrow\}$, thus:

    $$= \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(P_0 \cdot P_1) \cup \Theta(h)$$

  - If $\mathsf{T}_\Theta(P_0) = \emptyset$, then:

    $$\mathsf{R}_\Theta(P_0 \cdot P_1) = \mathsf{R}_\Theta(P_0)$$

    and by the induction hypothesis:

    $$\subseteq \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(P_0) \cup \Theta(h)$$

    by (5b), $\mathsf{T}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(P_0) = \mathsf{T}_\Theta(P_0) = \emptyset$, thus:

    $$= \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(P_0 \cdot P_1) \cup \Theta(h)$$

- if $P = P_0 + P_1$, trivial application of the induction hypothesis.
- if $P = \mu h'.\, P'$, there are two subcases.
  - If $h' = h$, let $\Theta' = \Theta\{\mathsf{T}_\Theta(h)/h\}$. Then:

    $$\begin{aligned}\mathsf{R}_\Theta(\mu h.\, P') &= \mathsf{R}_{\Theta\{\mathsf{T}_{\Theta\{\emptyset/h\}}(P')/h\}}(P') \\ &= \mathsf{R}_{\Theta'\{\mathsf{T}_{\Theta\{\emptyset/h\}}(P')/h\}}(P') \\ &= \mathsf{R}_{\Theta'\{\mathsf{T}_{\Theta'\{\emptyset/h\}}(P')/h\}}(P') \\ &= \mathsf{R}_{\Theta'}(\mu h.\, P') \\ &\subseteq \mathsf{R}_{\Theta'}(\mu h.\, P') \cup \Theta(h)\end{aligned}$$

– If $h' \neq h$, let $\Theta' = \Theta\{\mathsf{T}_{\Theta\{\emptyset/h'\}}(P')/h'\}$. Then:

$$\mathsf{R}_\Theta(\mu h.\, P') = \mathsf{R}_{\Theta'}(P')$$

by the induction hypothesis:

$$\subseteq \mathsf{R}_{\Theta'\{\mathsf{T}_{\Theta'}(h)/h\}}(P') \cup \Theta(h)$$

since $\Theta'(h) = \Theta(h)$ then $\mathsf{T}_{\Theta'}(h) = \mathsf{T}_\Theta(h)$, and so:

$$= \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(h)/h\}\{\mathsf{T}_{\Theta\{\emptyset/h'\}}(P')/h'\}}(P') \cup \Theta(h)$$

let $\Theta'' = \Theta\{\mathsf{T}_\Theta(h)/h\}$. Then, $\mathsf{T}_{\Theta''\{\emptyset/h'\}}(P') = \mathsf{T}_{\Theta\{\mathsf{T}_\Theta(h)/h\}\{\emptyset/h'\}}(P') = \mathsf{T}_{\Theta\{\emptyset/h'\}\{\mathsf{T}_{\Theta\{\emptyset/h'\}}(h)/h\}}(P') = \mathsf{T}_{\Theta\{\emptyset/h'\}}(P')$, and so:

$$= \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(h)/h\}\{\mathsf{T}_{\Theta''\{\emptyset/h'\}}(P')/h'\}}(P') \cup \Theta(h)$$
$$= \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(h)/h\}}(\mu h'.\, P') \cup \Theta(h)$$

For (5g), we have that:

$$\mathsf{R}_\Theta(\mu h.\, P) = \mathsf{R}_{\Theta\{\mathsf{T}_{\Theta\{\emptyset/h\}}(P)/h\}}(P)$$

by Lemma 5d:

$$\subseteq \mathsf{R}_{\Theta\{\mathsf{R}_{\Theta\{\emptyset/h\}}(P)/h\}}(P)$$

again, by Lemma 5d:

$$\subseteq \mathsf{R}_{\Theta\{\mathsf{R}_{\Theta\{\mathsf{T}_{\Theta\{\emptyset/h\}}(P)/h\}}(P)/h\}}(P)$$
$$= \mathsf{R}_{\Theta\{\mathsf{R}_\Theta(\mu h.\, P)/h\}}(P)$$

by Lemma 5f:

$$\subseteq \mathsf{R}_{\Theta\{\{\downarrow\}\cap\mathsf{R}_\Theta(\mu h.\, P)/h\}}(P) \cup \mathsf{R}_\Theta(\mu h.\, P)$$
$$= \mathsf{R}_{\Theta\{\mathsf{T}_\Theta(\mu h.\, P)/h\}}(P) \cup \mathsf{R}_\Theta(\mu h.\, P)$$

by Lemma A.4:

$$= \mathsf{R}_{\Theta\{\mathsf{T}_{\Theta\{\emptyset/h\}}(P)/h\}}(P) \cup \mathsf{R}_\Theta(\mu h.\, P)$$
$$= \mathsf{R}_\Theta(\mu h.\, P)$$

$\square$

Lemma A.6. For all closed strongly bound processes P:

(6a) $\eta \in [\![P]\!]^{op}(\mathcal{R}) \implies (\mathcal{R} \setminus \mathsf{R}_{\emptyset}(P)) \cap \mathsf{R}(\eta) = \emptyset$

(6b) $\mathcal{R} \subseteq \mathcal{R}' \implies [\![P]\!]^{op}(\mathcal{R}) \supseteq [\![P]\!]^{op}(\mathcal{R}')$

Proof. The item (a) follows by a simple inspection of the rules in Def. 2.2. The item (b) is straightforward by induction on the size of P. The only relevant rule is that for $\nu n. P$, for which we have:

$$\begin{aligned}
[\![\nu n. P]\!]^{op}(\mathcal{R}) &= \bigcup_{r \notin \mathcal{R}} [\![P\{r/n\}]\!]^{op}(\mathcal{R} \cup \{r\}) \\
&\supseteq \bigcup_{r \notin \mathcal{R}} [\![P\{r/n\}]\!]^{op}(\mathcal{R}' \cup \{r\}) \\
&\supseteq \bigcup_{r \notin \mathcal{R}'} [\![P\{r/n\}]\!]^{op}(\mathcal{R}' \cup \{r\}) \\
&= [\![\nu n. P]\!]^{op}(\mathcal{R}')
\end{aligned}$$

□

Lemma A.7. For all strongly bound processes P and P':

(7a) $P, \mathcal{R} \xrightarrow{\eta} P', \mathcal{R}' \implies P, \mathcal{R} \setminus \{r\} \xrightarrow{\eta} P', \mathcal{R}' \setminus \{r\}$ if $r \in \mathcal{R} \setminus \mathsf{R}(\eta)$

(7b) $P, \mathcal{R} \xrightarrow{\eta} P', \mathcal{R}' \implies P, \mathcal{R} \cup \{r\} \xrightarrow{\eta} P', \mathcal{R}' \cup \{r\}$ if $r \notin \mathcal{R}'$

(7c) $P\{r/n\}, \mathcal{R} \xrightarrow{\eta} \varepsilon, \mathcal{R}' \implies P\{r'/n\}, \mathcal{R} \xrightarrow{\eta} \varepsilon, \mathcal{R}'$ if $r \notin \mathsf{R}(\eta)$

(7d) $P\{r/n\}, \mathcal{R} \xrightarrow{\eta} \varepsilon, \mathcal{R}' \implies P\{r'/n\}, \mathcal{R}\{r'/r\} \xrightarrow{\eta} \varepsilon, \mathcal{R}'\{r'/r\}$ if $r \notin \mathsf{R}(\eta)$ and $r' \notin \mathcal{R}'$

Proof. The first three items are straightforward by induction on the number of transitions. For (7d), we proceed by induction on the number of transitions. In the base case there are zero transitions and the thesis trivially holds. Otherwise, let $\sigma = \{r'/r\}$. We consider the following exhaustive cases.

- if $P = \alpha(\rho)$, there are the following subcases:
  - If $\rho \in \mathsf{Res}$, there are two subcases. If $\rho = \bar{r}$ and $\bar{r} = r$, then $r \in \mathsf{R}(\eta)$, so contradicting the hypothesis. If $\rho = \bar{r}$ and $\bar{r} \neq r$, trivial.
  - if $\rho \in \mathsf{Nam}$, then $\rho = n$, otherwise the semantics of $P\{r/n\}$ is undefined. Thus, $\eta = \alpha(r)$, so contradicting $r \notin \mathsf{R}(\eta)$.
- if $P = !$, contradiction with the hypothesis.
- if $P = \nu m. \bar{P}$, then we consider two further subcases.

– if $n \neq m$, we have:

$$(\nu m.\, P)\{r/n\},\, \mathcal{R} \xrightarrow{\varepsilon} P\{r/n\}\{\bar{r}/m\},\, \mathcal{R} \cup \{\bar{r}\} \xrightarrow{\eta} \varepsilon,\, \mathcal{R}'$$

for some $\bar{r} \notin \mathcal{R}$. There are two further subcases:

∗ if $\bar{r} = r$, then by the induction hypothesis:

$$P\{\bar{r}/m\}\{r'/n\},\, (\mathcal{R} \cup \{\bar{r}\})\sigma \xrightarrow{\eta} \varepsilon,\, \mathcal{R}'\sigma$$

since $\bar{r} = r$:

$$P\{r'/n\}\{\bar{r}/m\},\, \mathcal{R}\sigma \cup \{r'\} \xrightarrow{\eta} \varepsilon,\, \mathcal{R}'\sigma$$

Since $\bar{r} = r \notin \mathsf{R}(\eta)$, then by (7c):

$$P\{r'/n\}\{r'/m\},\, \mathcal{R}\sigma \cup \{r'\} \xrightarrow{\eta} \varepsilon,\, \mathcal{R}'\sigma$$

Since $r' \notin \mathcal{R}' \supseteq \mathcal{R}$ and $r = \bar{r} \notin \mathcal{R}$, then $r' \notin \mathcal{R}\sigma$, so we conclude:

$$(\nu m.\, P)\{r'/n\},\, \mathcal{R}\sigma \xrightarrow{\varepsilon} P\{r'/m, r'/n\},\, \mathcal{R}\sigma \cup \{r'\}$$

∗ if $\bar{r} \neq r$, by the induction hypothesis:

$$P\{\bar{r}/m\}\{r'/n\},\, (\mathcal{R} \cup \{\bar{r}\})\sigma \xrightarrow{\eta} \varepsilon,\, \mathcal{R}'\sigma$$

and since $\bar{r} \neq r$:

$$P\{\bar{r}/m\}\{r'/n\},\, \mathcal{R}\sigma \cup \{\bar{r}\} \xrightarrow{\eta} \varepsilon,\, \mathcal{R}'\sigma$$

Since $\bar{r} \notin \mathcal{R}$ and $\bar{r} \neq r$ then $\bar{r} \notin \mathcal{R}\sigma$, so we conclude:

$$(\nu m.\, P)\{r'/n\},\, \mathcal{R}\sigma \xrightarrow{\varepsilon} P\{r'/n\}\{\bar{r}/m\},\, \mathcal{R}\sigma \cup \{\bar{r}\}$$

– If $n = m$, we have $(\nu m.\, P)\{r/n\} = \nu m.\, P$ and the proof is similar yet simpler of that for the case $n \neq m$.

- if $P = P_0 \cdot P_1$, then we can easily show that if $P\{r/n\}, \mathcal{R} \xrightarrow{\eta} \varepsilon, \mathcal{R}'$, then $\eta = \eta_0\eta_1$ for two shorter derivations:

$$P_0\{r/n\}, \mathcal{R} \xrightarrow{\eta_0} \varepsilon, \bar{\mathcal{R}} \qquad P_1\{r/n\}, \bar{\mathcal{R}} \xrightarrow{\eta_1} \varepsilon, \mathcal{R}'$$

Therefore, by the induction hypothesis:

$$P_0\{r'/n\}, \mathcal{R}\sigma \xrightarrow{\eta_0} \varepsilon, \bar{\mathcal{R}}\sigma \qquad P_1\{r'/n\}, \bar{\mathcal{R}}\sigma \xrightarrow{\eta_1} \varepsilon, \mathcal{R}'\sigma$$

which implies the thesis.

- if $P = P_0 + P_1$, the induction hypothesis suffices.
- if $P = \mu h.\bar{P}$, then $P\{r/n\} = \mu h.\,(\bar{P}\{r/n\})$, and thus the thesis follows directly from the induction hypothesis.

$\square$

Lemma A.8. For all strongly bound processes P, P':

$$P, \mathcal{R} \xrightarrow{a} P', \mathcal{R}' \implies [\![P]\!]^{op}(\mathcal{R}) \supseteq a \odot [\![P']\!]^{op}(\mathcal{R}')$$

Proof. By cases on the form of P. There are the following cases:

- if $P = \alpha(r)$ then $P' = \varepsilon$, $a = \alpha(r)$ and $\mathcal{R}' = \mathcal{R}$. Then:

$$[\![P]\!]^{op}(\mathcal{R}) = \{\alpha(r)\} = \alpha(r) \odot \{\varepsilon\} = \alpha(r) \odot [\![P']\!]^{op}(\mathcal{R}')$$

- if $P = !$, then $P' = !$ and $\mathcal{R}' = \mathcal{R}$. Thus:

$$[\![!]\!]^{op}(\mathcal{R}) = \{!\} = ! \odot \{!\} = ! \odot [\![!]\!]^{op}(\mathcal{R}')$$

- if $P = \nu n.\,\bar{P}$, then $P' = \bar{P}\{r/n\}$, for some $r \notin \mathcal{R}$, and $a = \varepsilon$, $\mathcal{R}' = \mathcal{R} \cup \{r\}$.

$$[\![P]\!]^{op}(\mathcal{R}) = \bigcup_{r' \notin \mathcal{R}} [\![\bar{P}\{r'/n\}]\!]^{op}(\mathcal{R} \cup \{r'\}) \supseteq [\![P']\!]^{op}(\mathcal{R}') = \varepsilon \odot [\![P']\!]^{op}(\mathcal{R}')$$

- if $P = P_0 \cdot P_1$, there are two further subcases. If $P_0 = \varepsilon$, then $P' = P_1$, $a = \varepsilon$ and $\mathcal{R}' = \mathcal{R}$. Thus, $[\![P]\!]^{op}_{\mathcal{R}} = [\![P']\!]^{op}_{\mathcal{R}'} = \varepsilon \odot [\![P']\!]^{op}_{\mathcal{R}'}$. Otherwise, if $P_0 \neq \varepsilon$, then $P_0, \mathcal{R} \xrightarrow{a} P_0', \mathcal{R}'$, and so $P' = P_0' \cdot P_1$. Thus:

$$\begin{aligned}
[\![P]\!]^{op}_{\mathcal{R}} &= \{\, \eta \mid P_0 \cdot P_1,\, \mathcal{R} \xrightarrow{\eta} \varepsilon,\, \bar{\mathcal{R}} \,\} \cup \{\, \eta! \mid P_0 \cdot P_1,\, \mathcal{R} \xrightarrow{\eta!} \bar{P},\, \bar{\mathcal{R}} \,\} \\
&\supseteq \{\, a\,\eta \mid P_0' \cdot P_1,\, \mathcal{R}' \xrightarrow{\eta} \varepsilon,\, \bar{\mathcal{R}} \,\} \cup \{\, a\,\eta\,! \mid P_0' \cdot P_1,\, \mathcal{R}' \xrightarrow{\eta!} \bar{P},\, \bar{\mathcal{R}} \,\} \\
&= a \odot [\![P_0' \cdot P_1]\!]^{op}(\mathcal{R}') \\
&= a \odot [\![P']\!]^{op}(\mathcal{R}')
\end{aligned}$$

- if $P = P_0 + P_1$, straightforward.
- if $P = \mu h.\, P''$, then $a = \varepsilon$, $P' = P''\{P/h\}$ and $\mathcal{R}' = \mathcal{R}$. Therefore:

$$\begin{aligned}
[\![P]\!]^{op}_{\mathcal{R}} &\supseteq \{\, \varepsilon\,\eta \mid P''\{P/h\},\, \mathcal{R}' \xrightarrow{\eta} \varepsilon,\, \bar{\mathcal{R}} \,\} \cup \{\, \varepsilon\,\eta\,! \mid P''\{P/h\},\, \mathcal{R}' \xrightarrow{\eta!} \bar{P},\, \bar{\mathcal{R}} \,\} \\
&= \varepsilon \odot [\![P']\!]^{op}(\mathcal{R}')
\end{aligned}$$

$\square$

We now introduce a further denotational semantics $[\![-]\!]^{sub}$ of strongly bound processes. The main difference from $[\![-]\!]^{s}$ of Def. 2.4 is the way the two semantics handle the case $\nu n.\,P$. In $[\![\nu n.\,P]\!]^{s}$, the freshly created resource r is used to extend the environment χ with the binding $\{r/n\}$. Instead, in $[\![\nu n.\,P]\!]^{sub}$ the substitution $\{r/n\}$ is performed directly on P — hence the environment χ can be omitted. Since substitutions are also used by $[\![-]\!]^{op}$, in the proof of full abstraction we shall conveniently use $[\![-]\!]^{sub}$ as a bridge between $[\![-]\!]^{op}$ and $[\![-]\!]^{s}$.

Definition A.9. Substitution semantics of strongly bound processes

The substitution semantics $[\![P]\!]^{sub}_{\theta}$ of a strongly bound process P such that $fn(P) = \emptyset$ is defined below. Let D_0 be the following cpo of sets of traces ordered by set inclusion: $D_0 = \{\, X \subseteq (\mathsf{Ev} \cup \{\,!\})^* \mid\, ! \in X \;\wedge\; \forall \eta \in X : \eta! \in X \,\}$. The set $\{\,!\}$ is the bottom element of D_0. Then, let $D_{sub} = \mathcal{P}_{\mathit{fin}}(\mathsf{Res}) \to D_0$ be the cpo of functions from the finite subsets of Res to D_0. Note that the bottom element $\perp$ of D_{sub} is $\lambda\mathcal{R}.\,\{\,!\}$. Then, the semantics of P (parametrized by θ) is a function in D_{sub}. The parameter θ is a function that maps each variable h to a function in D_{sub}. We require $dom(\theta) \supseteq fv(P)$. The semantics $[\![P]\!]^{sub}_{\theta}$ is inductively defined through the following equations.

$$
\begin{aligned}
[\![\varepsilon]\!]^{sub}_{\theta} &= \lambda\mathcal{R}.\,\{\,!, \varepsilon\} \\
[\![\alpha(\rho)]\!]^{sub}_{\theta} &= \lambda\mathcal{R}.\,\{\,!, \alpha(\rho), \alpha(\rho)\,!\} \quad \text{if } \rho \in \mathsf{Res} \\
[\![\nu n.\,P]\!]^{sub}_{\theta} &= \lambda\mathcal{R}.\,\textstyle\bigcup_{r \notin \mathcal{R}}\; [\![P\{r/n\}]\!]^{sub}_{\theta}(\mathcal{R} \cup \{r\}) \\
[\![P \cdot P']\!]^{sub}_{\theta} &= [\![P]\!]^{sub}_{\theta} \boxdot [\![P']\!]^{sub}_{\theta} \\
[\![P + P']\!]^{sub}_{\theta} &= [\![P]\!]^{sub}_{\theta} \sqcup [\![P']\!]^{sub}_{\theta} \\
[\![!]\!]^{sub}_{\theta} &= \perp \\
[\![h]\!]^{sub}_{\theta} &= \theta(h) \\
[\![\mu h.P]\!]^{sub}_{\theta} &= \bigsqcup_{i \geq 0} f^i(\perp) \;\; \text{where} \;\; f(Y) = [\![P]\!]^{sub}_{\theta\{Y/h\}}
\end{aligned}
$$

We first check that the above semantics is well-defined. Lemma A.10 proves that the image of the semantics function is indeed D_0. Lemma A.13 guarantees that the least upper bound in the last equation exists (since f is monotone). Also, since f is continuous, by the Knaster-Tarski theorem the semantics of $\mu h.\,P$ is the least fixed point of f.

Lemma A.10. For all strongly bound processes P, for all θ and $\mathcal{R}$, $! \in [\![P]\!]^{sub}_{\theta}(\mathcal{R})$.

Proof. Trivial. $\square$

Lemma A.11. The structure $(D_{sub}, \sqcup, \boxdot, id_{\sqcup}, id_{\boxdot})$, where $id_{\sqcup} = \bot$ and $id_{\boxdot} = \lambda\mathcal{R}.\{\,!, \varepsilon\}$ is a semi-ring.

Proof. It is easy to check that $\sqcup$ and $\boxdot$ are associative, $id_{\sqcup}, id_{\boxdot}$ are their identities, and that, for all $X, Y, Z \in D_{den}$, $(X \sqcup Y) \boxdot Z = X \boxdot Z \sqcup Y \boxdot Z$ and $Z \boxdot (X \sqcup Y) = (Z \boxdot X) \sqcup (Z \boxdot Y)$. □ □

Lemma A.12. Let $\{Y_i\}_i$ and $\{Z_i\}_i$ be subsets of D_{sub}. Then:

$$\bigsqcup_i (Y_i \boxdot Z_i) \;=\; (\bigsqcup_i Y_i) \boxdot (\bigsqcup_i Z_i)$$

Proof. We prove the following three facts. For all $W \in D_{sub}$:

(12a) $$W \boxdot \bigsqcup_i Z_i \;=\; \bigsqcup_i (W \boxdot Z_i)$$

(12b) $$(\bigsqcup_i Z_i) \boxdot W \;=\; \bigsqcup_i (Z_i \boxdot W)$$

For (12a), we have that:

$$\begin{aligned}
W \boxdot \bigsqcup_i Z_i &= \lambda\mathcal{R}.\ \bigcup \{\, \eta_0 \odot (\bigsqcup_i Z_i)(\mathcal{R} \cup \mathsf{R}(\eta_0)) \mid \eta_0 \in W(\mathcal{R})\,\} \\
&= \lambda\mathcal{R}.\ \bigcup \{\, \eta_0 \odot \bigcup_i Z_i(\mathcal{R} \cup \mathsf{R}(\eta_0)) \mid \eta_0 \in W(\mathcal{R})\,\} \\
&= \lambda\mathcal{R}.\ \bigcup \{ \bigcup_i \eta_0 \odot Z_i(\mathcal{R} \cup \mathsf{R}(\eta_0)) \mid \eta_0 \in W(\mathcal{R})\,\} \\
&= \lambda\mathcal{R}.\ \bigcup_i \bigcup \{\, \eta_0 \odot Z_i(\mathcal{R} \cup \mathsf{R}(\eta_0)) \mid \eta_0 \in W(\mathcal{R})\,\} \\
&= \bigsqcup_i \lambda\mathcal{R}.\ \bigcup \{\, \eta_0 \odot Z_i(\mathcal{R} \cup \mathsf{R}(\eta_0)) \mid \eta_0 \in W(\mathcal{R})\,\} \\
&= \bigsqcup_i (W \boxdot Z_i)
\end{aligned}$$

For (12b), we have that:

$$\begin{aligned}
(\bigsqcup_i Z_i) \boxdot W &= \lambda\mathcal{R}.\ \bigcup \{\, \eta_0 \odot W(\mathcal{R} \cup \mathsf{R}(\eta_0)) \mid \eta_0 \in (\bigsqcup_i Z_i)(\mathcal{R})\,\} \\
&= \lambda\mathcal{R}.\ \bigcup \{\, \eta_0 \odot W(\mathcal{R} \cup \mathsf{R}(\eta_0)) \mid \eta_0 \in \bigcup_i Z_i(\mathcal{R})\,\} \\
&= \lambda\mathcal{R}.\ \bigcup_i \bigcup \{\, \eta_0 \odot W(\mathcal{R} \cup \mathsf{R}(\eta_0)) \mid \eta_0 \in Z_i(\mathcal{R})\,\} \\
&= \bigsqcup_i \lambda\mathcal{R}.\ \bigcup \{\, \eta_0 \odot W(\mathcal{R} \cup \mathsf{R}(\eta_0)) \mid \eta_0 \in Z_i(\mathcal{R})\,\} \\
&= \bigsqcup_i (Z_i \boxdot W)
\end{aligned}$$

Summing up:

$$
\begin{aligned}
\big(\bigsqcup_i Y_i\big) \boxdot \big(\bigsqcup_i Z_i\big) &= \bigsqcup_j \big(\bigsqcup_i Y_i\big) \boxdot Z_j && \text{by (12a)} \\
&= \bigsqcup_j \bigsqcup_i (Y_i \boxdot Z_j) && \text{by (12b)} \\
&= \bigsqcup_i (Y_i \boxdot Z_i)
\end{aligned}
$$

which concludes the proof. □

Lemma A.13. For all strongly bound processes P such that $\mathit{fn}(P) = \emptyset$, and for all θ such that $\mathit{dom}(\theta) \cup \{h\} \supseteq \mathit{fv}(P)$, the function $f_P(Y) = [\![P]\!]^{sub}_{\theta\{Y/h\}}$ is continuous.

