Università di Pisa

Dipartimento di Informatica

Technical Report: TR-10-05

Hardness of some optimal oblivious routing generalizations

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March 4, 2010

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Abstract

A generalization of the robust network design problem with oblivious routing is investigated in [6], where the (uncertain) demands are served through two alternative routing templates. As indicated in [6], it is an open issue as to whether the proposed problem, called $(2-RND)$, is polynomially solvable or is NP-Hard. In this note we solve the issue by proving that $(2-RND)$, as well as some generalizations, are NP-Hard. The hardness result holds true also when some routing templates are given as input data. This strengthens the results in [6], where special $(2-RND)$ cases are devised which are tractable from a computational perspective.

Keywords: robust optimization, oblivious routing, mathematical models, NP-Hardness.

Introduction

Let $G=(V,E)$ be a directed network, with $|V|=n$ and $|E|=m$. Let K be a set of k origin-destination pairs, and c_{ij} denote the non-negative cost of installing a unit of capacity along (i,j), $\forall(i,j)\in E$. Let $\mathcal{D}$ be a bounded non empty polyhedron which describes the uncertain, non simultaneous, demands between the given origin-destination pairs.

The *robust network design problem* on G (RND) consists of determining a capacity allocation for the arcs of G, and choosing a routing for the origin-destination pairs, in such a way as to satisfy each demand in $\mathcal{D}$ at a minimum allocation cost. In the case of oblivious routing (RND) is polynomially solvable [1],[3]. The routing is said to be *oblivious* when the same routing template is used for each traffic demand in $\mathcal{D}$. More precisely, the routing specifies a unit flow for each origin-destination pair (s,t): if the demand d_{st} needs to be routed, then it is routed by simply scaling up the unit flow by d_{st}. Formally, let y denote a vector of routing variables, and y_{ij}^{st} be the fraction of the demand of (s,t) to be routed along the arc (i,j). Then y: $E\times K\rightarrow[0,1]$ is a *routing template* if it satisfies the following flow conservation constraints $\forall\ i\in V,\ (s,t)\in K$:

$$\sum_{(j,i)\in BS(i)} y_{ji}^{st} - \sum_{(i,j)\in FS(i)} y_{ij}^{st} = \phi_i^{st} \quad = \begin{cases} -1 & \text{if } i=s,\\ 1 & \text{if } i=t,\\ 0 & \text{otherwise.}\end{cases}$$

Equivalently, we can state $Hy^{st} = \phi^{st}, \forall (s,t) \in K$, where H is the node-arc incidence matrix of G and ϕ^{st} is the vector whose components are ϕ_i^{st}.

Let x be a vector of design variables such that x_{ij} denotes the amount of capacity to be allocated to the arc (i,j). The robust network design problem with oblivious routing can then be formulated according to the following semi-infinite mathematical programming model:

$$
\begin{aligned}
\min & \sum_{(i,j)\in E} c_{ij}x_{ij} & \\
& Hy^{st} = \phi^{st} & (s,t) \in K \\
& \sum_{(s,t)\in K} d_{st}y_{ij}^{st} \leq x_{ij} & (i,j) \in E, d \in \mathcal{D} \\
& x_{ij} \geq 0 & (i,j) \in E \\
& 0 \leq y_{ij}^{st} \leq 1 & (i,j) \in E,\ (s,t) \in K
\end{aligned}
$$

The above formulation is equivalent to:

$$
\begin{aligned}
\min & \sum_{(i,j)\in E} c_{ij}x_{ij} & \\
& Hy^{st} = \phi^{st} & (s,t) \in K \\
& \max\{\sum_{(s,t)\in K} d_{st}y_{ij}^{st} : d \in \mathcal{D}\} \leq x_{ij} & (i,j) \in E \\
& x_{ij} \geq 0 & (i,j) \in E \\
& 0 \leq y_{ij}^{st} \leq 1 & (i,j) \in E,\ (s,t) \in K
\end{aligned}
$$

By interpreting the $\{y_{ij}^{st}\}$ as constant values, by strong duality, it is thus possible to substitute each inner Linear Programming (LP) problem with its dual, thus obtaining a compact LP formulation for the robust network design problem with oblivious routing [1]. See [3] and [4] for special cases studied in the literature, which are based on special classes of demand polyhedra.