Proof. By induction on the size of P. Let $\{Y_i\}_i$ be a ω-chain of elements in D_{sub}. We have the following cases on P:

- the cases $P = \varepsilon$, $P = \alpha(r)$, $P = !$ are trivial.
- if $P = \nu n.\, P'$, then:

$$f_P(\bigsqcup_i Y_i) = \lambda\mathcal{R}.\ \bigcup_{r \notin \mathcal{R}} f_{P'\{r/n\}}(\bigsqcup_i Y_i)(\mathcal{R} \cup \{r\})$$

  Then, by the induction hypothesis:

$$
\begin{aligned}
&= \lambda\mathcal{R}.\ \bigcup_{r \notin \mathcal{R}} \big(\bigsqcup_i f_{P'\{r/n\}}(Y_i)\big)(\mathcal{R} \cup \{r\}) \\
&= \lambda\mathcal{R}.\ \bigcup_{r \notin \mathcal{R}} \bigcup_i f_{P'\{r/n\}}(Y_i)(\mathcal{R} \cup \{r\}) \\
&= \lambda\mathcal{R}.\ \bigcup_i \bigcup_{r \notin \mathcal{R}} f_{P'\{r/n\}}(Y_i)(\mathcal{R} \cup \{r\}) \\
&= \bigsqcup_i \lambda\mathcal{R}.\ \bigcup_{r \notin \mathcal{R}} f_{P'\{r/n\}}(Y_i)(\mathcal{R} \cup \{r\}) \\
&= \bigsqcup_i f_P(Y_i)
\end{aligned}
$$

- if $P = P' \cdot P''$, then:

$$
\begin{aligned}
f_P(\bigsqcup_i Y_i) &= [\![P' \cdot P'']\!]^{sub}_{\theta\{\bigsqcup_i Y_i/h\}} \\
&= [\![P']\!]^{sub}_{\theta\{\bigsqcup_i Y_i/h\}} \boxdot [\![P'']\!]^{sub}_{\theta\{\bigsqcup_i Y_i/h\}} \\
&= f_{P'}(\bigsqcup_i Y_i) \boxdot f_{P''}(\bigsqcup_i Y_i)
\end{aligned}
$$

and, by the induction hypothesis:

$$
\begin{aligned}
&= \bigsqcup_i f_{P'}(Y_i) \ \boxdot \ \bigsqcup_i f_{P''}(Y_i) \\
&= \bigsqcup_i [\![P']\!]^{sub}_{\theta\{Y_i/h\}} \boxdot \bigsqcup_i [\![P'']\!]^{sub}_{\theta\{Y_i/h\}}
\end{aligned}
$$

by Lemma A.12:

$$
\begin{aligned}
&= \bigsqcup_i \left([\![P']\!]^{sub}_{\theta\{Y_i/h\}} \boxdot [\![P'']\!]^{sub}_{\theta\{Y_i/h\}}\right) \\
&= \bigsqcup_i [\![P]\!]^{sub}_{\theta\{Y_i/h\}} \\
&= \bigsqcup_i f_P(Y_i)
\end{aligned}
$$

- if $P = P' + P''$, trivial.
- if $P = \mu h'. P'$, then:

$$
f_P(\bigsqcup_i Y_i) = \bigsqcup_{n\geq 0} (\lambda Y'. [\![P']\!]^{sub}_{\theta\{\bigsqcup_i Y_i/h\}\{Y'/h'\}})^n(\bot)
$$

if $h = h'$, then:

$$
\begin{aligned}
&= \bigsqcup_{n\geq 0} (\lambda Y'. [\![P']\!]^{sub}_{\theta\{Y'/h'\}})^n(\bot) \\
&= \bigsqcup_i \bigsqcup_{n\geq 0} (\lambda Y'. [\![P']\!]^{sub}_{\theta\{Y'/h'\}})^n(\bot) \\
&= \bigsqcup_i \bigsqcup_{n\geq 0} (\lambda Y'. [\![P']\!]^{sub}_{\theta\{Y_i/h\}\{Y'/h'\}})^n(\bot) \\
&= \bigsqcup_i f_P(Y_i)
\end{aligned}
$$

otherwise, if $h \neq h'$, then:

$$
= \bigsqcup_{n\geq 0} (\lambda Y'. [\![P']\!]^{sub}_{\theta\{Y'/h'\}\{\bigsqcup_i Y_i/h\}})^n(\bot)
$$

and by the induction hypothesis:

$$
\begin{aligned}
&= \bigsqcup_{n\geq 0} (\lambda Y'. \bigsqcup_i [\![P']\!]^{sub}_{\theta\{Y'/h'\}\{Y_i/h\}})^n(\bot) \\
&= \bigsqcup_{n\geq 0} (\bigsqcup_i \lambda Y'. [\![P']\!]^{sub}_{\theta\{Y'/h'\}\{Y_i/h\}})^n(\bot)
\end{aligned}
$$

by the induction hypothesis on P', $f_{P'}$ is continuous – and hence monotone – then $\{\lambda Y'.\, [\![P']\!]^{sub}_{\theta\{Y_i/h\}}\}_i$ is an ω-chain, and so:

$$
\begin{aligned}
&= \bigsqcup_i \bigsqcup_{n\geq 0} (\lambda Y'.\, [\![P']\!]^{sub}_{\theta\{Y'/h'\}\{Y_i/h\}})^n(\bot) \\
&= \bigsqcup_i \bigsqcup_{n\geq 0} (\lambda Y'.\, [\![P']\!]^{sub}_{\theta\{Y_i/h\}\{Y'/h'\}})^n(\bot) \\
&= \bigsqcup_i f_P(Y_i)
\end{aligned}
$$

□

Lemma A.14. We say $Y \in D_{sub}$ *anti-monotone* when $\mathcal{R} \subseteq \mathcal{R}'$ implies $Y(\mathcal{R}) \supseteq Y(\mathcal{R}')$ for all $\mathcal{R}, \mathcal{R}'$. For all P and anti-monotone θ, $[\![P]\!]^{sub}_\theta$ is anti-monotone.

Proof. By induction on the size of P. There are the following cases:

- if $P = \varepsilon$, $P = !$, $P = \alpha(r)$, trivial.
- if $P = h$, the thesis is implied by the anti-monotonicity of θ.
- if $P = P_0 \cdot P_1$ and $P = P_0 + P_1$, straightforward application of the induction hypothesis.
- if $P = \nu n.\, P'$, we have that:

  $$
  \begin{aligned}
  [\![P]\!]^{sub}_\theta(\mathcal{R}) &= \bigcup_{r\notin\mathcal{R}} [\![P\{r/n\}]\!]^{sub}_\theta(\mathcal{R}\cup\{r\}) \\
  &\subseteq \bigcup_{r\notin\mathcal{R}'} [\![P\{r/n\}]\!]^{sub}_\theta(\mathcal{R}\cup\{r\})
  \end{aligned}
  $$

  and by the induction hypothesis, since $\mathcal{R}'\cup\{r\} \subseteq \mathcal{R}\cup\{r\}$:

  $$
  \begin{aligned}
  &\subseteq \bigcup_{r\notin\mathcal{R}'} [\![P\{r/n\}]\!]^{sub}_\theta(\mathcal{R}'\cup\{r\}) \\
  &= [\![P]\!]^{sub}_\theta(\mathcal{R}')
  \end{aligned}
  $$

- if $P = \mu h.\, P'$, we prove that if Y is anti-monotone, then also $f(Y) = [\![P']\!]^{sub}_{\theta\{Y/h\}}$ is anti-monotone. Since θ and Y are anti-monotone, then $\theta\{Y/h\}$ is anti-monotone, so by the induction hypothesis:

  $$f(Y)(\mathcal{R}) = [\![P']\!]^{sub}_{\theta\{Y/h\}}(\mathcal{R}) \supseteq [\![P']\!]^{sub}_{\theta\{Y/h\}}(\mathcal{R}') = f(Y)(\mathcal{R}')$$

  From here it is simple to check that for all n, $f^n(\bot)$ is anti-monotone, and so this implies that $[\![P]\!]^{sub}_\theta$ is anti-monotone.

□

Definition A.15. For all $Y \in D_{sub}$, we define $\mathsf{R}(Y)$ and $\mathsf{T}(Y)$ as follows:

$$\mathsf{R}(Y) = \bigcap_{\mathcal{R}} \mathsf{R}(Y(\mathcal{R}))$$
$$\mathsf{T}(Y) = \{\downarrow\} \cap \mathsf{R}(Y)$$

Lemma A.16. For all $Y \in D_{sub}$:

$$\mathsf{T}(Y) = \begin{cases} \emptyset & \text{if } \forall \mathcal{R} : \eta \in Y(\mathcal{R}) \implies ! \in \eta \\ \{\downarrow\} & \text{otherwise} \end{cases}$$

Proof. First note that, by a straightforward inspection of the rules in Def. A.9, it follows that, for all $Y \in D_{sub}$:

(1) $\qquad \exists \mathcal{R} : \exists \eta \in Y(\mathcal{R}) \ : \ ! \notin \eta \iff \forall \mathcal{R} : \exists \eta \in Y(\mathcal{R}) \ : \ ! \notin \eta$

To prove the main statement,

$$\begin{aligned} \mathsf{T}(Y) = \emptyset &\iff \downarrow \notin \mathsf{R}(Y) \\ &\iff \exists \mathcal{R} : \downarrow \notin \mathsf{R}(Y(\mathcal{R})) \\ &\iff \exists \mathcal{R} : \forall \eta \in Y(\mathcal{R}) : \downarrow \notin \mathsf{R}(\eta) \\ &\iff \exists \mathcal{R} : \forall \eta \in Y(\mathcal{R}) : ! \in \eta \end{aligned}$$

by (1), used contrapositively, we obtain the thesis:

$$\iff \forall \mathcal{R} : \forall \eta \in Y(\mathcal{R}) : ! \in \eta$$

□

Example 7. The function R is not continuous.

Let $\{r_i\}_{i \in \omega}$ a set of distinct resources. Let $\{Y_i\}_{i \in \omega}$ a family of functions in D_{sub}, defined as

$$Y_i = \lambda \mathcal{R}. \{\, \alpha(r_k) \mid k > |\mathcal{R}| - i \,\}$$

where $|\mathcal{R}|$ denotes the cardinality of the finite set $\mathcal{R}$. This family is actually an ω-chain, since

$$k > |\mathcal{R}| - i \implies k > |\mathcal{R}| - (i+1)$$

Then,

$$\begin{aligned}
\mathsf{R}(\bigsqcup_i Y_i) &= \bigcap_{\mathcal{R}} \bigcup \{\, \mathsf{R}(\eta) \mid \eta \in (\bigsqcup_i Y_i)(\mathcal{R}) \,\} \\
&= \bigcap_{\mathcal{R}} \bigcup \{\, \mathsf{R}(\eta) \mid \eta \in \bigcup_i Y_i(\mathcal{R}) \,\} \\
&= \bigcap_{\mathcal{R}} \bigcup_i \bigcup \{\, \mathsf{R}(\eta) \mid \eta \in Y_i(\mathcal{R}) \,\} \\
&= \bigcap_{\mathcal{R}} \bigcup_i \{\, r_k \mid k > |\mathcal{R}| - i \,\} \\
&= \bigcap_{\mathcal{R}} \{\, r_k \mid k \in \omega \,\} \\
&= \{\, r_k \mid k \in \omega \,\}
\end{aligned}$$

However,

$$\begin{aligned}
\bigcup_i \mathsf{R}(Y_i) &= \bigcup_i \bigcap_{\mathcal{R}} \bigcup \{\, \mathsf{R}(\eta) \mid \eta \in Y_i(\mathcal{R}) \,\} \\
&= \bigcup_i \bigcap_{\mathcal{R}} \{\, r_k \mid k > |\mathcal{R}| - i \,\} \\
&= \bigcap_{\mathcal{R}} \emptyset \\
&= \emptyset
\end{aligned}$$

Therefore $\mathsf{R}(\bigsqcup_i Y_i) \neq \bigcup_i \mathsf{R}(Y_i)$, proving that $\mathcal{R}$ is not continuous.

Lemma A.17. For all monotone non-increasing Y, and for all ω-chain $\{\mathcal{R}_i\}_i$ such that $\bigcup_i \mathcal{R}_i = \mathsf{Res}$:

$$\mathsf{R}(Y) \;=\; \bigcap_i \bigcup \{\, \mathsf{R}(\eta) \mid \eta \in Y(\mathcal{R}_i) \,\}$$

Proof. The inclusion $\subseteq$ follows by set theory, while $\supseteq$ is implied by Lemma A.14. □

Lemma A.18. For all $Y \in D_{sub}$ and for all $\mathcal{R}$, $\mathsf{T}(Y) = \mathsf{T}(Y(\mathcal{R}))$.

Proof. Straightforward. □

Lemma A.19. For all strongly bound processes P with $fn(P) = \emptyset$:

$$\mathsf{R}([\![P]\!]^{sub}_{\theta}) = \mathsf{R}_{\mathsf{R}(\theta)}(P)$$

for all θ such that: $\forall \mathcal{R}\ \forall h \in fv(P) : \mathsf{R}(\theta(h)(\mathcal{R})) \subseteq \mathsf{R}(\theta(h)) \cup (\mathsf{Res} \setminus \mathcal{R})$.

Proof. We prove the following, stronger statement. For all P with $fn(P) = \emptyset$ and for all $\theta, \mathcal{R}$:

$$(2) \qquad \mathsf{R}_{\mathsf{R}(\theta)}(P) \subseteq \mathsf{R}(\llbracket P \rrbracket_\theta^{sub}(\mathcal{R})) \subseteq \mathsf{R}_{\mathsf{R}(\theta)}(P) \cup (\mathsf{Res} \setminus \mathcal{R})$$

First we show that the statement A.19 is implied by (2). We prove the double inclusion:

$$\begin{aligned}
\mathsf{R}_{\mathsf{R}(\theta)}(P) \subseteq \bigcap_{\mathcal{R}} \mathsf{R}(\llbracket P \rrbracket_\theta^{sub}(\mathcal{R})) = \mathsf{R}(\llbracket P \rrbracket_\theta^{sub})& \\
\subseteq \bigcap_{R} \big(\mathsf{R}_{\mathsf{R}(\theta)}(P) \cup (\mathsf{Res} \setminus \mathcal{R})\big)& \\
\subseteq \mathsf{R}_{\mathsf{R}(\theta)}(P) \cup \bigcap_{\mathcal{R}} (\mathsf{Res} \setminus \mathcal{R})& \\
= \mathsf{R}_{\mathsf{R}(\theta)}(P) \cup \emptyset& \\
= \mathsf{R}_{\mathsf{R}(\theta)}(P)&
\end{aligned}$$

To prove (2), we proceed by induction on the size of P.

- if $P = \varepsilon$, then $\mathsf{R}_{\mathsf{R}(\theta)}(\varepsilon) = \{\downarrow\} = \mathsf{R}(\{\,!, \varepsilon\}) = \mathsf{R}(\llbracket \varepsilon \rrbracket_\theta^{sub}(\mathcal{R}))$.
- if $P = \alpha(\rho)$, since $fn(P) = \emptyset$ then it suffices to consider the case $\rho = r$. We have that:

$$\mathsf{R}(\llbracket \alpha(r) \rrbracket_\theta^{sub}(\mathcal{R})) = \mathsf{R}(\{\,!, \alpha(r)\}) = \{r, \downarrow\} = \mathsf{R}_{\mathsf{R}(\theta)}(\alpha(r))$$

- if $P = h$, then:

$$\begin{aligned}
\mathsf{R}_{\mathsf{R}(\theta)}(h) &= \mathsf{R}(\theta)(h) \\
&= \mathsf{R}(\theta(h)) \\
&= \bigcap_{\mathcal{R}} \mathsf{R}(\theta(h)(\mathcal{R})) \\
&\subseteq \mathsf{R}(\theta(h)(\mathcal{R})) \\
&= \mathsf{R}(\llbracket h \rrbracket_\theta^{sub}(\mathcal{R})) \\
&= \mathsf{R}(\theta(h)(\mathcal{R}))
\end{aligned}$$

  and since by assumption $\mathsf{R}(\theta(h)(\mathcal{R})) \subseteq \mathsf{R}(\theta(h)) \cup (\mathsf{Res} \setminus \mathcal{R})$:

$$\begin{aligned}
&\subseteq \mathsf{R}(\theta(h)) \cup (\mathsf{Res} \setminus \mathcal{R}) \\
&= \mathsf{R}_{\mathsf{R}(\theta)}(h) \cup (\mathsf{Res} \setminus \mathcal{R})
\end{aligned}$$

- If $P = P_0 \cdot P_1$, there are two subcases.

– If $\mathsf{T}_{\mathsf{R}(\theta)}(P_0) = \emptyset$, then:

$$\mathsf{R}_{\mathsf{R}(\theta)}(P_0 \cdot P_1) = \mathsf{R}_{\mathsf{R}(\theta)}(P_0)$$

by the induction hypothesis:

$$\subseteq \mathsf{R}(\llbracket P_0 \rrbracket_\theta^{sub}(\mathcal{R}))$$

now, $\llbracket P_1 \rrbracket_\theta^{sub} \neq \emptyset$ by Lemma A.10. Since by the induction hypothesis and by Lemma A.18 $\mathsf{T}_{\mathsf{R}(\theta)}(P_0) = \mathsf{T}(\llbracket P_0 \rrbracket_\theta^{sub}) = \mathsf{T}(\llbracket P_0 \rrbracket_\theta^{sub}(\mathcal{R}))$:

$$\begin{aligned} &= \mathsf{R}(\llbracket P_0 \cdot P_1 \rrbracket_\theta^{sub}(\mathcal{R})) \\ &= \mathsf{R}(\llbracket P_0 \rrbracket_\theta^{sub}(\mathcal{R})) \end{aligned}$$

and by the induction hypothesis:

$$\begin{aligned} &\subseteq \mathsf{R}_{\mathsf{R}(\theta)}(P_0) \cup (\mathsf{Res} \setminus \mathcal{R}) \\ &= \mathsf{R}_{\mathsf{R}(\theta)}(P_0 \cdot P_1) \cup (\mathsf{Res} \setminus \mathcal{R}) \end{aligned}$$

– If $\mathsf{T}_{\mathsf{R}(\theta)}(P_0) = \{\downarrow\}$, we first prove that:

$$\mathsf{R}(\llbracket P \rrbracket_\theta^{sub}) = \mathsf{R}(\llbracket P_0 \rrbracket_\theta^{sub} \boxdot \llbracket P_1 \rrbracket_\theta^{sub})$$

$$= \bigcap_{\mathcal{R}} \bigcup \{\, \mathsf{R}(\eta_0 \odot \eta_1) \mid \begin{array}{l} \eta_0 \in \llbracket P_0 \rrbracket_\theta^{sub}(\mathcal{R}) \\ \eta_1 \in \llbracket P_1 \rrbracket_\theta^{sub}(\mathcal{R} \cup \mathsf{R}(\eta_0)) \end{array} \,\} \tag{3}$$

$$\supseteq \mathsf{R}(\llbracket P_0 \rrbracket_\theta^{sub}) \cup \mathsf{R}(\llbracket P_1 \rrbracket_\theta^{sub}) \tag{4}$$

Let $r \in (4)$. We have two further subcases.

* If $r \in \mathsf{R}(\llbracket P_0 \rrbracket_\theta^{sub})$, then for all $\mathcal{R}$, there exists $\eta_0 \in \llbracket P_0 \rrbracket_\theta^{sub}(\mathcal{R})$ such that $r \in \mathsf{R}(\eta_0)$. Then, by Lemma A.10, take an $\eta_1 \in \llbracket P_1 \rrbracket_\theta^{sub}(\mathcal{R} \cup \mathsf{R}(\eta_0)) \neq \emptyset$. Then $r \in \mathsf{R}(\eta_0) \subseteq \mathsf{R}(\eta_0 \odot \eta_1)$. This proves $r \in (3)$.
* If $r \in \mathsf{R}(\llbracket P_1 \rrbracket_\theta^{sub})$, then for all $\mathcal{R}'$ there exists $\eta_1 \in \llbracket P_1 \rrbracket_\theta^{sub}(\mathcal{R}')$ such that $r \in \mathsf{R}(\eta_1)$. To show $r \in (3)$, we need to prove that for all $\mathcal{R}$ there exists η_0, η_1 such that $\eta_0 \in \llbracket P_0 \rrbracket_\theta^{sub}(\mathcal{R})$, $\eta_1 \in \llbracket P_1 \rrbracket_\theta^{sub}(\mathcal{R} \cup \mathsf{R}(\eta_0))$, and $r \in \mathsf{R}(\eta_0 \odot \eta_1)$. Since $\mathsf{T}(\llbracket P_0 \rrbracket_\theta^{sub}) = \{\downarrow\}$, we can choose an $\eta_0 \in \llbracket P_0 \rrbracket_\eta^{sub}(\mathcal{R})$ with $! \notin \eta_0$. Then, by choosing $\mathcal{R}' = \mathcal{R} \cup \mathsf{R}(\eta_0)$, we obtain that there exists $\eta_1 \in \llbracket P_1 \rrbracket_\theta^{sub}(\mathcal{R} \cup \mathsf{R}(\eta_0))$ such that $r \in \mathsf{R}(\eta_1)$. We conclude the proof by $\mathsf{R}(\eta_0 \odot \eta_1) = \mathsf{R}(\eta_0) \cup \mathsf{R}(\eta_1)$, since $! \notin \eta_0$.

Therefore:

$$\mathsf{R}_{\mathsf{R}(\theta)}(P_0 \cdot P_1) = \mathsf{R}_{\mathsf{R}(\theta)}(P_0) \cup \mathsf{R}_{\mathsf{R}(\theta)}(P_1)$$

by the induction hypothesis:

$$= \mathsf{R}(\llbracket P_0 \rrbracket_\theta^{sub}) \cup \mathsf{R}(\llbracket P_1 \rrbracket_\theta^{sub})$$

since (4) $\subseteq$ (3):

$$\begin{aligned}
&\subseteq \mathsf{R}(\llbracket P_0 \cdot P_1 \rrbracket_\theta^{sub}) \\
&\subseteq \mathsf{R}(\llbracket P_0 \cdot P_1 \rrbracket_\theta^{sub}(\mathcal{R}))
\end{aligned}$$

For the other inclusion, we have that:

$$\mathsf{R}(\llbracket P_0 \cdot P_1 \rrbracket_\theta^{sub}(\mathcal{R})) = \bigcup \{\, \mathsf{R}(\eta_0 \odot \eta_1) \mid \eta_0 \in \llbracket P_0 \rrbracket_\theta^{sub}(\mathcal{R}),\ \eta_1 \in \llbracket P_1 \rrbracket_\theta^{sub}(\mathcal{R} \cup \mathsf{R}(\eta_0)) \,\}$$

by Lemma A.14:

$$\begin{aligned}
&\subseteq \bigcup \{\, \mathsf{R}(\eta_0 \odot \eta_1) \mid \eta_0 \in \llbracket P_0 \rrbracket_\theta^{sub}(\mathcal{R}),\ \eta_1 \in \llbracket P_1 \rrbracket_\theta^{sub}(\mathcal{R})) \,\} \\
&\subseteq \mathsf{R}(\llbracket P_0 \rrbracket_\theta^{sub})(\mathcal{R}) \cup \mathsf{R}(\llbracket P_1 \rrbracket_\theta^{sub})(\mathcal{R})
\end{aligned}$$

and by the induction hypothesis:

$$\begin{aligned}
&\subseteq \mathsf{R}_{\mathsf{R}(\theta)}(P_0) \cup \mathsf{R}_{\mathsf{R}(\theta)}(P_1) \cup (\mathsf{Res} \setminus \mathcal{R}) \\
&= \mathsf{R}_{\mathsf{R}(\theta)}(P_0 \cdot P_1) \cup (\mathsf{Res} \setminus \mathcal{R})
\end{aligned}$$

- If $P = P_0 + P_1$, we have that:

  $$\mathsf{R}_{\mathsf{R}(\theta)}(P_0 + P_1) = \mathsf{R}_{\mathsf{R}(\theta)}(P_0) \cup \mathsf{R}_{\mathsf{R}(\theta)}(P_1)$$

  by the induction hypothesis:

  $$\begin{aligned}
  &\subseteq \mathsf{R}(\llbracket P_0 \rrbracket_\theta^{sub}(\mathcal{R})) \cup \mathsf{R}(\llbracket P_1 \rrbracket_\theta^{sub}(\mathcal{R})) \\
  &= \bigcup \{\, \mathsf{R}(\eta) \mid \eta \in \llbracket P_0 \rrbracket_\theta^{sub}(\mathcal{R}) \,\} \cup \bigcup \{\, \mathsf{R}(\eta) \mid \eta \in \llbracket P_1 \rrbracket_\theta^{sub}(\mathcal{R}) \,\} \\
  &= \bigcup \{\, \mathsf{R}(\eta) \mid \eta \in \llbracket P_0 \rrbracket_\theta^{sub}(\mathcal{R}) \cup \llbracket P_1 \rrbracket_\theta^{sub}(\mathcal{R}) \,\} \\
  &= \mathsf{R}(\llbracket P_0 + P_1 \rrbracket_\theta^{sub}(\mathcal{R})) \\
  &= \mathsf{R}(\llbracket P_0 \rrbracket_\theta^{sub}(\mathcal{R})) \cup \mathsf{R}(\llbracket P_1 \rrbracket_\theta^{sub}(\mathcal{R}))
  \end{aligned}$$

  by the induction hypothesis:

  $$\begin{aligned}
  &\subseteq \mathsf{R}_{\mathsf{R}(\theta)}(P_0) \cup \mathsf{R}_{\mathsf{R}(\theta)}(P_1) \cup (\mathsf{Res} \setminus \mathcal{R}) \\
  &= \mathsf{R}_{\mathsf{R}(\theta)}(P_0 + P_1) \cup (\mathsf{Res} \setminus \mathcal{R})
  \end{aligned}$$

- If $P = \nu n.\, P'$, we have

$$\begin{aligned}
\mathsf{R}([\![P]\!]^{sub}_{\theta}(\mathcal{R})) &= \mathsf{R}(\bigcup_{r \notin \mathcal{R}} [\![P'\{r/n\}]\!]^{sub}_{\theta}(\mathcal{R} \cup \{r\})) \\
&= \bigcup_{r \notin \mathcal{R}} \mathsf{R}([\![P'\{r/n\}]\!]^{sub}_{\theta}(\mathcal{R} \cup \{r\})) \qquad (5)
\end{aligned}$$

Also, we have that:

$$\mathsf{R}_{\mathsf{R}(\theta)}(P) = \mathsf{R}_{\mathsf{R}(\theta)}(P')$$

using a simple inductive argument:

$$\subseteq \bigcup_{r \notin \mathcal{R}} \mathsf{R}_{\mathsf{R}(\theta)}(P'\{r/n\})$$

by the induction hypothesis:

$$\begin{aligned}
&\subseteq (5) \\
&\subseteq \bigcup_{r \notin \mathcal{R}} \big(\mathsf{R}_{\mathsf{R}(\theta)}(P'\{r/n\}) \cup (\mathsf{Res} \setminus (\mathcal{R} \cup \{r\}))\big)
\end{aligned}$$

by a simple inductive argument on P'

$$\begin{aligned}
&\subseteq \bigcup_{r \notin \mathcal{R}} \big(\mathsf{R}_{\mathsf{R}(\theta)}(P') \cup \{r\} \cup (\mathsf{Res} \setminus (\mathcal{R} \cup \{r\}))\big) \\
&= \mathsf{R}_{\mathsf{R}(\theta)}(P') \cup (\mathsf{Res} \setminus \mathcal{R}) \cup \bigcup_{r \notin \mathcal{R}} (\mathsf{Res} \setminus (\mathcal{R} \cup \{r\})) \\
&= \mathsf{R}_{\mathsf{R}(\theta)}(P') \cup (\mathsf{Res} \setminus \mathcal{R}) \cup (\mathsf{Res} \setminus \bigcap_{r \notin \mathcal{R}} (\mathcal{R} \cup \{r\})) \\
&= \mathsf{R}_{\mathsf{R}(\theta)}(P') \cup (\mathsf{Res} \setminus \mathcal{R}) \cup (\mathsf{Res} \setminus \mathcal{R}) \\
&= \mathsf{R}_{\mathsf{R}(\theta)}(P) \cup (\mathsf{Res} \setminus \mathcal{R})
\end{aligned}$$

- If $P = \mu h.\, P'$, first note that:

$$\mathsf{R}([\![P]\!]^{sub}_{\theta}(\mathcal{R})) = \mathsf{R}(\bigsqcup_{n \geq 0} f^n(\bot)(\mathcal{R})) = \bigcup_{n \geq 0} \mathsf{R}(f^n(\bot)(\mathcal{R})) \qquad (6)$$

We now prove the inclusion $\mathsf{R}_{\mathsf{R}(\Theta)}(P) \subseteq \mathsf{R}([\![P]\!]^{sub}_{\theta}(\mathcal{R}))$. We have that:

$$\mathsf{R}_{\mathsf{R}(\Theta)}(\mu h.\, P') = \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{T}_{\mathsf{R}(\theta)\{\emptyset/h\}}(P')/h\}}(P')$$

by Lemma 5d:

$$\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{R}_{\mathsf{R}(\theta)\{\emptyset/h\}}(P')/h\}}(P')$$

and by the induction hypothesis:

$$= \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{R}(\llbracket P' \rrbracket^{sub}_{\theta\{\bot/h\}})/h\}}(P')$$

by the induction hypothesis again:

$$\begin{aligned}
&\subseteq \mathsf{R}(\llbracket P' \rrbracket^{sub}_{\theta\{\llbracket P' \rrbracket^{sub}_{\theta\{\bot/h\}}/h\}}(\mathcal{R})) \\
&= \mathsf{R}(f^2(\bot)(\mathcal{R})) \\
&\subseteq \bigcup_{n\geq 0} \mathsf{R}(f^n(\bot)(\mathcal{R}))
\end{aligned}$$

We now show by induction on $n \geq 0$ that:

$$\mathsf{R}(f^n(\bot)(\mathcal{R})) \subseteq \mathsf{R}_{\mathsf{R}(\theta)}(P) \cup (\mathsf{Res} \setminus \mathcal{R})$$

The base case is trivial, since $\mathsf{R}(\bot(\mathcal{R})) = \mathsf{R}(\{\,!\,\}) = \emptyset$.

For the inductive case $n > 0$, we have that:

$$\mathsf{R}(f^n(\bot)(\mathcal{R})) = \mathsf{R}(\llbracket P' \rrbracket^{sub}_{\theta\{f^{n-1}(\bot)/h\}}(\mathcal{R}))$$

by the induction hypothesis on the size of P:

$$\begin{aligned}
&\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{R}(f^{n-1}(\bot))/h\}}(P') \cup (\mathsf{Res} \setminus \mathcal{R}) \\
&\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\bigcap_{\mathcal{R}} \mathsf{R}(f^{n-1}(\bot)(\mathcal{R}))/h\}}(P') \cup (\mathsf{Res} \setminus \mathcal{R})
\end{aligned}$$

by the induction hypothesis on n and Lemma 5d:

$$\begin{aligned}
&\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\bigcap_{\mathcal{R}} \left(\mathsf{R}_{\mathsf{R}(\theta)}(\mu h.\, P')(\mathcal{R}) \cup (\mathsf{Res}\setminus\mathcal{R})\right)/h\}}(P') \cup (\mathsf{Res} \setminus \mathcal{R}) \\
&= \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{R}_{\mathsf{R}(\theta)}(\mu h.\, P')/h\}}(P') \cup (\mathsf{Res} \setminus \mathcal{R})
\end{aligned}$$

and so by Lemma 5g:

$$= \mathsf{R}_{\mathsf{R}(\Theta)}(\mu h.\, P') \cup (\mathsf{Res} \setminus \mathcal{R})$$

Summing up:

$$\bigcup_{n\geq 0} \mathsf{R}(f^n(\bot)(\mathcal{R})) \subseteq \bigcup_{n\geq 0} \mathsf{R}_{\mathsf{R}(\Theta)}(\mu h.\, P') \cup (\mathsf{Res}\setminus\mathcal{R}) = \mathsf{R}_{\mathsf{R}(\Theta)}(\mu h.\, P') \cup (\mathsf{Res}\setminus\mathcal{R})$$

which concludes the proof.

$\square$

Lemma A.20. For all P, θ, $\mathcal{R}$, we have that:

$$\eta \in [\![P]\!]^{sub}_{\theta}(\mathcal{R}) \implies (\mathcal{R} \setminus \mathsf{R}_{\mathsf{R}(\theta)}(P)) \cap \mathsf{R}(\eta) = \emptyset$$

Proof. By induction on the size of P. We have the following cases:

- The cases ε, ! and h are trivial.
- If $P = \alpha(r)$, then $\mathsf{R}(\eta) = \{r\} \subseteq \mathsf{R}_{\mathsf{R}(\theta)}(P)$, which implies the thesis.
- If $P = \nu n.\, P'$, then $\eta \in [\![P'\{r/n\}]\!]^{sub}_{\theta}(\mathcal{R} \cup \{r\})$ for some $r \notin \mathcal{R}$. Then:

$$\begin{aligned}(\mathcal{R} \setminus \mathsf{R}_{\mathsf{R}(\theta)}(P)) \cap \mathsf{R}(\eta) &= (\mathcal{R} \setminus \mathsf{R}_{\mathsf{R}(\theta)}(P'\{r/n\}) \cap \mathsf{R}(\eta) \\ &\subseteq ((\mathcal{R} \cup \{r\}) \setminus \mathsf{R}_{\mathsf{R}(\theta)}(P'\{r/n\}) \cap \mathsf{R}(\eta) = \emptyset\end{aligned}$$

  where the last equation follows by the induction hypothesis.

- If $P = P_0 \cdot P_1$, then for all $\eta_0 \in [\![P_0]\!]^{sub}_{\theta}(\mathcal{R})$ and for all $\eta_1 \in [\![P_1]\!]^{sub}_{\theta}(\mathcal{R} \cup \mathsf{R}(\eta_0))$, the induction hypothesis of (a) gives:

$$\begin{aligned}&(\mathcal{R} \setminus \mathsf{R}_{\mathsf{R}(\theta)}(P_0)) \cap \mathsf{R}(\eta_0) = \emptyset \\ &((\mathcal{R} \cup \mathsf{R}(\eta_0)) \setminus \mathsf{R}_{\mathsf{R}(\theta)}(P_1)) \cap \mathsf{R}(\eta_1) = \emptyset\end{aligned}$$

  Let $\eta \in [\![P]\!]^{sub}_{\theta}(\mathcal{R})$, i.e. $\eta \in \eta_0 \odot [\![P_1]\!]^{sub}_{\theta}(\mathcal{R} \cup \mathsf{R}(\eta_0))$ for some $\eta_0 \in [\![P_0]\!]^{sub}_{\theta}(\mathcal{R})$. There are two subcases.
  If $! \in \eta_0$, then $\eta = \eta_0$, and:

$$(\mathcal{R} \setminus \mathsf{R}_{\mathsf{R}(\theta)}(P)) \cap \mathsf{R}(\eta) \subseteq (\mathcal{R} \setminus \mathsf{R}_{\mathsf{R}(\theta)}(P_0)) \cap \mathsf{R}(\eta_0) = \emptyset$$

  Otherwise, if $! \notin \eta_0$, then $\downarrow \in \mathsf{R}_{\mathsf{R}(\theta)}(P_0)$ by Lemma A.19(c), and so:

$$\begin{aligned}(\mathcal{R} \setminus \mathsf{R}_{\mathsf{R}(\theta)}(P)) \cap \mathsf{R}(\eta) &= (\mathcal{R} \setminus (\mathsf{R}_{\mathsf{R}(\theta)}(P_0) \cup \mathsf{R}_{\mathsf{R}(\theta)}(P_1)) \cap (\mathsf{R}(\eta_0) \cup \mathsf{R}(\eta_1)) \\ &= (\mathcal{R} \setminus \mathsf{R}_{\mathsf{R}(\theta)}(P_0)) \cap (\mathcal{R} \setminus \mathsf{R}_{\mathsf{R}(\theta)}(P_1)) \cap (\mathsf{R}(\eta_0) \cup \mathsf{R}(\eta_1)) \\ &\subseteq \big((\mathcal{R} \setminus \mathsf{R}_{\mathsf{R}(\theta)}(P_0)) \cap \mathsf{R}(\eta_0)\big) \cup \big((\mathcal{R} \setminus \mathsf{R}_{\mathsf{R}(\theta)}(P_1)) \cap \mathsf{R}(\eta_1)\big) \\ &\subseteq \emptyset \cup ((\mathcal{R} \cup \mathsf{R}(\eta_0)) \setminus \mathsf{R}_{\mathsf{R}(\theta)}(P_1)) \cap \mathsf{R}(\eta_1) \\ &= \emptyset\end{aligned}$$

- If $P = P_0 + P_1$, the thesis follows directly from the induction hypothesis.
- If $P = \mu h.\, P'$, let $\eta \in (\bigsqcup_i Y_i)(\mathcal{R})$, where $Y_0 = \bot$ and $Y_{i+1} = [\![P']\!]^{sub}_{\theta\{Y_i/h\}}$. Then, there exists k such that $\eta \in Y_k(\mathcal{R})$. If $k = 0$, then $\eta = !$ and the thesis follows trivially. Otherwise, we have that:

$$(\mathcal{R} \setminus \mathsf{R}([\![P]\!]^{sub}_{\theta})) \cap \mathsf{R}(\eta) = (\mathcal{R} \setminus \mathsf{R}(\bigsqcup_i Y_i)) \cap \mathsf{R}(\eta)$$

since R is monotone, then $\mathsf{R}(\bigsqcup_i Y_i) \supseteq \bigcup_i \mathsf{R}(Y_i)$, and so:

$$
\begin{aligned}
&\subseteq (\mathcal{R} \setminus \bigcup_i \mathsf{R}(Y_i)) \cap \mathsf{R}(\eta)\\
&= \bigcap_i ((\mathcal{R} \setminus \mathsf{R}(Y_i)) \cap \mathsf{R}(\eta))\\
&\subseteq (\mathcal{R} \setminus \mathsf{R}(Y_k)) \cap \mathsf{R}(\eta)\\
&\subseteq (\mathcal{R} \setminus \mathsf{R}([\![P']\!]^{sub}_{\theta\{Y_{k-1}/h\}})) \cap \mathsf{R}(\eta)
\end{aligned}
$$

by Lemma A.19:

$$
\begin{aligned}
&= (\mathcal{R} \setminus \mathsf{R}_{\mathsf{R}(\theta\{Y_{k-1}/h\})}(P')) \cap \mathsf{R}(\eta)\\
&= \emptyset
\end{aligned}
$$

where the last equality follows directly from the induction hypothesis.

□

Lemma A.21. Let P be a strongly bound processes with $fn(P) = \emptyset$, let θ be a function such that $dom(\theta) = \{h_1, \ldots, h_k\} \supseteq fv(P)$ and $\theta(h_i)(\bar{\mathcal{R}}) \subseteq [\![P_i]\!]^{op}(\bar{\mathcal{R}})$ for all $i \in 1..k$ and for all $\bar{\mathcal{R}} \supseteq \mathcal{R}$, and let $\mathcal{R} \supseteq \mathsf{R}_{\mathsf{R}(\theta)}(P)$ be a finite set of resources. Then:

$$[\![P]\!]^{sub}_{\theta}(\mathcal{R}) \subseteq [\![P\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R})$$

Proof. By induction on the size of P. Let $\eta \in [\![P]\!]^{sub}_{\theta}(\mathcal{R})$. If $\eta = !$, then by Def. 2.2 the thesis follows trivially. Otherwise, there are the following cases:

- if $P = \varepsilon$, then $\eta = \varepsilon \in [\![\varepsilon]\!]^{op}(\mathcal{R})$.

- if $P = !$, then $\eta = !$ – already considered.

- if $P = \alpha(r)$, there are two subcases.

  If $\eta = \alpha(r)$, since $\alpha(r), \mathcal{R} \xrightarrow{\alpha(r)} \varepsilon, \mathcal{R}$, then:

  $$\alpha(r) \in [\![P]\!]^{op}(\mathcal{R}) = [\![P\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R})$$

  If $\eta = \alpha(r)\,!$, similar to the case above.

- if $P = h$, then, by the hypothesis $dom(\theta) \supseteq fv(P)$, we have that $h = h_i$ for some $i \in 1..k$, and $[\![P]\!]^{sub}_{\theta} = [\![h_i]\!]^{sub}_{\theta} = \theta(h_i)$. By assumption, $\theta(h_i)(\mathcal{R}) \subseteq [\![P_i]\!]^{op}(\mathcal{R})$. Thus, $[\![P]\!]^{sub}_{\theta}(\mathcal{R}) \subseteq [\![h_i\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R}) = [\![P_i]\!]^{op}(\mathcal{R})$.

- if $P = \nu n.\, \bar{P}$, then $\eta \in [\![\bar{P}\{r/n\}]\!]^{sub}_{\theta}(\mathcal{R} \cup \{r\})$ for some $r \notin \mathcal{R}$. By the induction hypothesis:

$$[\![\bar{P}\{r/n\}]\!]^{sub}_{\theta}(\mathcal{R} \cup \{r\}) \subseteq [\![\bar{P}\{r/n\}\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R} \cup \{r\})$$

  Since $P\{P_1/h_1, \cdots, P_k/h_k\}, \mathcal{R} \xrightarrow{\varepsilon} \bar{P}\{P_1/h_1, \cdots, P_k/h_k\}\{r/n\}, \mathcal{R} \cup \{r\}$ then by Lemma A.8:

$$\varepsilon \odot [\![\bar{P}\{P_1/h_1, \cdots, P_k/h_k\}\{r/n\}]\!]^{op}(\mathcal{R} \cup \{r\}) \subseteq [\![P\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R})$$

  Since $[\![P_i]\!]^{sub}_{\theta}$ is defined, then $fn(P_i) = \emptyset$ for all $i \in 1..k$, and so:

$$\bar{P}\{r/n\}\{P_1/h_1, \cdots, P_k/h_k\} = \bar{P}\{P_1/h_1, \cdots, P_k/h_k\}\{r/n\}$$

  Summing up, we have proved that $\eta \in [\![P\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R})$.

- if $P = P' \cdot P''$, then $\eta \in ([\![P']\!]^{sub}_{\theta} \boxdot [\![P'']\!]^{sub}_{\theta})(\mathcal{R})$. By Def. 2.3, $\eta = \eta_0 \odot \eta_1$ for some $\eta_0 \in [\![P']\!]^{sub}_{\theta}(\mathcal{R})$ and $\eta_1 \in [\![P'']\!]^{sub}_{\theta}(\mathcal{R} \cup \mathsf{R}(\eta_0))$. By the induction hypothesis, $\eta_0 \in [\![P'\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R})$. By Lemma A.2, we have two cases.

  - We have $\eta_0 = \eta_0'\,!$ for some !-free η_0'. In this case $\eta = \eta_0$ by Def. 2.3. Since $P'\{P_1/h_1, \cdots, P_k/h_k\}, \mathcal{R} \xrightarrow{\eta_0'\,!} \bar{P}, \bar{\mathcal{R}}$ for some $\bar{P}$ and $\bar{\mathcal{R}}$. A simple inductive argument then yields:

$$P\{P_1/h_1, \cdots, P_k/h_k\}, \mathcal{R} \xrightarrow{\eta_0'\,!} \bar{P} \cdot (P''\{P_1/h_1, \cdots, P_k/h_k\}), \bar{\mathcal{R}}$$

    Therefore $\eta \in [\![P\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R})$, which is the thesis.

  - We have $! \notin \eta_0$, and so $\eta = \eta_0\eta_1$. Since $\eta_0 \in [\![P'\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R})$, then by definition:

$$P'\{P_1/h_1, \cdots, P_k/h_k\}, \mathcal{R} \xrightarrow{\eta_0} \varepsilon, \bar{\mathcal{R}}$$

    for some $\bar{\mathcal{R}} \supseteq \mathcal{R} \cup \mathsf{R}(\eta_0)$. Similarly, by the induction hypothesis on P'', we have that $\eta_1 \in [\![P''\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R} \cup \mathsf{R}\eta_0)$, so for some $\mathcal{R}'$:

$$P''\{P_1/h_1, \cdots, P_k/h_k\}, (\mathcal{R} \cup \mathsf{R}(\eta_0)) \xrightarrow{\eta_1} \varepsilon, \mathcal{R}'$$

    By simple set-theoretic arguments, we have:

$$\bar{\mathcal{R}} = \mathcal{R}_a \cup \mathcal{R}_b = (\bar{\mathcal{R}} \cap (\mathcal{R} \cup \mathsf{R}(\eta_0))) \cup (\bar{\mathcal{R}} \setminus (\mathcal{R} \cup \mathsf{R}(\eta_0)))$$

    Since $\bar{\mathcal{R}} \supseteq \mathcal{R} \cup \mathsf{R}(\eta_0)$, then $\mathcal{R}_a = \mathcal{R} \cup \mathsf{R}(\eta_0)$. We shall now construct a derivation:

$$P'\{P_1/h_1, \cdots, P_k/h_k\}, \mathcal{R} \xrightarrow{\eta_0} \varepsilon, \bar{\mathcal{R}}'$$

with $\bar{\mathcal{R}}' = \mathcal{R}_a \cup \mathcal{R}'_b$ for some $\mathcal{R}'_b$ disjoint from $\mathcal{R}'$. The intuition is that $\mathcal{R}_b$ contains only "garbage" resources, e.g. those vacuously generated by $\nu n. P$ with $n \notin P$. To construct this derivation, we α-convert the resources in $\mathcal{R}_b$ by repeatedly applying Lemma 7d. Take the set of resources $\mathcal{R}'_b$ such that $|\mathcal{R}'_b| = |\mathcal{R}_b|$ and $\mathcal{R}'_b \cap (\bar{\mathcal{R}} \cup \mathcal{R}') = \emptyset$. Let σ be a bijection mapping $\mathcal{R}_b$ into $\mathcal{R}'_b$. Moreover, let n be an arbitrary name: clearly, n is not free in the closed process $P'\{P_1/h_1, \cdots, P_k/h_k\}$.

Take a resource $r \in \mathcal{R}_b$: we now use Lemma 7d and α-convert it to $r\sigma \in \mathcal{R}'_b$. To this purpose, we note that $P'\{P_1/h_1, \cdots, P_k/h_k\} = P'\{P_1/h_1, \cdots, P_k/h_k\}\{r/n\}$. The hypoteses of the lemma are satisfied: $r \notin \mathsf{R}(\eta_0)$ since $\mathcal{R}_b \cap \mathsf{R}(\eta) = (\bar{\mathcal{R}} \setminus (\mathcal{R} \cup \mathsf{R}(\eta_0))) \cap \mathsf{R}(\eta_0) = \emptyset$, and $r\sigma \notin \bar{\mathcal{R}}$ since $\mathcal{R}'_b$ was chosen to this aim. Of course, we can return to the original process with $P'\{P_1/h_1, \cdots, P_k/h_k\}\{r\sigma/n\} = P'\{P_1/h_1, \cdots, P_k/h_k\}$. Therefore:

$$P'\{P_1/h_1, \cdots, P_k/h_k\}, \mathcal{R}\{r\sigma/r\} \xrightarrow{\eta_0} \varepsilon, \bar{\mathcal{R}}\{r\sigma/r\}$$

By a simple inductive argument, we similarly α-convert the other resources in $\mathcal{R}_b$: the hypotheses of Lemma 7d do not change, except that we must ensure $r\sigma \notin \bar{\mathcal{R}}\sigma'$ where σ' is the result of the previously performed α-conversions. This is however straightforward, since $r\sigma \notin \bar{\mathcal{R}}$, and $r\sigma \notin \mathit{ran}(\sigma')$. We have therefore constructed a derivation for:

$$P'\{P_1/h_1, \cdots, P_k/h_k\}, \mathcal{R}\sigma \xrightarrow{\eta_0} \varepsilon, \bar{\mathcal{R}}'$$

Since $\mathcal{R} \cap \mathit{dom}(\sigma) = \mathcal{R} \cap \mathcal{R}_b = \mathcal{R} \cap (\mathcal{R} \cup \mathsf{R}(\eta_0)) = \mathcal{R}$, so by definition of $\bar{\mathcal{R}}'$:

$$P'\{P_1/h_1, \cdots, P_k/h_k\}, \mathcal{R} \xrightarrow{\eta_0} \varepsilon, (\mathcal{R} \cup \mathsf{R}\eta_0) \cup \mathcal{R}'_b$$

To prove the thesis, we now exploit $\mathcal{R}'_b \cap \mathcal{R}' = \emptyset$ and apply Lemma 7b to every $r \in \mathcal{R}'_b$, obtaining:

$$P''\{P_1/h_1, \cdots, P_k/h_k\}, (\mathcal{R} \cup \mathsf{R}\eta_0) \cup \mathcal{R}'_b \xrightarrow{\eta_0} \varepsilon, \mathcal{R}' \cup \mathcal{R}'_b$$

We then can conclude by:

$$(P' \cdot P'')\{P_1/h_1, \cdots, P_k/h_k\}, \mathcal{R} \xrightarrow{\eta_0\eta_1} \varepsilon, \mathcal{R}' \cup \mathcal{R}'_b$$

which implies $\eta \in [\![P\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R})$.