While the assumption of oblivious routing leads to a tractable robust counterpart, according to the compact LP formulation above, the problem of determining the minimum cost capacity installation for G when each demand $d \in \mathcal{D}$ can be served by a different routing template, i.e., the so-called *robust network design problem with dynamic routing*, is also coNP-Hard in the special case of the single-source Hose model [2].

Intermediate scenarios have been studied in [6], where the following generalization of the robust network design problem with oblivious routing has been addressed: given G, K and $\mathcal{D}$, determine the minimum cost capacity installation for G in the case where the demands in $\mathcal{D}$ can be served by two (alternative) routing templates. Precisely, find a minimum cost capacity installation x, and find two routing templates, say y^1 and y^2, in such a way that each demand $d \in \mathcal{D}$ can be served either by y^1 or by y^2 (or possibly by both) by respecting the capacity constraints defined by x. This problem, referred to as the *robust network design problem with two routing templates*, or for short $(2 - RND)$, can be generalized to the case where a constant (or a polynomial) number of alternative routing templates can be used to satisfy the demands in $\mathcal{D}$. Thus, a hierarchy of robust network design problems is defined, with the oblivious routing case (polynomially solvable) at the bottom, and the dynamic routing case (coNP-Hard) at the top. Special $(2-RND)$ cases have been devised in [6], which are tractable from a computational perspective, and whose routing solutions can be less expensive (in terms of the cost of the required capacity allocation) than the optimal oblivious routing solution. However, it is outlined that it is an open issue as to whether problem $(2-RND)$, as well as its generalizations, are polynomially solvable or are NP-Hard. Time complexity issues concerning these problems are the subject of this paper. We prove in fact that $(2-RND)$ and its generalizations are NP-Hard. This is true also when some routing templates are given as input data, and when only a subset of $\mathcal{D}$ has to be routed. The obtained results thus strengthen the outcomes in [6], concerning special $(2-RND)$ cases which are tractable from a computational perspective.

The rest of the note is organized as follows. In Section 1 we prove that $(2-RND)$, in its decisional version, is NP-complete by reduction from the Partition problem. The reduction shows that the hardness statement is also true when some components of the routing templates y^1 and y^2, related to some origin-destination pairs, are given as input data, and when only a subset of $\mathcal{D}$ is required to be routed. As a consequence of the main result, also the problem generalizations where a constant (or a polynomial) number of alternative routing templates can be used to satisfy the demands in $\mathcal{D}$ are NP-Hard. Section 2 presents conclusions and suggestions for future research.

1. Hardness of $(2-RND)$

Let us start with some preliminary observations, which shed light on the hardness of $(2-RND)$ and of its generalizations.

Let Y denote the polyhedron of the routing templates related to G and K. In addition, as stated before, let x be a vector of design variables such that x_{ij} denotes the amount of capacity to be allocated to the arc (i,j). Given a routing template $y \in Y$ and given a vector of design variables x, we can define the subset of the demands of $\mathcal{D}$ which can served by y by respecting the capacity constraints defined by x:

$$\mathcal{D}(y,x) = \{d \in \mathcal{D} : \sum_{(s,t)\in K} d_{st} y_{ij}^{st} \leq x_{ij}, \forall (i,j) \in E\}. \tag{1}$$

Note that, for each given pair (y,x), $\mathcal{D}(y,x)$ is a polyhedron such that $\mathcal{D}(y,x) \subseteq \mathcal{D}$.