- if $P = P' + P''$, then $\eta \in [\![P']\!]^{sub}_\theta(\mathcal{R})$ or $\eta \in [\![P'']\!]^{sub}_\theta(\mathcal{R})$. In the first case, by induction hypothesis $\eta \in [\![P'\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R})$. Since $P\{P_1/h_1, \cdots, P_k/h_k\}$, $\mathcal{R} \xrightarrow{\varepsilon} P'\{P_1/h_1, \cdots, P_k/h_k\}$, $\mathcal{R}$ so by Lemma A.8:

$$\begin{aligned}[\![P'\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R}) &= \varepsilon \odot [\![P'\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R}) \\ &\subseteq [\![P\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R})\end{aligned}$$

which is the thesis. The other case is similar.

- if $P = \mu h.\, P'$, then $\eta \in f^n([\![!]\!]^{sub}_\theta)(\mathcal{R})$, for some $n \geq 0$. By induction on n, we prove that, for all $n \geq 0$:

  $$\forall \bar{\mathcal{R}} \supseteq \mathcal{R} \qquad f^n(\bot)\,(\bar{\mathcal{R}}) \;\subseteq\; [\![P\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R})$$

  The base case $n = 0$ is trivial, since $!$ is always included in the right-hand side. For the inductive case $n > 0$, we have that:

  $$\begin{aligned} f^{n+1}(\bot)\,(\bar{\mathcal{R}}) &= f(f^n(\bot))(\bar{\mathcal{R}}) \\ &= [\![P']\!]^{sub}_{\theta\{f^n(\bot)/h\}}\,(\bar{\mathcal{R}}) \end{aligned}$$

  by the induction hypothesis on n, $f^n(\bot)(\bar{\mathcal{R}}) \subseteq [\![P\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R})$ for all $\mathcal{R} \subseteq \bar{\mathcal{R}}$. Then, by the induction hypothesis on the size of P:

  $$\subseteq [\![P'\{P_1/h_1, \cdots, P_k/h_k, P\{P_1/h_1, \cdots, P_k/h_k\}/h\}]\!]^{op}(\bar{\mathcal{R}})$$

  and by Lemma 6b:

  $$\begin{aligned} &\subseteq [\![P'\{P_1/h_1, \cdots, P_k/h_k, P\{P_1/h_1, \cdots, P_k/h_k\}/h\}]\!]^{op}(\mathcal{R}) \\ &= \varepsilon \odot [\![P'\{P_1/h_1, \cdots, P_k/h_k, P\{P_1/h_1, \cdots, P_k/h_k\}/h\}]\!]^{op}(\mathcal{R}) \end{aligned}$$

  since $P\{P_1/h_1, \cdots, P_k/h_k\} \xrightarrow{\varepsilon} P'\{P_1/h_1, \cdots, P_k/h_k\}\{P\{P_1/h_1, \cdots, P_k/h_k\}/h\}$, then by Lemma A.8:

  $$\subseteq [\![P\{P_1/h_1, \cdots, P_k/h_k\}]\!]^{op}(\mathcal{R})$$

  which concludes the proof.

$\square$

Lemma A.22. For all strongly bound processes P, P' with $fn(P) = fn(P') = \emptyset$, for all θ such that $dom(\theta) \supseteq fv(P) \cup fv(P')$, and for all $h \notin fv(P')$:

$$[\![P\{P'/h\}]\!]^{sub}_\theta = [\![P]\!]^{sub}_{\theta\{[\![P']\!]^{sub}_\theta/h\}}$$

Proof. We proceed by induction on the size of P. Let $\theta' = \theta\{[\![P']\!]^{sub}_\theta/h\}$.

- If $P = \varepsilon$, $P = \alpha(r)$ or $P = !$, the thesis follows trivially, because the substitution is vacuous.
- If $P = h'$, there are two subcases. If $h' = h$, then:

  $$[\![h\{P'/h\}]\!]^{sub}_\theta = [\![P']\!]^{sub}_\theta = [\![h]\!]^{sub}_{\theta'}$$

  Otherwise, if $h' \neq h$, then $P\{P'/h\} = P$.

- If $P = \nu n.\, \bar{P}$, then:

$$[\![\nu n.\, \bar{P}\{P'/h\}]\!]^{sub}_{\theta} = \lambda\mathcal{R}.\, \bigcup_{r \notin \mathcal{R}} [\![\bar{P}\{P'/h\}\{r/n\}]\!]^{sub}_{\theta}(\mathcal{R} \cup \{r\})$$

since $fn(P') = \emptyset$ by hypothesis, then $P'\{r/n\} = P'$, and so:

$$= \lambda\mathcal{R}.\, \bigcup_{r \notin \mathcal{R}} [\![\bar{P}\{r/n\}\{P'/h\}]\!]^{sub}_{\theta}(\mathcal{R} \cup \{r\})$$

and, by the induction hypothesis:

$$\begin{aligned} &= \lambda\mathcal{R}.\, \bigcup_{r \notin \mathcal{R}} [\![\bar{P}\{r/n\}]\!]^{sub}_{\theta'}(\mathcal{R} \cup \{r\}) \\ &= [\![\nu n.\, \bar{P}]\!]^{sub}_{\theta'} \end{aligned}$$

- If $P = P_0 \cdot P_1$, then:

$$\begin{aligned} [\![(P_0 \cdot P_1)\{P'/h\}]\!]^{sub}_{\theta} &= [\![P_0\{P'/h\} \cdot P_1\{P'/h\}]\!]^{sub}_{\theta} \\ &= [\![P_0\{P'/h\}]\!]^{sub}_{\theta} \boxdot [\![P_1\{P'/h\}]\!]^{sub}_{\theta} \\ &= [\![P_0]\!]^{sub}_{\theta'} \boxdot [\![P_1]\!]^{sub}_{\theta'} \\ &= [\![P]\!]^{sub}_{\theta'} \end{aligned}$$

- If $P = P_0 + P_1$, then, by the induction hypothesis:

$$\begin{aligned} [\![(P_0 + P_1)\{P'/h\}]\!]^{sub}_{\theta} &= [\![P_0\{P'/h\}]\!]^{sub}_{\theta} \sqcup [\![P_1\{P'/h\}]\!]^{sub}_{\theta} \\ &= [\![P_0]\!]^{sub}_{\theta'} \sqcup [\![P_1]\!]^{sub}_{\theta'} \\ &= [\![P]\!]^{sub}_{\theta'} \end{aligned}$$

- If $P = \mu h'.\, \bar{P}$, there are two subcases. If $h' = h$, then $P\{P'/h\} = P$, and so the statement holds trivially. Otherwise $h' \neq h$, and so we have:

$$[\![\mu h'.\, \bar{P}\{P'/h\}]\!]^{sub}_{\theta} = \bigsqcup_{n \geq 0} \left(\lambda Y.\, [\![\bar{P}\{P'/h\}]\!]^{sub}_{\theta\{Y/h'\}}\right)^n(\perp)$$

and, by the induction hypothesis:

$$\begin{aligned} &= \bigsqcup_{n \geq 0} \left(\lambda Y.\, [\![\bar{P}]\!]^{sub}_{\theta'\{Y/h'\}}\right)^n(\perp) \\ &= [\![\mu h'.\, \bar{P}]\!]^{sub}_{\theta'} \end{aligned}$$

which concludes the proof.

$\square$

Lemma A.23 (Unfolding). For all strongly bound processes P, and for all θ:

$$[\![\mu h.\, P]\!]^{sub}_{\theta} \;\;=\;\; [\![P]\!]^{sub}_{\theta\{[\![\mu h.\, P]\!]^{sub}_{\theta}/h\}}$$

Proof. By Def. A.9, $[\![\mu h.\, P]\!]^{sub}_{\theta} = \bigsqcup_{i\geq 0} f^i(\bot)$, where $f(Y) = [\![P]\!]^{sub}_{\theta\{Y/h\}}$. Thus:

$$\begin{aligned}
[\![P\{\mu h.\, P/h\}]\!]^{sub}_{\theta} &= [\![P]\!]^{sub}_{\theta\{[\![\mu h.\, P]\!]^{sub}_{\theta}/h\}} && \text{by Lemma A.22}\\
&= f([\![\mu h.\, P]\!]^{sub}_{\theta}) && \text{by def. of } f\\
&= f(\bigsqcup_{i\geq 0} f^i(\bot))\\
&= \bigsqcup_{i\geq 0} f^i(\bot) && \bigsqcup_{i\geq 0} f^i(\bot) \text{ is a fixed point of } f\\
&= [\![\mu h.\, P]\!]^{sub}_{\theta}
\end{aligned}$$

$\square$

Lemma A.24. Let P, P' be closed, strongly bound process, let $\mathcal{R}, \mathcal{R}'$ be finite sets of resources, and let $\theta_0 = \emptyset$. Then:

$$P,\, \mathcal{R} \xrightarrow{a} P',\, \mathcal{R}' \implies [\![P]\!]^{sub}_{\theta_0}(\mathcal{R}) \supseteq a \odot [\![P']\!]^{sub}_{\theta_0}(\mathcal{R}')$$

Proof. We proceed by induction on the size of P. We first consider the case $P,\, \mathcal{R} \xrightarrow{!} !,\, \mathcal{R}$. In this case, $! \odot [\![\,!\,]\!]^{sub}_{\theta_0}(\mathcal{R}') = \{\,!\,\} \subseteq [\![P]\!]^{sub}_{\theta_0}(\mathcal{R})$. Otherwise, we consider the following exhaustive cases:

- If $P = \varepsilon$ or $P = !$, then the only applicable rule is $P, \mathcal{R} \xrightarrow{!} !, \mathcal{R}$, which we have already considered.
- If $P = \alpha(r)$, then $a = \alpha(r)$ and $P' = \varepsilon$. We have $[\![\alpha(r)]\!]^{sub}_{\theta_0}(\mathcal{R}) = \{\,!, \alpha(r), \alpha(r)\,!\,\} \supseteq \alpha(r) \odot \{\varepsilon,\, !\} = \alpha(r) \odot [\![\varepsilon]\!]^{sub}_{\theta_0}(\mathcal{R})$.
- If $P = \nu n.\, \bar{P}$, then $a = \varepsilon$, $P' = \bar{P}\{r/n\}$ for some $r \notin \mathcal{R}$, and $\mathcal{R}' = \mathcal{R} \cup \{r\}$. We have $[\![\nu n.\, \bar{P}]\!]_{\theta_0}(\mathcal{R}) = \bigcup_{r' \notin \mathcal{R}} [\![\bar{P}\{r'/n\}]\!]^{sub}_{\theta_0}(\mathcal{R} \cup \{r'\}) \supseteq [\![\bar{P}\{r/n\}]\!]_{\theta_0}(\mathcal{R} \cup \{r\}) = \varepsilon \odot [\![P']\!]_{\theta_0}(\mathcal{R}')$.
- If $P = P_0 + P_1$, then $P' = P_i$ with $i \in \{0, 1\}$, $\mathcal{R}' = \mathcal{R}$ and $a = \varepsilon$. We have $[\![P_0 + P_1]\!]^{sub}_{\theta_0}(\mathcal{R}) = [\![P_0]\!]^{sub}_{\theta_0}(\mathcal{R}) \sqcup [\![P_1]\!]^{sub}_{\theta_0}(\mathcal{R}) \supseteq \varepsilon \odot [\![P']\!]^{sub}_{\theta_0}(\mathcal{R})$.
- If, $P = P_0 \cdot P_1$ we have two subcases:

– If $P_0 = \varepsilon$ and $a = \varepsilon$, then $P' = P_1$ and $\mathcal{R}' = \mathcal{R}$. We have:

$$\begin{aligned}
[\![\varepsilon \cdot P_1]\!]^{sub}_{\theta_0}(\mathcal{R}) &= \big([\![\varepsilon]\!]^{sub}_{\theta_0} \boxdot [\![P_1]\!]^{sub}_{\theta_0}\big)(\mathcal{R}) \\
&= \big((\lambda\mathcal{R}'.\, \{\varepsilon, !\}) \boxdot [\![P_1]\!]^{sub}_{\theta_0}\big)(\mathcal{R}) \\
&= \{\varepsilon, !\} \odot [\![P_1]\!]^{sub}_{\theta_0}(\mathcal{R}) \\
&\supseteq \varepsilon \odot [\![P_1]\!]^{sub}_{\theta_0}(\mathcal{R})
\end{aligned}$$

– Otherwise, $P_0,\ \mathcal{R} \xrightarrow{a} P_0',\ \mathcal{R}'$ and $P' = P_0' \cdot P_1$. The induction hypothesis gives $[\![P_0]\!]^{sub}_{\theta_0}(\mathcal{R}) \supseteq a \odot [\![P_0']\!]^{sub}_{\theta_0}(\mathcal{R}')$, from which we obtain:

$$\begin{aligned}
[\![P_0 \cdot P_1]\!]^{sub}_{\theta_0}(\mathcal{R}) &= ([\![P_0]\!]^{sub}_{\theta_0} \boxdot [\![P_1]\!]^{sub}_{\theta_0})(\mathcal{R}) \\
&= \{\, \eta_0 \odot [\![P_1]\!]^{sub}_{\theta_0}(\mathcal{R} \cup \mathsf{R}(\eta_0)) \mid \eta_0 \in [\![P_0]\!]^{sub}_{\theta_0}(\mathcal{R}) \,\}
\end{aligned}$$

by the induction hypothesis, $[\![P_0]\!]^{sub}_{\theta_0}(\mathcal{R}) \supseteq a \odot [\![P_0']\!]^{sub}_{\theta_0}(\mathcal{R}')$, thus:

$$\supseteq \{\, \eta_0 \odot [\![P_1]\!]^{sub}_{\theta_0}(\mathcal{R} \cup \mathsf{R}(\eta_0)) \mid \eta_0 \in a \odot [\![P_0']\!]^{sub}_{\theta_0}(\mathcal{R}') \,\}$$

since $\mathcal{R} \subseteq \mathcal{R}'$, then by Lemma A.14:

$$\begin{aligned}
&\supseteq \{\, \eta_0 \odot [\![P_1]\!]^{sub}_{\theta_0}(\mathcal{R}' \cup \mathsf{R}(\eta_0)) \mid \eta_0 \in a \odot [\![P_0']\!]^{sub}_{\theta_0}(\mathcal{R}') \,\} \\
&= a \odot \{\, \eta_0 \odot [\![P_1]\!]^{sub}_{\theta_0}(\mathcal{R}' \cup \mathsf{R}(\eta_0)) \mid \eta_0 \in [\![P_0']\!]^{sub}_{\theta_0}(\mathcal{R}') \,\} \\
&= a \odot [\![P_0' \cdot P_1]\!]^{sub}_{\theta_0}(\mathcal{R}')
\end{aligned}$$

- If $P = \mu h.\, \bar{P}$, then $a = \varepsilon$, $\mathcal{R}' = \mathcal{R}$, and $P' = \bar{P}\{P/h\}$. Therefore:

$$[\![P]\!]^{sub}_{\theta_0}(\mathcal{R}) = \Big(\bigsqcup_{n \geq 0} f^n(\bot)\Big)(\mathcal{R})$$

since, by Lemma A.13, f is continuous, then by the Knaster-Tarski theorem:

$$\begin{aligned}
&= f\Big(\bigsqcup_{n \geq 0} f^n(\bot)\Big)(\mathcal{R}) \\
&= f([\![P]\!]^{sub}_{\theta_0})(\mathcal{R}) \\
&= [\![\bar{P}]\!]^{sub}_{\theta\{[\![P]\!]^{sub}_{\theta_0}/h\}}(\mathcal{R})
\end{aligned}$$

and, by Lemma A.22:

$$\begin{aligned}
&= [\![\bar{P}\{P/h\}]\!]^{sub}_{\theta_0}(\mathcal{R}) \\
&= \varepsilon \odot [\![P']\!]^{sub}_{\theta}(\mathcal{R})
\end{aligned}$$

which concludes the proof.

$\square$

Lemma A.25. Let P be a closed, strongly bound process, and let $\mathcal{R}$ be a finite set of resources. Then:

$$[\![P]\!]^{op}(\mathcal{R}) \;=\; [\![P]\!]^{sub}_{\emptyset}(\mathcal{R})$$

Proof. We first prove the inclusion $[\![P]\!]^{op}(\mathcal{R}) \subseteq [\![P]\!]^{sub}_{\emptyset}(\mathcal{R})$. Let $\eta \in [\![P]\!]^{op}_{\mathcal{R}}$. Then, $\eta = a_1 \cdots a_k$, and there exists a chain of transitions:

$$P, \mathcal{R} \xrightarrow{a_1} P_1, \mathcal{R}_1 \xrightarrow{a_2} \cdots \xrightarrow{a_k} P_k, \mathcal{R}_k$$

where $\mathcal{R}' = \mathcal{R}_k \setminus \mathcal{R}$, and either $P_k = \varepsilon$, or $a_k = !$ (in the second case, Lemma A.2 implies that $a_i \neq !$ for all $i < k$). Using Lemma A.24 for $k-1$ times, we obtain:

$$a_1 \odot a_2 \odot \cdots \odot a_k \odot [\![P_k]\!]^{sub}_{\emptyset}(\mathcal{R}_k) \subseteq [\![P]\!]^{sub}_{\emptyset}(\mathcal{R})$$

Now we consider the two cases $P_k = \varepsilon$ and $a_k = !$. If $P_k = \varepsilon$, then $[\![P_k]\!]^{sub}_{\emptyset}(\mathcal{R}_k) = \{\varepsilon\}$, and so $\eta \odot \{\varepsilon\} = \{\eta\}$. Thus, in both cases we have:

$$\{\eta\} = \eta \odot [\![P_k]\!]^{sub}_{\emptyset}(\mathcal{R}_k) \subseteq [\![P]\!]^{sub}_{\emptyset}(\mathcal{R})$$

which proves the inclusion $[\![P]\!]^{op}(\mathcal{R}) \subseteq [\![P]\!]^{sub}_{\emptyset}(\mathcal{R})$. The other inclusion follows directly from Lemma A.21 (note that $fv(P) = \emptyset$ by hypothesis). $\square$

Lemma A.26. For all strongly bound P, for all $\mathcal{R}$, θ and χ such that $fn(P\chi) = \emptyset$:

$$[\![P\chi]\!]^{sub}_{\theta}(\mathcal{R}) = [\![P]\!]^{s}_{\theta}(\chi)(\mathcal{R})$$

Proof. By induction on the size of P. There are the following cases:

- if $P = \varepsilon$, $P = !$, $P = h$ or $P = \alpha(\rho)$, trivial.
- if $P = \nu n.\, P'$, then:

  $$[\![P\chi]\!]^{sub}_{\theta}(\mathcal{R}) = \bigcup_{r \notin \mathcal{R}} [\![P'\chi\{r/n\}]\!]^{sub}_{\theta}(\mathcal{R} \cup \{r\})$$

  and by the induction hypothesis:

  $$\begin{aligned} &= \bigcup_{r \notin \mathcal{R}} [\![P']\!]^{s}_{\theta}(\chi\{r/n\})(\mathcal{R} \cup \{r\}) \\ &= [\![P]\!]^{s}_{\theta}(\chi)(\mathcal{R}) \end{aligned}$$

- if $P = P_0 \cdot P_1$, then:

$$\begin{aligned}
[\![P\chi]\!]^{sub}_{\theta}(\mathcal{R}) &= ([\![P_0\chi]\!]^{sub}_{\theta} \boxdot [\![P_1\chi]\!]^{sub}_{\theta})(\mathcal{R}) \\
&= \{\, \eta_0 \odot \eta_1 \mid \eta_0 \in [\![P_0\chi]\!]^{sub}_{\theta}(\mathcal{R}),\ \eta_1 \in [\![P_1\chi]\!]^{sub}_{\theta}(\mathcal{R} \cup \mathsf{R}(\eta_0)) \,\}
\end{aligned}$$

and by the induction hypothesis:

$$\begin{aligned}
&= \{\, \eta_0 \odot \eta_1 \mid \eta_0 \in [\![P_0]\!]^{s}_{\theta}(\chi)(\mathcal{R}),\ \eta_1 \in [\![P_1]\!]^{s}_{\theta}(\chi)(\mathcal{R} \cup \mathsf{R}(\eta_0)) \,\} \\
&= ([\![P_0]\!]^{s}_{\theta} \boxdot [\![P_1]\!]^{s}_{\theta})(\chi)(\mathcal{R}) \\
&= [\![P]\!]^{s}_{\theta}(\chi)(\mathcal{R})
\end{aligned}$$

- if $P = P_0 + P_1$, straightforward by the induction hypothesis.
- if $P = \mu h.\, P'$, then let:

$$[\![P\chi]\!]^{sub}_{\theta}(\mathcal{R}) = \bigcup_{n \geq 0} (\lambda Z.\, [\![P'\chi]\!]^{sub}_{\theta\{Z/h\}})^n(\bot_{D_{sub}})\,(\mathcal{R})$$

by the induction hypothesis, since $fn(P'\chi) = fn(P\chi) = \emptyset$:

$$\begin{aligned}
&= \bigcup_{n \geq 0} (\lambda Z.\, \lambda \bar{\mathcal{R}}.\, [\![P']\!]^{s}_{\theta\{Z/h\}}(\chi)(\bar{\mathcal{R}}))^n(\bot_{D_{sub}})\,(\mathcal{R}) \\
&= [\![P]\!]^{s}_{\theta}(\chi)(\mathcal{R})
\end{aligned}$$

which concludes the proof.

□

Theorem 2.5 (Full abstraction). For all closed strongly bound processes P, and for all finite sets of resources $\mathcal{R}$:

$$[\![P]\!]^{op}(\mathcal{R}) \;=\; [\![P]\!]^{s}_{\emptyset}(\mathcal{R})(\emptyset)$$

Proof. By Lemmata A.25 and A.26, the hypotheses of which are trivially satisfied:

$$[\![P]\!]^{op}(\mathcal{R}) \;=\; [\![P]\!]^{sub}_{\emptyset}(\mathcal{R}) \;=\; [\![P]\!]^{s}_{\emptyset}(\emptyset)(\mathcal{R})$$

□

B Proofs: weakly bound processes

In this Appendix we prove some intermediate results about weakly bound processes. These will be exploited in App. C and in App. D to show the correctness of the bindify transformation and of the trace inclusion preorder, respectively.

Lemma B.1. For all weakly bound processes p:

$$fn(p) \cap bn^{\Box}(p) = \emptyset \tag{1a}$$

$$Fn(p) \supseteq fn(p) \cup bn^{\diamond}(p) \tag{1b}$$

Proof. Straightforward by induction on the structure of p. $\Box$

The following lemma ensures that the condition $\mathcal{R} \supseteq ran(\chi)$ is always respected by the intermediate results of Def. 3.6. Therefore, in what follows we shall always omit to explicitly check this condition.

Lemma B.2. For all weakly bound processes p, and for all $\mathcal{R}$, χ:

$$(\eta, \chi') \in [\![p]\!]^w_\theta(\chi)(\mathcal{R}) \wedge \mathcal{R} \supseteq ran(\chi) \implies \mathcal{R} \cup \mathsf{R}(\eta) \supseteq ran(\chi')$$

Proof. The only rule that widens $ran(\chi)$ is that for $new(n)$. In that case, $\chi(n) \in \mathsf{R}(\eta)$, so the condition of the Lemma is satisfied. Also, note that in the case $p = \mu h.\, p'$, the operator set_χ sets $\chi' = \chi$. $\Box$

Definition B.3. For all $Y \in D_w$, we define $\mathsf{T}(Y)$ and $\mathsf{R}(Y)$ as follows:

$$\mathsf{R}(Y) = \bigcap_{\mathcal{R},\chi} \mathsf{R}(fst(Y(\chi)(\mathcal{R})))$$

$$\mathsf{T}(Y) = \{\downarrow\} \cap \mathsf{R}(Y)$$

Lemma B.4. For all $Y \in D_w$:

$$\mathsf{T}(Y) = \begin{cases} \emptyset & \text{if } \forall \mathcal{R}, \chi.\, (\eta, \chi') \in Y(\chi)(\mathcal{R}) \implies ! \in \eta \\ \{\downarrow\} & \text{otherwise} \end{cases}$$

Proof. Similar to the proof of Lemma A.16. $\Box$

Definition B.5. We say that $Y \in D_w$ is *sane* if and only if, for all $\mathcal{R}, \chi$:

(5a) $Y(\chi)(\mathcal{R}) \neq \emptyset$

(5b) $Y(\chi)(\mathcal{R}) \supseteq Y(\chi)(\mathcal{R}')$ if $\mathcal{R} \subseteq \mathcal{R}'$

(5c) $(\eta, \chi') \in Y(\bar{\chi})(\mathcal{R}) \implies \chi' \supseteq \bar{\chi}$

(5d) $(\eta, \chi'\{r/n\}) \in Y(\chi\{r/n\})(\mathcal{R}) \wedge r \notin \eta \implies (\eta, \chi') \in Y(\chi)(\mathcal{R})$

(5e) if $r \notin \mathsf{R}(\eta) \wedge n \notin dom(\chi')$:

$$(\eta, \chi') \in Y(\chi)(\mathcal{R}) \implies (\eta, \chi'\{r/n\}) \in Y(\chi\{r/n\})(\mathcal{R} \cup \{r\})$$

(5f) $(\eta, \chi') \in Y(\chi)(\mathcal{R}) \implies \mathsf{R}(Y) \subseteq \mathsf{R}(\eta) \subseteq \mathcal{R}(Y) \cup ran(\chi) \cup (\mathsf{Res} \setminus \mathcal{R})$

We say that $Z \in D_{sub}$ is *sane* if $\lambda\mathcal{R}, \chi.set_\chi Z(\mathcal{R})$ is sane.
We say that θ is *sane* if $\theta(h)$ is sane for all $h \in dom(\theta)$.

Lemma B.6. For all weakly bound processes p and sane θ:

(6a) $[\![p]\!]^w_\theta$ is sane

(6b) $f(Z) = \lambda\bar{\mathcal{R}}.\,fst([\![p]\!]^w_{\theta\{Z/h\}}(\chi)(\bar{\mathcal{R}}))$ is sane, for all χ and sane $Z \in D_{sub}$

Proof. The item (5a) follows trivially from Def. 3.6. In the case $p = new(n)$ with $n \notin dom(\chi)$, since $\mathcal{R}$ is finite we can always choose $r \notin \mathcal{R}$.

For (5b), let $\mathcal{R} \subseteq \mathcal{R}'$. By induction on p, there are the following cases:

- if $p = \varepsilon$, $p = !$, trivial.
- if $p = \alpha(\rho)$, the only non-trivial case is when $\mathcal{R}$ affects the result, i.e. when $\alpha = new$ and $\rho = n \notin dom(\chi)$. In this case, we have that:

$$\begin{aligned}
[\![new(n)]\!]^w_\theta(\chi)(\mathcal{R}) &= \{(\,!,\chi)\} \cup \bigcup_{r\notin\mathcal{R}} set_{\chi\{r/n\}}\{new(r), new(r)\,!\} \\
&\supseteq \{(\,!,\chi)\} \cup \bigcup_{r\notin\mathcal{R}'} set_{\chi\{r/n\}}\{new(r), new(r)\,!\} \\
&= [\![new(n)]\!]^w_\theta(\chi)(\mathcal{R}')
\end{aligned}$$

- if $p = h$, the thesis is implied by the sanity of θ.
- if $p = p_0 \cdot p_1$ and $p = p_0 + p_1$, straightforward application of the induction hypothesis.
- if $p = \mu h.\,p'$, it suffices to prove that if $Z \in D_{sub}$ is anti-monotone, then also $f(Z) = \lambda\bar{\mathcal{R}}.\,fst([\![p']\!]^w_{\theta\{Z/h\}}(\chi')(\bar{\mathcal{R}}))$ is anti-monotone. This follows directly from item (6b).

For (5c), let $(\eta, \chi') \in [\![p]\!]^w_\theta(\chi)(\mathcal{R})$. There are the following cases:

- if $p = \varepsilon$ or $p = !$, trivial.
- if $p = \alpha(\rho)$, the only non-trivial case is when $\chi' \neq \chi$, i.e. when $\alpha = new$ and $\rho = n \notin dom(\chi)$. In this case, we have that:

  $$(\eta, \chi') \in [\![new(n)]\!]^w_\theta(\chi)(\mathcal{R}) = \{(\,!,\chi)\} \cup \bigcup_{r\notin\mathcal{R}} set_{\chi\{r/n\}}\{new(r), new(r)\,!\}$$

  Thus, either $\chi' = \chi$ or $\chi' = \chi\{r/n\} \supseteq \chi$.

- if $p = h$, the thesis is implied by the sanity of θ.

- if $p = p_0 \cdot p_1$, then $(\eta, \chi') = (\eta_0, \chi_0) \odot (\eta_1, \chi_1)$, where:

$$(\eta_0, \chi_0) \in [\![p_0]\!]^w_\theta(\chi)(\mathcal{R})$$
$$(\eta_1, \chi_1) \in [\![p_1]\!]^w_\theta(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0))$$

  By the induction hypothesis on p_0, $\chi_0 \supseteq \chi$, which implies the thesis in the case $! \in \eta_0$. If $! \notin \eta_0$, then the induction hypothesis on p_1 gives $\chi_1 \supseteq \chi_0$, and summing up $\chi_1 \supseteq \chi$.

- if $p = p_0 + p_1$, straightforward application of the induction hypothesis.

- if $p = \mu h.\, p'$, the thesis follows directly from item (6b).

For (5d), let $(\eta, \chi'\{r/n\}) \in [\![p]\!]^w_\theta(\chi\{r/n\})(\mathcal{R})$, with $r \notin \eta$. There are the following cases:

- if $p = \varepsilon$ or $p = !$, trivial.

- if $p = \alpha(\rho)$, there are two subcases. If $\rho \neq n$ the binding $\{r/n\}$ is immaterial, and so the thesis follows directly. The case $\rho = n$ is also trivial, because $\eta = \alpha(r)$ contradicts the hypothesis $r \notin \mathsf{R}(\eta)$.

- if $p = h$, the thesis is implied by the sanity of θ.

- if $p = p_0 \cdot p_1$, then $(\eta, \chi'\{r/n\}) = (\eta_0, \chi_0) \odot (\eta_1, \chi_1)$, where:

$$(\eta_0, \chi_0) \in [\![p_0]\!]^w_\theta(\chi\{r/n\})(\mathcal{R})$$
$$(\eta_1, \chi_1) \in [\![p_1]\!]^w_\theta(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0))$$

  Since $r \notin \eta$ implies $r \notin \eta_0$, and by (5c) $\chi_0 \supseteq \chi\{r/n\}$ (i.e. $\chi_0 = \bar{\chi}_0\{r/n\}$, for some $\bar{\chi}_0$) then by the induction hypothesis on p_0:

$$(\eta_0, \bar{\chi}_0) \in [\![p_0]\!]^w_\theta(\chi)(\mathcal{R})$$

  If $! \in \eta_0$, then $\chi' = \bar{\chi}_0$ proves the thesis. Otherwise, $! \notin \eta_0$, and by the induction hypothesis on p_1:

$$(\eta_1, \chi') \in [\![p_1]\!]^w_\theta(\bar{\chi}_0)(\mathcal{R})$$

  Summing up, we have the thesis $(\eta, \chi') \in [\![p_0 \cdot p_1]\!]^w_\theta(\chi)(\mathcal{R})$.

- if $p = p_0 + p_1$, straightforward application of the induction hypothesis.

- if $p = \mu h.\, p'$, the thesis follows directly from item (6b).

For (5e), let $(\eta, \chi') \in [\![p]\!]^w_\theta(\chi)(\mathcal{R})$ let $r \notin \mathsf{R}(\eta)$, and $n \notin dom(\chi')$. There are the following cases:

- if $p = \varepsilon$ or $p = !$, trivial.

- if $p = \alpha(\rho)$, the only non-trivial case is when $\alpha = \mathit{new}$ and $\rho = n \notin \mathit{dom}(\chi)$. We have that $\eta = \alpha(r')$ and $\chi' = \chi\{r'/n\}$ for some $r' \notin \mathcal{R}$. Since $r \notin \mathcal{R} \cup \mathsf{R}(\eta) = \mathcal{R} \cup \{r'\}$, then also $r' \notin \mathcal{R} \cup \{r\}$, which implies the thesis.

- if $p = h$, the thesis is implied by the sanity of θ.

- if $p = p_0 \cdot p_1$, then $(\eta, \chi') = (\eta_0, \chi_0) \odot (\eta_1, \chi_1)$, where:

$$
\begin{aligned}
&(\eta_0, \chi_0) \in [\![p_0]\!]^w_\theta(\chi)(\mathcal{R}) \\
&(\eta_1, \chi_1) \in [\![p_1]\!]^w_\theta(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0))
\end{aligned}
$$

  By the induction hypothesis applied on p_0 and on p_1:

$$
\begin{aligned}
&(\eta_0, \chi_0\{r/n\}) \in [\![p_0]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r'\}) && \forall r' \notin \mathcal{R} \cup \mathsf{R}(\eta_0) \\
&(\eta_1, \chi_1\{r/n\}) \in [\![p_1]\!]^w_\theta(\chi_0\{r/n\})(\mathcal{R} \cup \mathsf{R}(\eta_0) \cup \{r'\}) && \forall r' \notin \mathcal{R} \cup \mathsf{R}(\eta_0) \cup \mathsf{R}(\eta_1)
\end{aligned}
$$

  If $! \in \eta_0$, the first item above suffices. Otherwise, $\mathsf{R}(\eta) = \mathsf{R}(\eta_0) \cup \mathsf{R}(\eta_1)$, and summing up, $(\eta, \chi'\{r/n\}) \in [\![p]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r'\})$ for all $r' \notin \mathcal{R} \cup \mathsf{R}(\eta)$.

- if $p = p_0 + p_1$, straightforward application of the induction hypothesis.

- if $p = \mu h.\, p'$, the thesis follows directly from item (6b).

For the item 5f, the proof proceeds similarly to the proof of Lemma A.19. □

Lemma B.7. For all weakly bound processes p and sane θ:

$$
[\![p]\!]^w_\theta(\chi)(\mathcal{R}) \subseteq \bigcup_{r \notin \mathcal{R}} [\![p]\!]^w_\theta(\chi)(\mathcal{R} \cup \{r\})
$$

Proof. We prove the following, stronger statement:

(7) $\quad (\eta, \chi') \in [\![p]\!]^w_\theta(\chi)(\mathcal{R}) \implies (\eta, \chi') \in \bigcap_{r \notin \mathcal{R} \cup \mathsf{R}(\eta)} [\![p]\!]^w_\theta(\chi)(\mathcal{R} \cup \{r\})$

Let $(\eta, \chi') \in [\![p]\!]^w_\theta(\chi)(\mathcal{R})$, let $r \notin \mathsf{R}(\eta)$, and choose $n \notin \mathit{dom}(\chi')$. By Lemma B.6, $[\![p]\!]^w_\theta(\chi)(\mathcal{R})$ is sane, and thus by (5e):

$$
(\eta, \chi'\{r/n\}) \in [\![p]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\})
$$

Therefore, by (5d):

$$
(\eta, \chi') \in [\![p]\!]^w_\theta(\chi)(\mathcal{R} \cup \{r\})
$$

□

Lemma B.8. For all weakly bound processes p and sane θ, $\mathsf{R}([\![p]\!]^w_\theta) = \mathsf{R}_{\mathsf{R}(\theta)}(p)$.

Proof. The thesis follows from the following stronger statement, implied by Lemma B.6:

$$\mathsf{R}_{\mathsf{R}(\theta)}(p) \;\subseteq\; \mathsf{R}([\![p]\!]^{w}_{\theta}(\chi))(\mathcal{R}) \;\subseteq\; \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi) \cup (\mathsf{Res} \setminus \mathcal{R})$$

□

Lemma B.9. For all weakly bound processes p, if $(\eta, \chi') \in [\![p]\!]^{w}_{\theta}(\chi)(\mathcal{R})$ then:

(9a) $\quad ran(\chi') \subseteq ran(\chi) \cup \mathsf{R}(\eta)$

(9b) $\quad dom(\chi') \subseteq dom(\chi) \cup bn^{\diamond}(p)$

(9c) $\quad \forall n \notin p \;:\; (\eta, \chi'|_{dom(\chi')\setminus\{n\}}) \in [\![p]\!]^{w}_{\theta}(\chi|_{dom(\chi)\setminus\{n\}})(\mathcal{R})$

(9d) $\quad \forall n \in bn^{\square}(p) \;:\; ! \in \eta \;\vee\; \chi'(n) \in \mathsf{R}(\eta)$

Proof. The first three items are straightforward by induction on the size of p. For (9d), by induction on the size of p there are the following cases:

- if $p = \varepsilon$, $p = \alpha(\rho)$ with $\alpha \neq new$, $p = !$, $p = h$, or $p = \mu h.\, p'$, the precondition is false.
- if $p = new(n)$, immediate by definition.
- if $p = p_0 \cdot p_1$, then $n \in bn^{\square}(p_0)$ or $n \in bn^{\square}(p_1)$. If $! \in \eta$, there is nothing to prove. Otherwise, $\eta = \eta_0 \odot \eta_1$, where $(\eta_0, \chi_0) \in [\![p_0]\!]^{w}_{\theta}(\chi)(\mathcal{R})$ and $(\eta_1, \chi') \in [\![p_1]\!]^{w}_{\theta}(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0))$. There are two subcases.

  If $n \in bn^{\square}(p_0)$, then by the induction hypothesis $\chi_0(n) \in \mathsf{R}(\eta_0)$. The thesis follows by $\mathsf{R}(\eta) \supseteq \mathsf{R}(\eta_0)$ and by Def. 5c, which implies $\chi'(n) = \chi_0(n)$.

  If $n \in bn^{\square}(p_1)$, then by the induction hypothesis $\chi'(n) \in \mathsf{R}(\eta_1)$, which implies the thesis.
- if $p = p_0 + p_1$, then $n \in bn^{\square}(p_0)$ and $n \in bn^{\square}(p_1)$. The thesis then follows by the induction hypothesis.

□

C Proofs: correctness of bindification

In this Appendix we shall establish in Theorem 4.3 the correctness of bindification, i.e. that $[\![p]\!]^w_\emptyset = [\![bindify(p)]\!]^s_\emptyset$ for each well-bound p. Some intermediate results and definitions precede the proof of the main theorem.

Definition C.1. For all η, χ and $n \in \mathsf{Nam}$, we define:

$$flt_n((\eta,\chi)) = \begin{cases} (\eta,\chi) & \text{if } \chi(n) \in \mathsf{R}(\eta) \\ (\eta, \chi|_{dom(\chi)\setminus\{n\}}) & \text{otherwise} \end{cases}$$

Definition C.2. We say $Y \in D_w$ is *anticipating on* n if, for all χ such that $n \notin dom(\chi)$ and $\mathcal{R} \supseteq \mathsf{R}(Y)$ such that $Y(\chi)(\mathcal{R})$ is defined:

$$Y(\chi)(\mathcal{R}) \;\; = \;\; flt_n(\bigcup_{r\notin\mathcal{R}} Y(\chi\{r/n\})(\mathcal{R}\cup\{r\}))$$

We say that $Y \in D_{sub}$ is *anticipating* if $\lambda\mathcal{R},\chi.\, set_\chi Y(\mathcal{R})$ is anticipating on Nam.

Lemma C.3. For all weakly bound processes p, for all anticipating θ such that $dom(\theta) \supseteq fv(p)$, and for all $n \notin fn(p)$, $[\![p]\!]^w_\theta$ is anticipating on n.

Proof. Let $\mathcal{R} \supseteq \mathsf{R}([\![p]\!]^w_\theta) = \mathsf{R}_{\mathsf{R}(\theta)}(p)$ (by Lemma B.8) and let χ be such that $n \notin dom(\chi) \supseteq fn(p)$. By induction on the size of p, there are the following cases:

- If $p = \varepsilon$, $p = !$, trivial.
- If $p = \alpha(r')$ – where possibly $\alpha = new$ – then:

$$\begin{aligned} & flt_n(\bigcup_{r\notin\mathcal{R}} [\![\alpha(r')]\!]^w_\theta(\chi\{r/n\})(\mathcal{R}\cup\{r\})) \\ &= flt_n(\bigcup_{r\notin\mathcal{R}} set_{\chi\{r/n\}}\{\,!, \alpha(r'), \alpha(r')\,!\}) \\ &= \bigcup_{r\notin\mathcal{R}} flt_n(set_{\chi\{r/n\}}\{\,!, \alpha(r'), \alpha(r')\,!\}) \end{aligned}$$

  since $r \notin \mathcal{R} \supseteq \mathsf{R}_{\mathsf{R}(\theta)}(\alpha(r')) \ni r'$, and $n \notin dom(\chi)$:

$$\begin{aligned} &= \bigcup_{r\notin\mathcal{R}} set_\chi\{\,!, \alpha(r'), \alpha(r')\,!\} \\ &= [\![\alpha(r')]\!]^w_\theta(\chi)(\mathcal{R}) \end{aligned}$$

- If $p = \alpha(n')$, with $\alpha \neq \mathit{new}$, then $n' \in \mathit{fn}(p) \not\ni n$. Therefore, $n' \neq n$, so:

$$\begin{aligned}
&\mathit{flt}_n(\bigcup_{r \notin \mathcal{R}} [\![\alpha(n')]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\})) \\
&= \mathit{flt}_n(\bigcup_{r \notin \mathcal{R}} \mathit{set}_{\chi\{r/n\}}\{\,!, \alpha(\chi(n')), \alpha(\chi(n'))\,!\}) \\
&= \bigcup_{r \notin \mathcal{R}} \mathit{flt}_n(\mathit{set}_{\chi\{r/n\}}\{\,!, \alpha(\chi(n')), \alpha(\chi(n'))\,!\})
\end{aligned}$$

since $\chi(n') \in \mathit{ran}(\chi) \subseteq \mathcal{R} \not\ni r$ and $n \notin \mathit{dom}(\chi)$, then:

$$\begin{aligned}
&= \bigcup_{r \notin \mathcal{R}} \mathit{set}_\chi\{\,!, \alpha(\chi(n')), \alpha(\chi(n'))\,!\} \\
&= [\![\alpha(\chi(n'))]\!]^w_\theta(\chi)(\mathcal{R})
\end{aligned}$$

- If $p = \mathit{new}(n)$, the thesis follows by definition, since:

$$\begin{aligned}
&\{(\,!, \chi), (\mathit{new}(r), \chi\{r/n\}), (\mathit{new}(r)\,!, \chi\{r/n\})\} \\
&= \mathit{flt}_n(\mathit{set}_{\chi\{r/n\}}\{\,!, \mathit{new}(r), \mathit{new}(r)\,!\})
\end{aligned}$$

- If $p = h$, then:

$$\mathit{flt}_n(\bigcup_{r \notin \mathcal{R}} [\![h]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\})) = \mathit{flt}_n(\bigcup_{r \notin \mathcal{R}} \mathit{set}_{\chi\{r/n\}}\theta(h)(\mathcal{R} \cup \{r\}))$$

since $\lambda\mathcal{R}, \chi.\, \mathit{set}_\chi\theta(h)(\mathcal{R})$ is anticipating:

$$\begin{aligned}
&= \mathit{set}_\chi\theta(h)(\mathcal{R}) \\
&= [\![h]\!]^w_\theta(\chi)(\mathcal{R})
\end{aligned}$$

- If $p = p_0 \cdot p_1$, we prove the two inclusions separately. For $\subseteq$, let:

$$(\eta, \chi') \in [\![p]\!]^w_\theta(\chi)(\mathcal{R}) = \{\, (\eta_0, \chi_0) \odot (\eta_1, \chi_1) \mid \begin{array}{l} (\eta_0, \chi_0) \in [\![p_0]\!]^w_\theta(\chi)(\mathcal{R}) \\ (\eta_1, \chi_1) \in [\![p_1]\!]^w_\theta(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0)) \end{array} \}$$

Since $n \notin \mathit{fn}(p) \supseteq \mathit{fn}(p_0)$, and $\mathcal{R} \supseteq \mathsf{R}_{\mathsf{R}(\theta)}(p) \supseteq \mathsf{R}_{\mathsf{R}(\theta)}(p_0)$, then by the induction hypothesis on p_0 we have that:

$$(\eta_0, \bar{\chi_0}) \in [\![p_0]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\})$$

for some r, $\bar{\chi_0}$ such that $(\eta_0, \chi_0) = \mathit{flt}_n((\eta_0, \bar{\chi_0}))$. There are two subcases:

– If $r \in \mathsf{R}(\eta_0)$, then $\mathsf{R}(\eta_0) \cup \{r\} = \mathsf{R}(\eta_0)$ and $\bar{\chi_0} = \chi_0$. Then:

$$(\eta_1, \chi_1) \in [\![p_1]\!]^w_\theta(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0) \cup \{r\})$$

and thus we conclude:

$$(\eta, \chi') = (\eta_0, \chi_0) \odot (\eta_1, \chi_1) \in \bigcup_{r \notin \mathcal{R}} [\![p_0 \cdot p_1]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\})$$

Since $r \in \mathsf{R}(\eta)$, the thesis follows from $\mathit{flt}_n((\eta, \chi')) = (\eta, \chi')$.

– If $r \notin \mathsf{R}(\eta_0)$, then $(\eta_0, \chi_0) = \mathit{flt}_n((\eta_0, \bar{\chi_0}))$ implies that $n \notin \mathit{dom}(\chi_0)$. We consider two further subcases.

  ∗ if $! \in \eta_0$, then $(\eta_0, \bar{\chi_0}) \in [\![p_0]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\})$ implies that:

  $$(\eta_0, \bar{\chi_0}) \in [\![p_0 \cdot p_1]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\})$$

  The thesis follows from $\mathit{flt}_n((\eta_0, \bar{\chi_0})) = (\eta_0, \chi_0)$.

  ∗ if $! \notin \eta_0$, then $\mathcal{R} \supseteq \mathsf{R}_{\mathsf{R}(\theta)}(p_0 \cdot p_1) = \mathsf{R}_{\mathsf{R}(\theta)}(p_0) \cup \mathsf{R}_{\mathsf{R}(\theta)}(p_1)$. Since $n \notin \mathit{fn}(p) = \mathit{fn}(p_0) \cup (\mathit{fn}(p_1) \setminus \mathit{bn}^\Box(p_0))$, then $n \notin \mathit{fn}(p_1)$ or $n \in \mathit{bn}^\Box(p_0)$. The case $n \in \mathit{bn}^\Box(p_0)$ is not possible, since by Lemma 9d this would imply $n \in \mathit{dom}(\chi_0)$. Therefore, $n \notin \mathit{fn}(p_1)$, so by the induction hypothesis on p_1:

  $$(\eta_1, \bar{\chi_1}) \in [\![p_1]\!]^w_\theta(\chi_0\{r'/n\})(\mathcal{R} \cup \mathsf{R}(\eta_0) \cup \{r'\})$$

  for some $r' \notin \mathcal{R} \cup \mathsf{R}(\eta_0)$ and $\bar{\chi_1}$ such that $\mathit{flt}_n((\eta_1, \bar{\chi_1})) = (\eta_1, \chi_1)$. Since $[\![p_0]\!]^w_\theta$ is sane and $n \notin \mathit{dom}(\chi_0)$, then by Def. 5e:

  $$(\eta_0, \chi_0\{r'/n\}) \in [\![p_0]\!]^w_\theta(\chi\{r'/n\})(\mathcal{R} \cup \{r'\})$$

  where $r' \notin \mathsf{R}(\eta_0)$. Summing up:

  $$(\eta_0, \chi_0) \odot (\eta_1, \bar{\chi_1}) = (\eta_0, \chi_0\{r'/n\}) \odot (\eta_1, \bar{\chi_1}) \in [\![p_0 \cdot p_1]\!]^w_\theta(\chi\{r'/n\})(\mathcal{R} \cup \{r'\})$$

  Since $r' \notin \mathsf{R}(\eta_0)$, then the thesis follows from:

  $$\begin{aligned}\mathit{flt}_n((\eta_0, \chi_0) \odot (\eta_1, \bar{\chi_1})) &= \mathit{flt}_n((\eta_0\eta_1, \bar{\chi_1})) \\ &= (\eta_0, \chi_0) \odot \mathit{flt}_n((\eta_1, \bar{\chi_1})) \\ &= (\eta_0, \chi_0) \odot (\eta_1, \chi_1) \\ &= (\eta, \chi')\end{aligned}$$

To prove $\supseteq$, let $(\eta, \bar{\chi}') \in [\![p_0 \cdot p_1]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\})$ for some $r \notin \mathcal{R}$ and $n \notin \mathit{dom}(\chi)$. Thus $(\eta, \bar{\chi}') = (\eta_0, \bar{\chi_0}) \odot (\eta_1, \bar{\chi_1})$, where:

$$(\eta_0, \bar{\chi_0}) \in [\![p_0]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\}) \qquad (\eta_1, \bar{\chi_1}) \in [\![p_1]\!]^w_\theta(\bar{\chi_0})(\mathcal{R} \cup \mathsf{R}(\eta_0) \cup \{r\})$$

By the induction hypothesis on p_0, we have that $\mathit{flt}_n((\eta_0, \bar{\chi_0})) \in [\![p_0]\!]^w_\theta(\chi)(\mathcal{R})$. There are two subcases.

– If $r \in \mathsf{R}(\eta_0)$, then, $\mathit{flt}_n((\eta_0, \bar{\chi_0})) = (\eta_0, \bar{\chi_0})$. By hypothesis:

$$(\eta_1, \bar{\chi_1}) \in [\![p_1]\!]^w_\theta(\bar{\chi_0})(\mathcal{R} \cup \mathsf{R}(\eta_0) \cup \{r\})$$

then, since $r \in \mathsf{R}(\eta_0)$:

$$(\eta_1, \bar{\chi_1}) \in [\![p_1]\!]^w_\theta(\bar{\chi_0})(\mathcal{R} \cup \mathsf{R}(\eta_0))$$

Summing up,

$$(\eta_0, \bar{\chi_0}) \odot (\eta_1, \bar{\chi_1}) \in [\![p_0 \cdot p_1]\!]^w_\theta(\chi)(\mathcal{R})$$

To prove the thesis, we show that $(\eta_0, \bar{\chi_0}) \odot (\eta_1, \bar{\chi_1}) = \mathit{flt}_n((\eta_0, \bar{\chi_0}) \odot (\eta_1, \bar{\chi_1}))$. There are two subcases.

  * if $! \in \eta_0$, then $\eta = \eta_0$, $\bar{\chi}' = \bar{\chi_0}$, and $\mathit{flt}_n((\eta_0, \bar{\chi_0})) = (\eta_0, \bar{\chi_0})$.
  * if $! \notin \eta_0$, then since $r \in \mathsf{R}(\eta_0\eta_1)$ and $\bar{\chi_1}(n) = r$ (Def. 5c):

$$\mathit{flt}_n((\eta_0, \bar{\chi_0}) \odot (\eta_1, \bar{\chi_1})) = \mathit{flt}_n((\eta_0\eta_1, \bar{\chi_1})) = (\eta_0\eta_1, \bar{\chi_1})$$

– If $r \notin \mathsf{R}(\eta_0)$, then $\mathit{flt}_n((\eta_0, \bar{\chi_0})) = (\eta_0, \chi_0)$ where $\chi_0 = \bar{\chi_0}|_{dom(\bar{\chi_0}) \setminus \{n\}}$ (by Def. 5c). This implies $n \notin dom(\chi_0)$. There are two subcases.

  * If $! \in \eta_0$, then $(\eta_0, \chi_0) = \mathit{flt}_n((\eta_0, \bar{\chi_0})) \in [\![p_0]\!]^w_\theta(\chi)(\mathcal{R})$ implies $(\eta_0, \chi_0) \in [\![p_0 \cdot p_1]\!]^w_\theta(\chi)(\mathcal{R})$. To prove the thesis, just recall that $\mathit{flt}_n((\eta_0, \bar{\chi_0})) = (\eta_0, \chi_0)$.
  * If $! \notin \eta_0$ then $\mathcal{R} \supseteq \mathsf{R}_{\mathsf{R}(\theta)}(p_0 \cdot p_1) = \mathsf{R}_{\mathsf{R}(\theta)}(p_0) \cup \mathsf{R}_{\mathsf{R}(\theta)}(p_1)$, and $n \notin \mathit{fn}(p_1)$ (see the proof of the analogous case for the inclusion $\subseteq$). The induction hypothesis on p_1 then gives:

$$\mathit{flt}_n((\eta_1, \bar{\chi_1})) \in [\![p_1]\!]^w_\theta(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0))$$

Let $(\eta_1, \chi_1) = \mathit{flt}_n((\eta_1, \bar{\chi_1}))$. Then:

$$(\eta_0, \chi_0) \odot (\eta_1, \chi_1) \in [\![p_0 \cdot p_1]\!]^w_\theta(\chi)(\mathcal{R})$$

To prove the thesis, since $\bar{\chi_1}(n) = r$ by Def. 5c, and $r \notin \mathsf{R}(\eta_0)$:

$$\begin{aligned}
\mathit{flt}_n((\eta_0, \bar{\chi_0}) \odot (\eta_1, \bar{\chi_1})) &= (\eta_0, \bar{\chi_0}) \odot \mathit{flt}_n((\eta_1, \bar{\chi_1})) \\
&= (\eta_0, \bar{\chi_0}) \odot (\eta_1, \chi_1) \\
&= (\eta_0\eta_1, \chi_1) \\
&= (\eta_0, \chi_0) \odot (\eta_1, \chi_1)
\end{aligned}$$

- If $p = p_0 + p_1$, then, by the induction hypothesis:

$$\begin{aligned}
[\![(p_0 + p_1)]\!]^w_\theta(\chi)(\mathcal{R}) &= [\![p_0]\!]^w_\theta(\chi)(\mathcal{R}) \cup [\![p_1]\!]^w_\theta(\chi)(\mathcal{R}) \\
&= \Big(\bigcup_{r \notin \mathcal{R}} [\![p_0]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\})\Big) \cup \Big(\bigcup_{r \notin \mathcal{R}} [\![p_1]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\})\Big) \\
&= \bigcup_{r \notin \mathcal{R}} \Big([\![p_0]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\}) \cup [\![p_1]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\})\Big) \\
&= \bigcup_{r \notin \mathcal{R}} [\![p_0 + p_1]\!]^w_\theta(\chi\{r/n\})(\mathcal{R} \cup \{r\})
\end{aligned}$$

- If $p = \mu h.\, p'$, then let:

$$f(Z) = \lambda \bar{\mathcal{R}}.\, \mathit{fst}(\llbracket p' \rrbracket^w_{\theta\{Z/h\}}(\chi')(\bar{\mathcal{R}}))$$

where $\chi' = \chi|_{dom(\chi)\setminus bn^\diamond(p')}$. We first show that, for all $Z \in D_{sub}$:

$$(8) \qquad \mathsf{R}(Z) \subseteq \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi') \implies \mathsf{R}(f(Z)) \subseteq \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi')$$

We have that:

$$\mathsf{R}(f(Z)) = \bigcap_{\bar{\mathcal{R}}} \mathsf{R}(\mathit{fst}(\llbracket p' \rrbracket^w_{\theta\{Z/h\}}(\chi')))(\bar{\mathcal{R}})$$

and by Lemma B.8:

$$\begin{aligned}
&\subseteq \bigcap_{\bar{\mathcal{R}}} \big(\mathsf{R}_{\mathsf{R}(\theta\{Z/h\})}(p') \cup ran(\chi') \cup (\mathsf{Res} \setminus \bar{\mathcal{R}})\big) \\
&= \mathsf{R}_{\mathsf{R}(\theta\{Z/h\})}(p') \cup ran(\chi') \\
&= \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{R}(Z)/h\}}(p') \cup ran(\chi')
\end{aligned}$$

since $\mathsf{R}(Z) \subseteq \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi')$ by hypothesis, then by Lemma 5d:

$$\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi')/h\}}(p') \cup ran(\chi')$$

and by Lemma 5f:

$$\begin{aligned}
&\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\{\downarrow\} \cap (\mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi'))/h\}}(p') \cup \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi') \cup ran(\chi') \\
&\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{T}_{\mathsf{R}(\theta)}(p)/h\}}(p') \cup \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi')
\end{aligned}$$

and by Lemma 5d:

$$\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{R}_{\mathsf{R}(\theta)}(p)/h\}}(p') \cup \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi')$$

and by Lemma 5g:

$$\begin{aligned}
&= \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi') \\
&= \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi')
\end{aligned}$$

which proves (8). We now prove that, for all $Z \in D_{sub}$ such that $\lambda \mathcal{R}, \chi.set_\chi Z(\mathcal{R})$ is sane:

$$\begin{aligned}
(9) \qquad & \mathsf{R}(Z) \subseteq \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi') \\
&\implies set_\chi f(Z)(\mathcal{R}) = \mathit{flt}_n\Big(\bigcup_{r \notin \mathcal{R}} set_{\chi'\{r/n\}} f(Z)(\mathcal{R} \cup \{r\})\Big)
\end{aligned}$$

We show the two inclusions separately.

For $\subseteq$, let $(\eta, \chi) \in set_\chi f(Z)(\mathcal{R})$. Then, $(\eta, \bar{\chi}) \in f(Z)(\mathcal{R})$ for some $\bar{\chi}$, i.e.:

$$(\eta, \bar{\chi}) \in [\![p']\!]^w_{\theta\{Z/h\}}(\chi')(\mathcal{R})$$

By Lemma B.7, there exists $r \notin \mathcal{R}$ such that:

$$(\eta, \bar{\chi}) \in [\![p']\!]^w_{\theta\{Z/h\}}(\chi')(\mathcal{R} \cup \{r\})$$

which implies:

$$(\eta, \chi\{r/n\}) \in set_{\chi\{r/n\}}[\![p']\!]^w_{\theta\{Z/h\}}(\chi')(\mathcal{R} \cup \{r\})$$

To conclude, we shall prove that $flt_n((\eta, \chi\{r/n\})) = (\eta, \chi)$. To do that, it suffices to show $r \notin \mathsf{R}(\eta)$, which we prove as follows. By Lemma B.8:

$$\begin{aligned}\mathsf{R}(\eta) &\subseteq \mathsf{R}_{\mathsf{R}(\theta\{Z/h\})}(p') \cup ran(\chi') \cup (\mathsf{Res} \setminus (\mathcal{R} \cup \{r\})) \\ &= \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{R}(Z)/h\}}(p') \cup ran(\chi') \cup (\mathsf{Res} \setminus (\mathcal{R} \cup \{r\}))\end{aligned}$$

since $\mathsf{R}(Z) \subseteq \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi')$ by hypothesis, then by Lemma 5d:

$$\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi')/h\}}(p') \cup ran(\chi') \cup (\mathsf{Res} \setminus (\mathcal{R} \cup \{r\}))$$

and by Lemma 5f:

$$\begin{aligned}&\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\{\downarrow\} \cap (\mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi'))/h\}}(p') \cup \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi') \cup (\mathsf{Res} \setminus (\mathcal{R} \cup \{r\})) \\ &\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{T}_{\mathsf{R}(\theta)}(p)/h\}}(p') \cup \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi') \cup (\mathsf{Res} \setminus (\mathcal{R} \cup \{r\}))\end{aligned}$$

and by Lemma 5d:

$$\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{R}_{\mathsf{R}(\theta)}(p)/h\}}(p') \cup \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi') \cup (\mathsf{Res} \setminus (\mathcal{R} \cup \{r\}))$$

and by Lemma 5g:

$$\begin{aligned}&= \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi') \cup (\mathsf{Res} \setminus (\mathcal{R} \cup \{r\})) \\ &= \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup ran(\chi') \cup (\mathsf{Res} \setminus (\mathcal{R} \cup \{r\}))\end{aligned}$$

since by hypothesis $\mathsf{R}_{\mathsf{R}(\theta)}(p) \subseteq \mathcal{R} \supseteq ran(\chi')$:

$$\subseteq \mathcal{R} \cup (\mathsf{Res} \setminus (\mathcal{R} \cup \{r\}))$$

Since $r \notin \mathcal{R}$, this concludes the proof of the $\subseteq$ inclusion of (9).

For $\supseteq$, let $(\eta, \bar{\chi}) \in flt_n(set_{\chi\{r/n\}}[\![p']\!]^w_{\theta\{r/n\}}(\chi)(\mathcal{R} \cup \{r\}))$. Then, $(\eta, \bar{\chi}') \in [\![p']\!]^w_{\theta\{r/n\}}(\chi)(\mathcal{R} \cup \{r\})$ for some $\bar{\chi}'$. Similarly to the inclusion $\subseteq$, it follows that $r \notin \mathsf{R}(\eta)$. So, $\mathit{flt}_n((\eta, \chi\{r/n\})) = (\eta, \chi)$, which implies $\bar{\chi} = \chi$ and concludes the proof of (9).

Back to the main proof, from (8) and (9) it follows that, for all $i \geq 0$:

$$set_\chi f^i(\bot_{D_{sub}})(\mathcal{R}) = \mathit{flt}_n(\bigcup_{r \notin \mathcal{R}} set_{\chi\{r/n\}} f^i(\bot_{D_{sub}})(\mathcal{R} \cup \{r\}))$$

which trivially implies the thesis.

$\square$

Lemma C.4. For all weakly bound processes p such that $wb(p)$:

(4a) $\quad fn(p) = Fn(bindify(p))$

(4b) $\quad \mathsf{R}_\Theta(p) = \mathsf{R}_\Theta(bindify(p))$

(4c) $\quad (\eta, \chi') \in [\![p]\!]^w_\theta(\chi)(\mathcal{R}) \ \wedge\ dom(\chi) \supseteq bn^\diamond(p) \implies \chi = \chi'$

Proof. For (4a), first note that $Fn(bindify(p)) = Fn(\nu\, bn^\diamond(p).\, \beta(p)) = Fn(\beta(p)) \setminus bn^\diamond(p)$. We then proceed by induction on the structure of p. All the cases except for $p = p_0 \cdot p_1$ are straightforward. If $p = p_0 \cdot p_1$, we have that:

$$fn(p) = fn(p_0) \cup (fn(p_1) \setminus bn^\Box(p_0))$$

by the induction hypothesis, applied twice:

$$\begin{aligned}
&= (Fn(\beta(p_0)) \setminus bn^\diamond(p_0)) \cup ((Fn(\beta(p_1)) \setminus bn^\diamond(p_1)) \setminus bn^\Box(p_0)) \\
&= (Fn(\beta(p_0)) \setminus bn^\diamond(p_0)) \cup (Fn(\beta(p_1)) \setminus (bn^\diamond(p_1) \cup bn^\Box(p_0)))
\end{aligned}$$

Since $wb(p)$, then $fn(p) \cap bn^\diamond(p) = \emptyset$, and so $fn(p) \setminus bn^\diamond(p) = fn(p)$:

$$= \big((Fn(\beta(p_0)) \setminus bn^\diamond(p_0)) \cup (Fn(\beta(p_1)) \setminus (bn^\diamond(p_1) \cup bn^\Box(p_0)))\big) \setminus bn^\diamond(p)$$

and since $bn^\Box(p_0) \subseteq bn^\diamond(p_0) \subseteq bn^\diamond(p) \supseteq bn^\diamond(p_1)$:

$$\begin{aligned}
&= (Fn(\beta(p_0)) \cup Fn(\beta(p_1))) \setminus bn^\diamond(p) \\
&= Fn(\beta(p)) \setminus bn^\diamond(p) \\
&= Fn(bindify(p))
\end{aligned}$$

For (4b), first note that $\mathsf{R}_\Theta(bindify(p)) = \mathsf{R}_\Theta(\beta(p))$. Then, the proof is trivial by induction on the structure of p.

For (4c), the only rule that updates χ is that for $new(n)$, when $n \notin dom(\chi)$. However, it is never applicable, since $bn^\diamond(new(n)) = \{n\} \subseteq dom(\chi)$. $\square$

Lemma C.5. For all weakly bound processes p such that $wb(p)$, and for all $\mathcal{R}, \chi, \theta$ such that $dom(\chi) \supseteq fn(p)$, $dom(\theta) \supseteq fv(p)$, and $\mathcal{R} \supseteq \mathsf{R}_{\mathsf{R}(\theta)}(p)$:

$$
\begin{array}{llcll}
\text{(5a)} & [\![p]\!]^w_\theta(\mathcal{R})(\chi) & & \text{is defined} & \\
\text{(5b)} & [\![bindify(p)]\!]^s_\theta(\chi)(\mathcal{R}) & = & fst([\![p]\!]^w_\theta(\chi)(\mathcal{R})) & \text{if } dom(\chi) \cap bn^\diamond(p) = \emptyset \\
\text{(5c)} & [\![\beta(p)]\!]^s_\theta(\chi)(\mathcal{R}) & = & fst([\![p]\!]^w_\theta(\chi)(\mathcal{R})) & \text{if } dom(\chi) \supseteq bn^\diamond(p)
\end{array}
$$

Proof. We proceed by induction on the size of p. For the first item, it is easy to prove that $wb(p)$ implies $fn(p) \cap bn^\diamond(p) = \emptyset$, and furthermore this property is also preserved under the recursions μh. Therefore, each free name in p is always bound in the environment χ, and so the semantics $[\![p]\!]^w_\theta(\mathcal{R})(\chi)$ is defined.

We now show that (5c) implies (5b).

$$[\![bindify(p)]\!]^s_\theta(\chi)(\mathcal{R}) = [\![\nu\, bn^\diamond(p).\, \beta(p)]\!]^s_\theta(\chi)(\mathcal{R})$$

by definition of the semantics of ν (and with a slight abuse of notation):

$$= \bigcup_{\substack{\mathcal{R} \cap \mathcal{R}' = \emptyset \\ |\mathcal{R}'| = |bn^\diamond(p)|}} [\![\beta(p)]\!]^s_\theta(\chi\{\mathcal{R}'/bn^\diamond(p)\})(\mathcal{R} \cup \mathcal{R}')$$

since $dom(\chi\{\mathcal{R}'/bn^\diamond(p)\}) \supseteq bn^\diamond(p)$, then by (5c):

$$= \bigcup_{\substack{\mathcal{R} \cap \mathcal{R}' = \emptyset \\ |\mathcal{R}'| = |bn^\diamond(p)|}} fst([\![p]\!]^w_\theta(\chi\{\mathcal{R}'/bn^\diamond(p)\})(\mathcal{R} \cup \mathcal{R}'))$$

now, let $n \in bn^\diamond(p)$. By hypothesis, we have that $n \notin dom(\chi)$. Also, since $dom(\chi) \supseteq Fn(p)$, Lemma B.1 implies that $n \notin fn(p)$. By Lemma C.3 applied for all $n \in bn^\diamond(p)$:

$$= fst([\![p]\!]^w_\theta(\chi)(\mathcal{R}))$$

which proves that (5c) implies (5b). For (5c), there are the following cases:

- if $p = \varepsilon$, $p = !$, $p = \alpha(\rho)$ it directly follows by the definition of the semantics. Note that in the case $p = new(n)$, we have $n \in dom(\chi)$ by hypothesis, so no fresh resource is generated by the weak semantics.
- if $p = h$, then:

  $$[\![\beta(h)]\!]^s_\theta(\chi)(\mathcal{R}) = [\![h]\!]^s_\theta(\chi)(\mathcal{R}) = \theta(h)(\mathcal{R}) = fst(set_\chi \theta(h)(\mathcal{R})) = fst([\![h]\!]^w_\theta(\chi)(\mathcal{R}))$$

- if $p = p_0 \cdot p_1$, then there are two subcases.

– If $\downarrow \notin \mathsf{R}_{\mathsf{R}(\theta)}(p_0)$, then:

$$\begin{aligned}\llbracket \beta(p) \rrbracket^s_\theta(\chi)(\mathcal{R}) &= (\llbracket \beta(p_0) \rrbracket^s_\theta \boxdot \llbracket \beta(p_1) \rrbracket^s_\theta)(\chi)(\mathcal{R}) \\ &= \llbracket \beta(p_0) \rrbracket^s_\theta(\chi)(\mathcal{R})\end{aligned}$$

since $bn^\diamond(p_0) \subseteq bn^\diamond(p) \subseteq dom(\chi)$, $wb(p_0)$, $dom(\chi) \supseteq Fn(p) \supseteq Fn(p_0)$, and $\mathcal{R} \supseteq \mathsf{R}_{\mathsf{R}(\theta)}(p) = \mathsf{R}_{\mathsf{R}(\theta)}(p_0)$, then by the induction hypothesis:

$$= \mathit{fst}(\llbracket p_0 \rrbracket^w_\theta(\chi)(\mathcal{R}))$$

by Lemma B.8, since $\downarrow \notin \mathsf{R}_{\mathsf{R}(\theta)}(p_0) \implies\ ! \in \eta_0$ for all $(\eta_0, \chi_0) \in \llbracket p_0 \rrbracket^w_\theta(\chi)(\mathcal{R})$:

$$= \mathit{fst}(\llbracket p_0 \cdot p_1 \rrbracket^w_\theta(\chi)(\mathcal{R}))$$

– Otherwise, if $\downarrow \in \mathsf{R}_{\mathsf{R}(\theta)}(p_0)$, then:

$$\begin{aligned}\llbracket \beta(p) \rrbracket^s_\theta(\chi)(\mathcal{R}) &= (\llbracket \beta(p_0) \rrbracket^s_\theta \boxdot \llbracket \beta(p_1) \rrbracket^s_\theta)(\chi)(\mathcal{R}) \\ &= \{\, \eta_0 \odot \eta_1 \mid \eta_0 \in \llbracket \beta(p_0) \rrbracket^s_\theta(\chi)(\mathcal{R}),\, \eta_1 \in \llbracket \beta(p_1) \rrbracket^s_\theta(\chi)(\mathcal{R} \cup \mathsf{R}(\eta_0)) \,\}\end{aligned}$$

since $bn^\diamond(p_0) \subseteq bn^\diamond(p) \subseteq dom(\chi)$, $wb(p_0)$, $dom(\chi) \supseteq Fn(p) \supseteq Fn(p_0)$, and $\mathcal{R} \supseteq \mathsf{R}_{\mathsf{R}(\theta)}(p) \supseteq \mathsf{R}_{\mathsf{R}(\theta)}(p_0)$, then by the induction hypothesis on p_0:

$$= \{\, \eta_0 \odot \eta_1 \mid \eta_0 \in \mathit{fst}(\llbracket p_0 \rrbracket^w_\theta(\chi)(\mathcal{R})),\, \eta_1 \in \llbracket \beta(p_1) \rrbracket^s_\theta(\chi)(\mathcal{R} \cup \mathsf{R}(\eta_0)) \,\}$$

since $bn^\diamond(p_1) \subseteq bn^\diamond(p) \subseteq dom(\chi)$, $wb(p_1)$, $dom(\chi) \supseteq Fn(p) \supseteq Fn(p_1)$, and $\mathcal{R} \cup \mathsf{R}(\eta_0) \supseteq \mathcal{R} \supseteq \mathsf{R}_{\mathsf{R}(\theta)}(p) \supseteq \mathsf{R}_{\mathsf{R}(\theta)}(p_1)$, then by the induction hypothesis on p_1:

$$\begin{aligned}&= \{\, \eta_0 \odot \eta_1 \mid \eta_0 \in \mathit{fst}(\llbracket p_0 \rrbracket^w_\theta(\chi)(\mathcal{R})),\, \eta_1 \in \mathit{fst}(\llbracket p_1 \rrbracket^w_\theta(\chi)(\mathcal{R} \cup \mathsf{R}(\eta_0))) \,\} \\ &= \{\, \eta_0 \odot \eta_1 \mid (\eta_0, \chi_0) \in \llbracket p_0 \rrbracket^w_\theta(\chi)(\mathcal{R}),\, (\eta_1, \chi_1) \in \llbracket p_1 \rrbracket^w_\theta(\chi)(\mathcal{R} \cup \mathsf{R}(\eta_0)) \,\}\end{aligned}$$

since $dom(\chi) \supseteq bn^\diamond(p)$, then by Lemma 4c $\chi_0 = \chi$, and:

$$\begin{aligned}&= \{\, \eta_0 \odot \eta_1 \mid (\eta_0, \chi_0) \in \llbracket p_0 \rrbracket^w_\theta(\chi)(\mathcal{R}),\, (\eta_1, \chi_1) \in \llbracket p_1 \rrbracket^w_\theta(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0)) \,\} \\ &= \mathit{fst}(\llbracket p_0 \rrbracket^w_\theta \boxdot \llbracket p_1 \rrbracket^w_\theta(\chi)(\mathcal{R})) \\ &= \mathit{fst}(\llbracket p_0 \cdot p_1 \rrbracket^w_\theta(\chi)(\mathcal{R}))\end{aligned}$$

- if $p = p_0 + p_1$, we have, for all $i \in \{0, 1\}$, $wb(p_i)$, $dom(\chi) \supseteq fn(p_i)$, $dom(\theta) \supseteq fv(p_i)$, and $\mathcal{R} \supseteq \mathsf{R}_{\mathsf{R}(\theta)}(p_i)$, and $dom(\chi) \supseteq bn^\diamond(p_i)$. Therefore, we conclude by the induction hypothesis and $\beta(p_0 + p_1) = \beta(p_0) + \beta(p_1)$.

- if $p = \mu h.\, p'$, we have $\beta(p) = \mu h.\, \mathit{bindify}(p')$. Let:

$$\chi' = \chi|_{dom(\chi)\setminus bn^{\diamond}(p')} \qquad f(Z) = \lambda\bar{\mathcal{R}}.\ [\![\mathit{bindify}(p')]\!]^{s}_{\theta\{Z/h\}}(\chi')(\bar{\mathcal{R}})$$

We first prove that, for all Z:
(10)
$\mathsf{R}(Z) \subseteq \mathcal{R} \wedge \mathsf{T}(Z) \subseteq \mathsf{T}_{\mathsf{R}(\theta)}(p) \implies \mathsf{R}(f(Z)) \subseteq \mathcal{R} \wedge \mathsf{T}(f(Z)) \subseteq \mathsf{T}_{\mathsf{R}(\theta)}(p)$

To prove (10), we have that:

$$\mathsf{R}(f(Z) = \mathsf{R}(\lambda\bar{\mathcal{R}}.\ [\![\mathit{bindify}(p')]\!]^{s}_{\theta\{Z/h\}}(\chi')(\bar{\mathcal{R}}))$$

by Lemma 4a, $Fn(\mathit{bindify}(p')\chi') = \mathit{fn}(p'\chi') = \emptyset$. Then, by Lemma A.26:

$$= \mathsf{R}([\![\mathit{bindify}(p')\chi']\!]^{sub}_{\theta\{Z/h\}})$$

by Lemma A.19:

$$= \mathsf{R}_{\mathsf{R}(\theta\{Z/h\})}(\mathit{bindify}(p')\chi')$$

by Lemma 5e:

$$\subseteq \mathsf{R}_{\mathsf{R}(\theta\{Z/h\})}(\mathit{bindify}(p')) \cup \mathit{ran}(\chi')$$

by Lemma 4b:

$$= \mathsf{R}_{\mathsf{R}(\theta\{Z/h\})}(p') \cup \mathit{ran}(\chi')$$
$$= \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{R}(Z)/h\}}(p') \cup \mathit{ran}(\chi')$$

by Lemma 5f:

$$\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{T}(Z)/h\}}(p') \cup \mathsf{R}(Z) \cup \mathit{ran}(\chi')$$

since $\mathsf{T}(Z) \subseteq \mathsf{T}_{\mathsf{R}(\theta)}(p)$ by hypothesis, then by Lemma 5a:

$$\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{T}_{\mathsf{R}(\theta)}(p)/h\}}(p') \cup \mathsf{R}(Z) \cup \mathit{ran}(\chi')$$

by Lemma 5d:

$$\subseteq \mathsf{R}_{\mathsf{R}(\theta)\{\mathsf{R}_{\mathsf{R}(\theta)}(p)/h\}}(p') \cup \mathsf{R}(Z) \cup \mathit{ran}(\chi')$$

by Lemma 5g:

$$\subseteq \mathsf{R}_{\mathsf{R}(\theta)}(p) \cup \mathsf{R}(Z) \cup \mathit{ran}(\chi')$$

Since, by hypothesis, all the three components are included in $\mathcal{R}$, we conclude that $\mathsf{R}(f(Z)) \subseteq \mathcal{R}$ (recall that the semantics $[\![p]\!]^w_\theta(\chi)(\mathcal{R})$ is undefined if $\mathcal{R} \not\supseteq ran(\chi)$). Also, by the hypothesis $\mathsf{T}(Z) \subseteq \mathsf{T}_{\mathsf{R}(\theta}(p)$, we have:

$$\begin{aligned} \mathsf{T}(f(Z)) &= \mathsf{R}(f(Z)) \cap \{\downarrow\} \\ &\subseteq (\mathsf{R}_{\mathsf{R}(\theta)}(p) \cup \mathsf{R}(Z) \cup ran(\chi')) \cap \{\downarrow\} \\ &= \mathsf{T}_{\mathsf{R}(\theta)}(p) \cup \mathsf{T}(Z) \\ &= \mathsf{T}_{\mathsf{R}(\theta)}(p) \end{aligned}$$

which concludes the proof of (10). Back to the main statement, we have:

$$\begin{aligned} [\![\beta(p)]\!]^s_\theta(\chi)(\mathcal{R}) &= [\![\mu h.\, bindify(p')]\!]^s_\theta(\chi)(\mathcal{R}) \\ &= \bigcup_{i\geq 0} (\lambda Z, \bar{\mathcal{R}}.\, [\![bindify(p')]\!]^s_{\theta\{Z/h\}}(\chi)(\bar{\mathcal{R}}))^i(\bot_{D_{sub}})(\mathcal{R}) \end{aligned}$$

since $bindify(p') = \nu\, bn^\diamond(p').\, \beta(p')$, then $\chi|_{bn^\diamond(p')}$ has no influence on the semantics $[\![bindify(p')]\!]^s_{\theta\{Z/h\}}(\chi)(\bar{\mathcal{R}})$, and so:

$$= \bigcup_{i\geq 0} (\lambda Z, \bar{\mathcal{R}}.\, [\![bindify(p')]\!]^s_{\theta\{Z/h\}}(\chi')(\bar{\mathcal{R}}))^i(\bot_{D_{sub}})(\mathcal{R})$$

by (10), in the equation above $\bar{\mathcal{R}} \supseteq \mathsf{R}_{\mathsf{R}(\theta\{Z/h\})}(p')$ for all i. Also, $dom(\chi') \supseteq fn(p')$. Indeed, $dom(\chi') = dom(\chi) \setminus bn^\diamond(p') \supseteq Fn(p) \setminus bn^\diamond(p') \supseteq (fn(p) \cup bn^\diamond(p)) \setminus bn^\diamond(p') = fn(p') \setminus bn^\diamond(p') = fn(p')$, where we have used Lemma B.1 and the hypothesis $wb(p')$. We can then apply the induction hypothesis (5b):

$$\begin{aligned} &= \bigcup_{i\geq 0} (\lambda Z, \bar{\mathcal{R}}.\, fst([\![p']\!]^w_{\theta\{Z/h\}}(\chi')(\bar{\mathcal{R}})))^i(\bot_{D_{sub}})(\mathcal{R}) \\ &= fst(set_\chi \bigcup_{i\geq 0} (\lambda Z, \bar{\mathcal{R}}.\, fst([\![p']\!]^w_{\theta\{Z/h\}}(\chi')(\bar{\mathcal{R}})))^i(\bot_{D_{sub}})(\mathcal{R})) \\ &= fst([\![\mu h.\, p']\!]^w_\theta(\chi)(\mathcal{R})) \end{aligned}$$

□

Theorem 4.3. For all closed, weakly bound processes p such that $wb(p)$, $[\![p]\!]^w_\emptyset(\emptyset)(\emptyset)$ is defined, and:

$$[\![bindify(p)]\!]^s_\emptyset(\emptyset)(\emptyset) \quad = \quad fst([\![p]\!]^w_\emptyset(\emptyset)(\emptyset))$$

Proof. Direct from Lemma 5b, the hypotheses of which are trivially satisfied. □

D Proofs: equational theories and trace inclusion

In this Appendix we prove the main results from Sect. 5, i.e. that the preorder for strongly bound processes preserves trace inclusion (Theorem 5.3), and that the same happens for weakly bound processes (Theorem 5.6). Finally, in Theorem 5.7 we show that the equational theory of weakly bound processes preserve well-boundedness, and that processes smaller (w.r.t. $\precsim_{\mathcal{N}}$) than well-bound processes are well-bound.

Definition D.1. The *names* $\mathsf{N}(p)$ of a weakly bound process p are defined as follows:

$$\mathsf{N}(p) = \mathit{fn}(p) \cup bn^{\diamond}(p)$$

Definition D.2. Let $Y, Z \in D_w$. We write $Y \precsim_{\mathcal{N}} Z$ when for all $\mathcal{R}, \chi$ such that $\mathcal{R} \cap \mathcal{N} = \chi \cap \mathcal{N} = \emptyset$:

$$\{\, (\eta, \chi'|_{dom(\chi') \setminus \mathcal{N}}) \mid (\eta, \chi') \in Y(\chi)(\mathcal{R} \cup (\mathcal{N} \cap \mathsf{Res}))\,\} \subseteq Z(\chi)(\mathcal{R})$$

Let $Y, Z \in D_{sub}$, we write $Y \preceq_{\mathcal{N}} Z$ whenever $\lambda\mathcal{R}, \chi.\, set_\chi Y(\mathcal{R}) \precsim_{\mathcal{N}} \lambda\mathcal{R}, \chi.\, set_\chi Z(\mathcal{R})$. We write $Y \preceq Z$ when $Y \preceq_{\emptyset} Z$.

Lemma D.3. For all strongly bound processes P, and for all θ, θ':

$$\theta \preceq \theta' \implies [\![P]\!]^{sub}_{\theta} \preceq [\![P]\!]^{sub}_{\theta'}$$

Proof. Straightforward by induction on the structure of P. □

Theorem 5.3. For all closed, strongly bound processes P and P':

- if $P = P'$, then $[\![P]\!]^{s}_{\emptyset} = [\![P']\!]^{s}_{\emptyset}$.
- if $P \preceq P'$ then $[\![P]\!]^{s}_{\emptyset}(\chi)(\mathcal{R}) \subseteq [\![P']\!]^{s}_{\emptyset}(\chi)(\mathcal{R})$, for all $\mathcal{R}$ and χ.

Proof. The first item follows from Lemma A.11 and Lemma A.23. For the second item, since $P \preceq P'$ implies $P\chi \preceq P'\chi$, then by Lemma A.26 is suffices to prove that $[\![P]\!]^{sub}_{\emptyset}(\mathcal{R}) \subseteq [\![P']\!]^{sub}_{\emptyset}(\mathcal{R})$, for all $\mathcal{R}$. We proceed by induction on the size of the derivation of $P \preceq P'$:

- $P = P'$. The thesis follows from the first item.
- $P' = P + P''$. Then, for all $\mathcal{R}$:

$$[\![P]\!]^{sub}_{\theta}(\mathcal{R}) \subseteq [\![P]\!]^{sub}_{\theta}(\mathcal{R}) \cup [\![P'']\!]^{sub}_{\theta}(\mathcal{R}) = [\![P']\!]^{sub}_{\theta}(\mathcal{R})$$

- $P \preceq P''$ and $P'' \preceq P'$. Straightforward by transitivity of $\subseteq$.
- $P = \mathcal{C}(\bar{P}) \preceq \mathcal{C}(\bar{P}')$ and $\bar{P} \preceq \bar{P}'$. There are the following cases:

- $\mathcal{C} = P'' \cdot \bullet$. Let $\eta \in [\![P'' \cdot \bar{P}]\!]^{sub}_{\theta}(\mathcal{R})$. Then, $\eta = \eta_0 \odot \eta_1$, with $\eta_0 \in [\![P'']\!]^{sub}_{\theta}(\mathcal{R})$ and $\eta_1 \in [\![\bar{P}]\!]^{sub}_{\theta}(\mathcal{R} \cup \mathsf{R}(\eta_0))$. There are two subcases.

  If $! \in \eta_0$, then $\eta = \eta_0 \in [\![P'' \cdot \bar{P}']\!]^{sub}_{\theta}(\mathcal{R})$.

  If $! \notin \eta_0$, then $\eta = \eta_0\eta_1$. Since $\bar{P} \preceq \bar{P}'$, then by the induction hypothesis: $\eta_1 \in [\![\bar{P}']\!]^{sub}_{\theta}(\mathcal{R} \cup \mathsf{R}(\eta_0))$, which implies the thesis $\eta_0\eta_1 \in [\![P']\!]^{sub}_{\theta}(\mathcal{R})$.
- $\mathcal{C} = \bullet \cdot P''$. Let $\eta \in [\![\bar{P} \cdot P'']\!]^{sub}_{\theta}(\mathcal{R})$. Then, $\eta = \eta_0 \odot \eta_1$, with $\eta_0 \in [\![\bar{P}]\!]^{sub}_{\theta}(\mathcal{R})$ and $\eta_1 \in [\![P'']\!]^{sub}_{\theta}(\mathcal{R} \cup \mathsf{R}(\eta_0))$. By the induction hypothesis, $\eta_0 \in [\![\bar{P}']\!]^{sub}_{\theta}(\chi)(\mathcal{R})$. There are the following two subcases.

  If $! \in \eta_0$, then $\eta_0 \odot \eta_1 = \eta_0$, which implies the thesis.

  If $! \notin \eta_0$, then $\eta_0 \odot \eta_1 = \eta_0\eta_1$, from which the thesis $\eta_0 \odot \eta_1 \in [\![P']\!]^{sub}_{\theta}(\mathcal{R})$.
- $\mathcal{C} = \nu n. \bullet$. The thesis follows directly from the induction hypothesis.
- $\mathcal{C} = \mu h. \bullet$. By definition, $[\![P]\!]^{sub}_{\theta} = \bigsqcup_{i \geq 0} f^i(\bot)$ and $[\![P']\!]^{sub}_{\theta} = \bigsqcup_{i \geq 0} g^i(\bot)$, where $f(Y) = [\![\bar{P}]\!]^{sub}_{\theta\{Y/h\}}$ and $g(Y) = [\![\bar{P}']\!]^{sub}_{\theta\{Y/h\}}$. It suffices proving that, for all $i \geq 0$, $f^i(\bot) \subseteq g^i(\bot)$. The base case $i = 0$ is trivial. For the inductive case, we have:

$$\begin{aligned}
f^{i+1}(\bot) &= f(f^i(\bot)) \\
&= [\![\bar{P}]\!]^{sub}_{\theta\{f^i(\bot)/h\}} \\
&\subseteq [\![\bar{P}]\!]^{sub}_{\theta\{g^i(\bot)/h\}} && \text{by } f^i(\bot) \preceq g^i(\bot) \text{ and Lemma D.3} \\
&\subseteq [\![\bar{P}']\!]^{sub}_{\theta\{g^i(\bot)/h\}} && \text{by the induction hypothesis on } \bar{P} \preceq \bar{P}' \\
&= g(g^i(\bot)) \\
&= g^{i+1}(\bot)
\end{aligned}$$

- the other contexts, $\bullet + P''$ and $P'' + \bullet$, are trivial.

□

The following two lemmata are straightforward.

Lemma D.4. For all weakly bound processes p and q such that $p \approx q$:

(4a) $\quad fn(p) = fn(q)$

(4b) $\quad bn^{\Box}(p) = bn^{\Box}(q)$

(4c) $\quad bn^{\diamond}(p) = bn^{\diamond}(q)$

Lemma D.5. For all weakly bound processes p and q such that $p \precsim_{\mathcal{N}} q$:

$$fn(p) \subseteq fn(q) \tag{5a}$$

$$bn^{\diamond}(p) \subseteq bn^{\diamond}(q) \cup \mathcal{N} \tag{5b}$$

$$bn^{\Box}(p) \supseteq bn^{\Box}(q) \tag{5c}$$

Definition D.6. The *captures* $cpt_{\mathcal{N}}(p, h)$ of h in a weakly bound process p are defined inductively as follows:

$$cpt_{\mathcal{N}}(\varepsilon, h) = cpt_{\mathcal{N}}(\,!, h) = cpt_{\mathcal{N}}(\alpha(\rho), h) = \emptyset \qquad cpt_{\mathcal{N}}(h', h) = \begin{cases} \mathcal{N} & \text{if } h = h' \\ \emptyset & \text{otherwise} \end{cases}$$

$$cpt_{\mathcal{N}}(p_0 \cdot p_1, h) = cpt_{\mathcal{N}}(p_0 + p_1, h) = cpt_{\mathcal{N}}(p_0, h) \cup cpt_{\mathcal{N}}(p_1, h)$$

$$cpt_{\mathcal{N}}(\mu h'.p', h) = \begin{cases} cpt_{\mathcal{N} \cup bn^{\diamond}(p')}(p', h) & \text{if } h \neq h' \\ \emptyset & \text{otherwise} \end{cases}$$

We say $p\{p'/h\}$ *capture-avoiding* if $bn^{\diamond}(p') = \emptyset = cpt_{bn^{\diamond}(p)}(p, h) \cap fn(p')$.

Lemma D.7 (Substitution)**.** If $p\{p'/h\}$ is capture-avoiding, then:

$$[\![p\{p'/h\}]\!]^w_\theta(\chi)(\mathcal{R}) \quad = \quad [\![p]\!]^w_{\theta\{\lambda\bar{\mathcal{R}}.\,fst([\![p']\!]^w_\theta(\chi)(\bar{\mathcal{R}}))/h\}}(\chi)(\mathcal{R})$$

Proof. By induction on the structure of p. If the substitution is vacuous, then the thesis follows trivially. Otherwise, we note that $fn(p') \subseteq dom(\chi)$ must hold for the semantics of the both sides to be defined.

There are the following cases:

- if $p = h$, the hypothesis $bn^{\diamond}(p') = \emptyset$ suffices.
- if $p = p_0 \cdot p_1$, then $(\eta, \chi') \in [\![p\{p'/h\}]\!]^w_\theta(\chi)(\mathcal{R})$ if and only if there exist (η_0, χ_0) and (η_1, χ_1) such that $(\eta, \chi') = (\eta_0, \chi_0) \odot (\eta_1, \chi_1)$ and:

  $$(\eta_0, \chi_0) \in [\![p_0\{p'/h\}]\!]^w_\theta(\chi)(\mathcal{R})$$
  $$(\eta_1, \chi_1) \in [\![p_1\{p'/h\}]\!]^w_\theta(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0))$$

  Since both $p_0\{p'/h\}$ and $p_1\{p'/h\}$ are capture-avoiding, then by the induction hypothesis on p_0 and on p_1, the above is equivalent to:

  $$(\eta_0, \chi_0) \in [\![p_0]\!]^w_{\theta\{\lambda\bar{\mathcal{R}}.\,fst([\![p']\!]^w_\theta(\chi)(\bar{\mathcal{R}}))/h\}}(\chi)(\mathcal{R})$$
  $$(\eta_1, \chi_1) \in [\![p_1]\!]^w_{\theta\{\lambda\bar{\mathcal{R}}.\,fst([\![p']\!]^w_\theta(\chi_0)(\bar{\mathcal{R}}))/h\}}(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0))$$

  Since $fn(p') \subseteq dom(\chi)$ and $bn^{\diamond}(p') = \emptyset$, a simple structural induction shows that $fst([\![p']\!]^w_\theta(\chi_0)(\bar{\mathcal{R}})) = fst([\![p']\!]^w_\theta(\chi)(\bar{\mathcal{R}}))$ – recall that χ_0 and χ agree on $dom(\chi)$. Therefore, the second item above can be restated as:

  $$(\eta_1, \chi_1) \in [\![p_1]\!]^w_{\theta\{\lambda\bar{\mathcal{R}}.\,fst([\![p']\!]^w_\theta(\chi)(\bar{\mathcal{R}}))/h\}}(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0))$$

Summing up, all the above is equivalent to:

$$(\eta_0, \chi_0) \odot (\eta_1, \chi_1) \in [\![p]\!]^w_{\theta\{\lambda\bar{\mathcal{R}}.\, fst([\![p']\!]^w_\theta(\chi)(\bar{\mathcal{R}}))/h\}}(\chi)(\mathcal{R})$$

- if $p = p_0 + p_1$, straightforward by the induction hypothesis.
- if $p = \mu h'.\, \bar{p}$ with $h' \neq h$, then:

$$[\![p\{p'/h\}]\!]^w_\theta(\chi)(\mathcal{R}) = [\![\mu h'.\, \bar{p}\{p'/h\}]\!]^w_\theta(\chi)(\mathcal{R}) \;=\; set_\chi \bigcup_{i \geq 0} f^i(\bot_{D_{sub}})(\mathcal{R})$$

where $f(Z) = \lambda\bar{\mathcal{R}}.\, fst([\![\bar{p}\{p'/h'\}]\!]^w_{\theta\{Z/h'\}}(\chi')(\bar{\mathcal{R}}))$, where $\chi' = \chi|_{dom(\chi)\setminus bn^\diamond(\bar{p})}$. Since $p\{p'/h\}$ is capture-avoiding, then:

$$\begin{aligned} cpt_{bn^\diamond(\bar{p})}(\bar{p}, h) \cap fn(p') &= cpt_{bn^\diamond(p)\cup bn^\diamond(\bar{p})}(\bar{p}, h) \cap fn(p') \\ &= cpt_{bn^\diamond(p)}(p, h) \cap fn(p') = \emptyset \end{aligned}$$

Therefore the substitution $\bar{p}\{p'/h\}$ is capture-avoiding, and so by the induction hypothesis:

$$f(Z) = \lambda\bar{\mathcal{R}}.\, fst([\![\bar{p}]\!]^w_{\theta\{Z/h',\, \lambda\bar{\mathcal{R}}.\, fst([\![p']\!]^w_\theta(\chi')(\bar{\mathcal{R}}))/h\}}(\chi')(\bar{\mathcal{R}}))$$

Since $bn^\diamond(\bar{p}) \cap fn(p') = \emptyset$, and $bn^\diamond(p') = \emptyset$, then χ and $\chi' = \chi|_{dom(\chi)\setminus bn^\diamond(\bar{p})}$ agree on the names in p'. This implies that:

$$f(Z) = \lambda\bar{\mathcal{R}}.\, fst([\![\bar{p}]\!]^w_{\theta\{Z/h',\, \lambda\bar{\mathcal{R}}.\, fst([\![p']\!]^w_\theta(\chi)(\bar{\mathcal{R}}))/h\}}(\chi')(\bar{\mathcal{R}}))$$

which concludes the proof.

□

Definition D.8. Let χ be a function from Nam to Res, and let $n, m \in \mathsf{Nam}$. We define $\chi[m/n]$ as the function:

$$\chi[m/n](n') = \begin{cases} \chi(n) & \text{if } n' = m \\ \chi(n') & \text{if } n' \neq m \text{ and } n' \neq n \end{cases}$$

Lemma D.9. For all weakly bound processes p such that $wb(p)$, $\mathcal{R}, \chi$ and sane θ:

(9a) if $n \notin bn^{\diamond}(p)$ and $r \in \mathcal{R}$:
$(\eta, \chi') \in [\![p\{r/n\}]\!]^{w}_{\theta}(\chi)(\mathcal{R}) \implies (\eta, \chi'\{r/n\}) \in [\![p]\!]^{w}_{\theta}(\chi\{r/n\})(\mathcal{R})$

(9b) if $n \notin bn^{\diamond}(p) \cup dom(\chi)$ and $m \notin \mathsf{N}(p)$:
$(\eta, \chi') \in [\![p\{m/n\}]\!]^{w}_{\theta}(\chi)(\mathcal{R}) \implies (\eta, \chi'[n/m]) \in [\![p]\!]^{w}_{\theta}(\chi[n/m])(\mathcal{R})$

(9c) if $n \notin fn(p) \cup dom(\chi)$ and $r \notin \mathsf{N}(p) \cup \mathcal{R} \cup ran(\chi)$, then:
$(\eta, \chi') \in [\![p\{r/n\}]\!]^{w}_{\theta}(\chi)(\mathcal{R} \cup \{r\}) \implies (\eta, \chi'') \in [\![p]\!]^{w}_{\theta}(\chi)(\mathcal{R})$
where $\chi'' = \chi'$ if $r \notin \mathsf{R}(\eta)$, otherwise $\chi'' = \chi'\{r/n\}$.

(9d) if $n \notin fn(p) \cup dom(\chi)$ and $m \notin \mathsf{N}(p) \cup dom(\chi)$, then:
$(\eta, \chi') \in [\![p\{m/n\}]\!]^{w}_{\theta}(\chi)(\mathcal{R}) \implies (\eta, \chi'[n/m]) \in [\![p]\!]^{w}_{\theta}(\chi)(\mathcal{R})$

Proof. The first two items are straightforward by structural induction. For (9c), we proceed by induction on the structure of p. There are the following cases:

- if $p = new(n')$, there are two further subcases.

  If $n' \neq n$, then the substitution is vacuous, and $\eta = new(r')$, $\chi' = \chi\{r'/n'\}$ for some $r' \notin \mathcal{R}$. Then, $(\eta, \chi') \in [\![new(n')]\!]^{w}_{\theta}(\chi)(\mathcal{R})$.

  If $n' = n$, then $\eta = new(r)$ and $\chi' = \chi$. Then, $r \in \mathsf{R}(\eta)$, and $(\eta, \chi\{r/n\}) \in [\![new(n)]\!]^{w}_{\theta}(\chi)(\mathcal{R})$.

- if $p = p_0 \cdot p_1$, then $(\eta, \chi') = (\eta_0, \chi_0) \odot (\eta_1, \chi_1)$, where:

$$(\eta_0, \chi_0) \in [\![p_0\{r/n\}]\!]^{w}_{\theta}(\chi)(\mathcal{R} \cup \{r\})$$
$$(\eta_1, \chi_1) \in [\![p_1\{r/n\}]\!]^{w}_{\theta}(\chi_0)(\mathcal{R} \cup \{r\} \cup \mathsf{R}(\eta_0))$$

  By the induction hypothesis on p_0, it follows that:

$$(\eta_0, \chi''_0) \in [\![p_0]\!]^{w}_{\theta}(\chi)(\mathcal{R}) \qquad \chi''_0 = \begin{cases} \chi_0 & \text{if } r \notin \mathsf{R}(\eta_0) \\ \chi_0\{r/n\} & \text{otherwise} \end{cases}$$

  If $! \in \eta_0$, this implies the thesis. Otherwise, there are two subcases.

  - $r \notin \mathsf{R}(\eta_0)$. In this case, we first prove that $n \notin fn(p_1)$. Since by hypothesis $n \notin fn(p) = fn(p_0) \cup (fn(p_1) \setminus bn^{\square}(p_0))$, it suffices to prove that $n \notin bn^{\square}(p_0)$. By contradiction, assume $n \in bn^{\square}(p_0)$. Then, by Lemma 9d, $\chi''_0(n) = \chi_0(n) \in \mathsf{R}(\eta_0)$. By Lemma 9b, $n \in dom(\chi_0) \subseteq dom(\chi) \cup bn^{\diamond}(p_0\{r/n\})$. This is a contradiction, because $n \notin dom(\chi)$ by hypothesis, and clearly $n \notin bn^{\diamond}(p_0\{r/n\})$. This proves $n \notin fn(p_1)$. Lemma 9b implies that

$dom(\chi_0) \subseteq dom(\chi) \cup bn^\diamond(p_0\{r/n\})$, and so as above it follows that $n \notin dom(\chi_0)$. By Lemma 9a, $ran(\chi_0) \subseteq ran(\chi) \cup (\mathsf{Res} \setminus (\mathcal{R} \cup \{r\}))$, so $r \notin ran(\chi_0)$. We have then verified that all the hypothesis of the Lemma hold for p_1, so by the induction hypothesis:

$$(\eta_1, \chi_1'') \in [\![p_1]\!]^w_\theta(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0)) \qquad \chi_1'' = \begin{cases} \chi_1 & \text{if } r \notin \mathsf{R}(\eta_1) \\ \chi_1\{r/n\} & \text{otherwise} \end{cases}$$

Since in this case $r \in \mathsf{R}(\eta_1) \iff r \in \mathsf{R}(\eta_0\eta_1)$ we have that:

$$(\eta_0\eta_1, \chi_1'') \in [\![p_0 \cdot p_1]\!]^w_\theta(\chi)(\mathcal{R}) \qquad \chi_1'' = \begin{cases} \chi_1 & \text{if } r \notin \mathsf{R}(\eta) \\ \chi_1\{r/n\} & \text{otherwise} \end{cases}$$

- $r \in \mathsf{R}(\eta_0)$. Since $\chi_0'' = \chi\{r/n\}$, Lemma 9b implies that $n \in bn^\diamond(p_0)$, and since $wb(p)$, then $n \notin bn^\diamond(p_1)$. We are therefore under the hypotheses of item (a) of the current Lemma, which implies:

$$(\eta_1, \chi_1\{r/n\}) \in [\![p_1]\!]^w_\theta(\chi_0\{r/n\})(\mathcal{R} \cup \{r\} \cup \mathsf{R}(\eta_0))$$

Summing up, since $r \in \mathsf{R}(\eta) \supseteq \mathsf{R}(\eta_0)$:

$$(\eta_0\eta_1, \chi_1\{r/n\}) \in [\![p_0 \cdot p_1]\!]^w_\theta(\chi)(\mathcal{R})$$

- if $p = p_0 + p_1$, straightforward application of the induction hypothesis.
- if $p = \mu h.\, p'$, then $p\{r/n\} = p$, since $n \notin fn(p)$ and the substitution is vacuous if $n \in bn^\diamond(p')$. Also note that $r \notin \mathsf{R}(\eta) \subseteq \mathsf{R}(p) \cup ran(\chi) \cup (\mathsf{Res} \setminus (\mathcal{R} \cup \{r\}))$. By sanity (5b), $(\eta, \chi') \in [\![p]\!]^w_\theta(\chi)(\mathcal{R} \cup \{r\})$ implies $(\eta, \chi') \in [\![p]\!]^w_\theta(\chi)(\mathcal{R})$.

For (9d), we proceed by induction on the size of p. There are the following cases:

- if $p = new(n')$, there are two further subcases.
  - If $n' \neq n$, then $\eta = new(r')$ and $\chi' = \chi\{r'/n'\}$ for some $r' \notin \mathcal{R}$. Then, $(new(r'), \chi'[n/m]) \in [\![new(n')]\!]^w_\theta(\chi)(\mathcal{R})$ since $n, m \notin dom(\chi)$ implies $\chi'[n/m] = \chi\{r'/n'\}[n/m] = \chi'$.
  - If $n' = n$, then $\eta = new(r')$ and $\chi' = \chi\{r'/m\}$ for some $r' \notin \mathcal{R}$. Then, $(new(r'), \chi'[n/m]) \in [\![new(n)]\!]^w_\theta(\chi)(\mathcal{R})$ since $m \notin dom(\chi)$ implies $\chi'[n/m] = \chi\{r'/m\}[n/m] = \chi\{r'/n\}$.
- if $p = p_0 \cdot p_1$, then $(\eta, \chi') = (\eta_0, \chi_0) \odot (\eta_1, \chi_1)$, where:

$$(\eta_0, \chi_0) \in [\![p_0\{m/n\}]\!]^w_\theta(\chi)(\mathcal{R})$$
$$(\eta_1, \chi_1) \in [\![p_1\{m/n\}]\!]^w_\theta(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0))$$

By the induction hypothesis on p_0, it follows that:

$$(\eta_0, \chi_0[n/m]) \in [\![p_0]\!]^w_\theta(\chi)(\mathcal{R})$$

If $! \in \eta_0$, this implies the thesis. Otherwise, there are two subcases.

- $n \in bn^\diamond(p_1)$. Since $wb(p)$, then $wb(p_1)$ and so $n \notin fn(p_1)$ and $n \notin bn^\diamond(p_0) \cup fn(p_0)$. This also implies $n \notin dom(\chi_0)$, because by Lemma 9b, $dom(\chi_0) \subseteq dom(\chi) \cup bn^\diamond(p_0\{m/n\}) \not\ni n$. Also, $m \notin dom(\chi_0)$, because by Lemma 9b, $dom(\chi_0[n/m]) \subseteq dom(\chi) \cup bn^\diamond(p_0) \not\ni n$ (see Def. D.8). All the hypoteses of the Lemma are safisfied, and so by the induction hypothesis on p_1 it follows that:

  $$(\eta_1, \chi_1[n/m]) \in [\![p_1]\!]^w_\theta(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0))$$

  Summing up:

  $$(\eta_0\eta_1, \chi_1[n/m]) \in [\![p_0 \cdot p_1]\!]^w_\theta(\chi)(\mathcal{R})$$

- $n \notin bn^\diamond(p_1)$. In this case the hypothesis of item (b) of the current Lemma are satisfied, so:

  $$(\eta_1, \chi_1[n/m]) \in [\![p_1]\!]^w_\theta(\chi_0[n/m])(\mathcal{R} \cup \mathsf{R}(\eta_0))$$

  Summing up:

  $$(\eta_0\eta_1, \chi_1[n/m]) \in [\![p_0 \cdot p_1]\!]^w_\theta(\chi)(\mathcal{R})$$

- if $p = p_0 + p_1$, straightforward application of the induction hypothesis.
- if $p = \mu h.\, p'$, then $p\{m/n\} = p$, since $n \notin fn(p)$ and the substitution is vacuous if $n \in bn^\diamond(p')$. Since $n, m \notin dom(\chi)$, then $\chi'[n/m] = \chi'$, which implies the thesis.

□

Lemma D.10. If $X \precsim_{\mathcal{N}} Y$, $Y \precsim_{\mathcal{N}'} Z$ and $\mathcal{N} \cap \mathcal{N}' = \emptyset$, then $X \precsim_{\mathcal{N} \cup \mathcal{N}'} Z$.

Proof. Let $\mathcal{N}'' = \mathcal{N} \cup \mathcal{N}'$, and let $\mathcal{R}, \chi$ be such that $\mathcal{R} \cap \mathcal{N}'' = \chi \cap \mathcal{N}'' = \emptyset$. Let $(\eta, \chi') \in X(\chi)(\mathcal{R} \cup (\mathcal{N}'' \cap \mathsf{Res}))$. Since $(\mathcal{R} \cup (\mathcal{N}' \cap \mathsf{Res})) \cap \mathcal{N} = \chi \cap \mathcal{N} = \emptyset$ and $X \precsim_{\mathcal{N}} Y$, then there exists $(\eta', \chi'|_{dom(\chi') \setminus \mathcal{N}}) \in Y(\chi)(\mathcal{R} \cup (\mathcal{N}' \cap \mathsf{Res}))$ and $\eta \precsim \eta'$. Since $\mathcal{R} \cap \mathcal{N}' = \chi \cap \mathcal{N}' = \emptyset$, and $Y \precsim_{\mathcal{N}'} Z$, then there exists $(\eta'', \chi'|_{dom(\chi') \setminus \mathcal{N}''}) \in Z(\chi)(\mathcal{R})$ and $\eta' \precsim \eta''$. This proves the thesis, because the relation $\precsim$ on traces is transitive. □

Lemma D.11. For all weakly bound processes p, p' and for all $\mathcal{N}$ and θ:

$$p \approx p' \implies [\![p]\!]^w_\theta \precsim_\emptyset [\![p']\!]^w_\theta \tag{11a}$$

$$p \precsim_{\mathcal{N}} p' \implies [\![p]\!]^w_\theta \precsim_{\mathcal{N}} [\![p']\!]^w_\theta \tag{11b}$$

Proof. The first item is straightforward by induction on the size of the proof of $p \approx p'$. For the second item, we proceed by induction on the size of the proof of $p \precsim_{\mathcal{N}} p'$. There are the following exhaustive cases:

- if $p = \mathcal{C}(\bar{p}) \precsim_{\mathcal{N}} \mathcal{C}(\bar{p}')$ and $\bar{p} \precsim_{\mathcal{N}} \bar{p}'$, with $\mathcal{N} \cap \mathsf{N}(\mathcal{C}) = \emptyset$, there are the following exhaustive cases:
  - if $\mathcal{C} = p'' \cdot \bullet$, then $\mathcal{N} \cap \mathsf{N}(p'') = \emptyset$. Let $(\eta, \chi') \in [\![p'' \cdot \bar{p}]\!]^w_\theta(\chi)(\mathcal{R} \cup (\mathcal{N} \cap \mathsf{Res}))$. Then, $(\eta, \chi') = (\eta_0, \chi_0) \odot (\eta_1, \chi_1)$, with:

    $$(\eta_0, \chi_0) \in [\![p'']\!]^w_\theta(\chi)(\mathcal{R} \cup (\mathcal{N} \cap \mathsf{Res}))$$
    $$(\eta_1, \chi_1) \in [\![\bar{p}]\!]^w_\theta(\chi_0)(\mathcal{R} \cup (\mathcal{N} \cap \mathsf{Res}) \cup \mathsf{R}(\eta_0))$$

    There are two subcases.
    If $! \in \eta_0$, then $(\eta, \chi') = (\eta_0, \chi_0)$. Since $\mathcal{N} \cap \mathsf{N}(p'') = \emptyset$, then by Lemma 9c applied $|\mathcal{N}|$ times, it follows that:

    $$(\eta_0, \chi_0|_{dom(\chi_0)\setminus\mathcal{N}}) \in [\![p'']\!]^w_\theta(\chi|_{dom(\chi)\setminus\mathcal{N}})(\mathcal{R} \cup (\mathcal{N} \cap \mathsf{Res}))$$

    and since $\mathcal{N} \cap dom(\chi) = \emptyset$:

    $$(\eta_0, \chi_0|_{dom(\chi_0)\setminus\mathcal{N}}) \in [\![p'']\!]^w_\theta(\chi)(\mathcal{R} \cup (\mathcal{N} \cap \mathsf{Res}))$$

    by Def. 5b:

    $$(\eta_0, \chi_0|_{dom(\chi_0)\setminus\mathcal{N}}) \in [\![p'']\!]^w_\theta(\chi)(\mathcal{R})$$

    and finally, since $! \in \eta_0$:

    $$(\eta_0, \chi_0|_{dom(\chi_0)\setminus\mathcal{N}}) \in [\![p'' \cdot \bar{p}']\!]^w_\theta(\chi)(\mathcal{R})$$

    If $! \notin \eta_0$, then $(\eta, \chi') = (\eta_0\eta_1, \chi_1)$. We have that:

    1. $\mathcal{N} \cap (\mathcal{R} \cup \mathsf{R}(\eta_0)) = \emptyset$, since $\mathcal{N}$ is disjoint from $\mathcal{R}$ by hypothesis, and by Lemma B.8, $\mathsf{R}(\eta_0) \subseteq \mathsf{R}_{\mathsf{R}(\theta)}(p'') \cup ran(\chi) \cup (\mathsf{Res} \setminus (\mathcal{R} \cup \mathcal{N}))$, and both $ran(\chi)$ and $\mathsf{R}_{\mathsf{R}(\theta)}(p'') \subseteq \mathsf{N}(p'')$ are disjoint from $\mathcal{N}$ by hypothesis.
    2. $\mathcal{N} \cap \chi_0 = \emptyset$. By Lemma 9b, $dom(\chi_0) \subseteq dom(\chi) \cup bn^\diamond(p'')$ and both $dom(\chi)$ and $bn^\diamond(p'') \subseteq \mathsf{N}(p'')$ are disjoint from $\mathcal{N}$. Also, by Lemma 9a, $ran(\chi_0) \subseteq ran(\chi) \cup \mathsf{R}(\eta_0)$, and $ran(\chi)$ is disjoint from $\mathcal{N}$ by hypothesis, while we have proved in the previous item that $\mathsf{R}(\eta_0) \cap \mathcal{N} = \emptyset$.

    By the induction hypothesis:

    $$(\eta_1, \chi_1|_{dom(\chi_1)\setminus\mathcal{N}}) \in [\![\bar{p}']\!]^w_\theta(\chi_0)(\mathcal{R} \cup \mathsf{R}(\eta_0))$$

    from which we obtain the thesis:

    $$(\eta_0\eta_1, \chi_1|_{dom(\chi_1)\setminus\mathcal{N}}) \in [\![p']\!]^w_\theta(\chi)(\mathcal{R} \cup (\mathcal{N} \cap \mathsf{Res}))$$

- if $\mathcal{C} = \bullet \cdot p''$, then $\mathcal{N} \cap \mathsf{N}(p'') = \emptyset$. Let $(\eta, \chi') \in [\![\bar{p} \cdot p'']\!]^w_\theta(\chi)(\mathcal{R} \cup (\mathcal{N} \cap \mathsf{Res}))$. Then, $(\eta, \chi') = (\eta_0, \chi_0) \odot (\eta_1, \chi_1)$, with:

$$(\eta_0, \chi_0) \in [\![\bar{p}]\!]^w_\theta(\chi)(\mathcal{R} \cup (\mathcal{N} \cap \mathsf{Res}))$$
$$(\eta_1, \chi_1) \in [\![p'']\!]^w_\theta(\chi_0)(\mathcal{R} \cup (\mathcal{N} \cap \mathsf{Res}) \cup \mathsf{R}(\eta_0))$$

By the induction hypothesis, there exists $(\eta_0, \chi_0|_{dom(\chi_0)\setminus\mathcal{N}}) \in [\![\bar{p}']\!]^w_\theta(\chi)(\mathcal{R})$ By Def. 5b it follows that:

$$(\eta_1, \chi_1) \in [\![p'']\!]^w_\theta(\chi_0)(\mathcal{R} \cup (\mathcal{N} \cap \mathsf{Res}) \cup \mathsf{R}(\eta_0))$$

and so by Lemma 9c, applied $|\mathcal{N}|$ times:

$$(\eta_1, \chi_1|_{dom(\chi_1)\setminus\mathcal{N}}) \in [\![p'']\!]^w_\theta(\chi_0|_{dom(\chi_0)\setminus\mathcal{N}})(\mathcal{R} \cup (\mathcal{N} \cap \mathsf{Res}) \cup \mathsf{R}(\eta_0))$$

Summing up:

$$(\eta_0, \chi_0|_{dom(\chi_0)\setminus\mathcal{N}}) \odot (\eta_1, \chi_1|_{dom(\chi_1)\setminus\mathcal{N}}) \in [\![p']\!]^w_\theta(\chi)(\mathcal{R})$$

There are two subcases.

  * If $! \in \eta_0$, then:

$$(\eta_0, \chi_0|_{dom(\chi_0)\setminus\mathcal{N}}) \odot (\eta_1, \chi_1|_{dom(\chi_1)\setminus\mathcal{N}}) = (\eta_0, \chi_0|_{dom(\chi_0)\setminus\mathcal{N}})$$

  * Otherwise, if $! \notin \eta_0$, then:

$$(\eta_0, \chi_0|_{dom(\chi_0)\setminus\mathcal{N}}) \odot (\eta_1, \chi_1|_{dom(\chi_1)\setminus\mathcal{N}}) = (\eta_0\eta_1, \chi_1|_{dom(\chi_1)\setminus\mathcal{N}})$$

- the context $\mu h.\bullet$ does not exist, by Def. 5.5.
- the other contexts, $\bullet + p''$ and $p'' + \bullet$, are trivial.

- if $p' = p + p''$, then $\mathcal{N} = \emptyset$. Let $(\eta, \chi') \in [\![p]\!]^w_\theta(\chi)(\mathcal{R} \cup (\emptyset \cap \mathsf{Res}))$. Then, $(\eta, \chi'|_{dom(\chi)\setminus\emptyset}) \in [\![p']\!]^w_\theta(\chi)(\mathcal{R})$.
- if $p \precsim_{\mathcal{N}'} p''$ and $p'' \precsim_{\mathcal{N}''} p'$, with $\mathcal{N} = \mathcal{N}' \cup \mathcal{N}''$ and $\mathcal{N}' \cap \mathcal{N}'' = \emptyset$, the thesis follows by Lemma D.10.
- if $p \approx p'$, the thesis follows from Lemma 11a.
- if $p = \bar{p}\vartheta\{p'/h\}$ and $p' = \mu h.\, \bar{p}$, then $\mathcal{N} = ran(\vartheta)$, $dom(\vartheta) = bn^\diamond(p)$ and $ran(\vartheta) \cap \mathsf{N}(\bar{p}) = \emptyset$.

$$(\eta, \chi') \in [\![\bar{p}\vartheta\{p'/h\}]\!]^w_\theta(\chi)(\mathcal{R} \cup (ran(\vartheta) \cap \mathsf{Res}))$$

Since $\bar{p}\vartheta\{p'/h\}$ is capture-avoiding, then by the Substitution Lemma (D.7):

$$(\eta, \chi') \in [\![\bar{p}\vartheta]\!]^w_{\theta\{\lambda\bar{\mathcal{R}}.\, fst([\![p']\!]^w_\theta(\chi)(\bar{\mathcal{R}}))/h\}}(\chi)(\mathcal{R} \cup (ran(\vartheta) \cap \mathsf{Res}))$$

By repeated applications of Lemma D.9(c,d), whose conditions remain satisfied after each application, there exists χ'' such that:

$$(\eta, \chi'') \in [\![\bar{p}]\!]^w_{\theta\{\lambda\bar{\mathcal{R}}.\, fst([\![p']\!]^w_\theta(\chi)(\bar{\mathcal{R}}))/h\}}(\chi)(\mathcal{R})$$

Therefore:

$$\begin{aligned}
&\eta \in fst([\![\bar{p}]\!]^w_{\theta\{\lambda\bar{\mathcal{R}}.\, fst([\![p']\!]^w_\theta(\chi)(\bar{\mathcal{R}}))/h\}}(\chi)(\mathcal{R})) \\
&\quad = fst([\![\bar{p}]\!]^w_{\theta\{\lambda\bar{\mathcal{R}}.\, fst([\![p']\!]^w_\theta(\chi|_{dom(\chi)\setminus bn^\diamond(\bar{p})})(\bar{\mathcal{R}}))/h\}}(\chi)(\mathcal{R})) \\
&\quad = f(\lambda\bar{\mathcal{R}}.\, fst([\![p']\!]^w_\theta(\chi|_{dom(\chi)\setminus bn^\diamond(\bar{p})})(\bar{\mathcal{R}})))(\mathcal{R})
\end{aligned}$$

where $f(Z) = \lambda\bar{\mathcal{R}}.\, fst([\![\bar{p}]\!]^w_{\theta\{Z/h\}}(\chi|_{dom(\chi)\setminus bn^\diamond(\bar{p})})(\bar{\mathcal{R}}))$.

$$\begin{aligned}
&\quad = f\big(\lambda\bar{\mathcal{R}}.\, fst(set_{\chi|_{dom(\chi)\setminus bn^\diamond(\bar{p})}} \bigcup_{i\geq 0} f^i(\bot_{D_{sub}})(\bar{\mathcal{R}}))\big)(\mathcal{R}) \\
&\quad = f\big(\lambda\bar{\mathcal{R}}.\, \bigcup_{i\geq 0} f^i(\bot_{D_{sub}})(\bar{\mathcal{R}})\big)(\mathcal{R})
\end{aligned}$$

and since f is a fixed point:

$$= fst([\![p']\!]^w_\theta(\chi)(\mathcal{R}))$$

Therefore, $(\eta, \chi) \in [\![p']\!]^w_\theta(\chi)(\mathcal{R})$. Since $(\eta, \chi') \in [\![\bar{p}\vartheta\{p'/h\}]\!]^w_\theta$ and $bn^\diamond(\bar{p}\vartheta\{p'/h\}) = bn^\diamond(\bar{p}\vartheta)$, then by (5c) and by Lemma 9b:

$$dom(\chi) \subseteq dom(\chi') \subseteq dom(\chi) \cup bn^\diamond(\bar{p}\vartheta) = dom(\chi) \cup ran(\vartheta)$$

Therefore, $dom(\chi') \setminus ran(\vartheta) = dom(\chi) \setminus ran(\theta)$, and since $dom(\chi) \cap ran(\vartheta) = \emptyset$, then $dom(\chi') \setminus ran(\vartheta) = dom(\chi)$. To conclude the proof, it suffices to note that $\chi'|_{dom(\chi')\setminus\mathcal{N}} = \chi'|_{dom(\chi')\setminus ran(\vartheta)} = \chi'|_{dom(\chi)} = \chi$, where the last equation follows from 5c.

□

Theorem 5.6. For all closed, weakly bound processes p and q:

- if $p \approx q$, then $[\![p]\!]^w_\emptyset = [\![q]\!]^w_\emptyset$.
- if $p \precsim_{\mathcal{N}} q$, then $fst([\![p]\!]^w_\emptyset(\chi)(\mathcal{R})) \subseteq fst([\![q]\!]^w_\emptyset(\chi)(\mathcal{R}))$, for all $\mathcal{R}$ and χ such that $dom(\chi) \cap \mathcal{N} = \emptyset$ and both the semantics are defined.

Proof. The first item follows from Lemma 11a. The second item follows from Lemma 11b and by Def. D.2. □

Theorem 5.7. For all weakly bound processes p and q:

- if $p \approx q$, then $wb(p)$ if and only if $wb(q)$.
- if $p \approx q$ and $wb(p)$, then $bindify(p) = bindify(q)$.
- if $p \precsim_{\mathcal{N}} q$ and $wb(q)$, then $wb(p)$.

Proof. The first items is straightforward by Lemma D.4 and by induction on the derivation of $p \approx q$. The second item is by induction on the structure of p, and it exploits Lemma 4c to yield $bn^{\diamond}(p) = bn^{\diamond}(q)$. The last item is straightforward by Lemma D.5 and by induction on the derivation of $p \precsim_{\mathcal{N}} q$. □