Based on definition (1), problem $(2-RND)$ can be restated as follows: find a minimum cost capacity installation x, and find two routing templates, say y^1 and y^2 in Y (possibly $y^1 = y^2$), such that $\mathcal{D}(y^1,x) \cup \mathcal{D}(y^2,x) = \mathcal{D}$. Mathematically:

$$\min \sum_{(i,j)\in E} c_{ij} x_{ij} \tag{2}$$

$$\text{s.t. } y^1, y^2 \in Y \tag{3}$$

$$x_{ij} \geq 0 \qquad \forall (i,j) \in E \tag{4}$$

$$\mathcal{D}(y^1,x) \cup \mathcal{D}(y^2,x) = \mathcal{D}. \tag{5}$$

In the formulation above, the logical constraint $\mathcal{D}(y^1, x) \cup \mathcal{D}(y^2, x) = \mathcal{D}$ is satisfied if and only if $\mathcal{D}(y^1, x) \cup \mathcal{D}(y^2, x)$ is a convex set equal to $\mathcal{D}$.

Now, consider the special case of $(2 - RND)$ where y^1 and y^2 are given. That is, assume that the alternative unit flows to be used to satisfy the demands of the origin-destination pairs in K are given as an input data. It is possible to show that, also in this special case, the set of vectors x which are feasible for $(2 - RND)$, i.e. the set of vectors x such that $\mathcal{D}(y^1, x) \cup \mathcal{D}(y^2, x) = \mathcal{D}$, may be not convex. This suggests that the problem of verifying whether a certain capacity vector x is feasible for $(2 - RND)$ can be hard from a computational perspective. As an example, consider an instance of $(2 - RND)$ where two alternative routing templates for an origin-destination pair (s, t) are given, which use the arcs (i, j) and (u, v), where i, j, u and v are distinct nodes of the network. The first routing template is such that the demand of (s, t) is evenly split between the two arcs, i.e. $y^{1st}_{ij} = y^{1st}_{uv} = 1/2$. The latter routing template is such that $y^{2st}_{ij} = 1/4$ while $y^{2st}_{uv} = 3/4$. Assume that each other origin-destination pair of the considered instance can not use either (i, j) or (u, v) (for example, no path linking the other origin-destination pairs includes either (i, j) or (u, v)). Let $\mathcal{D}$ be a polyhedron such that the only constraint involving the demand of (s, t) is $d_{st} \leq 4$, and consider the two capacity vectors, say x^1 and x^2, which satisfy $x^1_{ij} = x^1_{uv} = 2$, $x^2_{ij} = 1$, $x^2_{uv} = 3$, while $x^1_{rw} = x^2_{rw} = M$ for each other arc (r, w) of the given network, where M denotes a very high value. It is easy to show that both x^1 and x^2 are feasible capacity vectors for the considered instance of $(2 - RND)$, that is, $\mathcal{D}(y^1, x^1) \cup \mathcal{D}(y^2, x^1) = \mathcal{D}$ and $\mathcal{D}(y^1, x^2) \cup \mathcal{D}(y^2, x^2) = \mathcal{D}$. In fact, x^1 is feasible if the demand of (s, t) is routed via the routing template y^1, while x^2 is feasible if the demand of (s, t) is routed via the routing template y^2. On the other hand, no proper convex combination of x^1 and x^2 is a feasible capacity vector. In fact, each capacity vector x must satisfy the disjunctive constraint ($x_{ij} \geq 2$ and $x_{uv} \geq 2$) or ($x_{ij} \geq 1$ and $x_{uv} \geq 3$), which is violated by each proper convex combination.

Now we are ready to state the main result. We shall prove that $(2 - RND)$, in its decisional version, is NP-complete by reduction from the Partition problem [5], which can be stated as follows: Given q positive integer coefficients a_i, $i = 1, \ldots, q$, is there a partition of these coefficients into two subsets such that the sum of the coefficients in each subset is $\frac{\sum_{i=1}^q a_i}{2}$?

The decisional version of $(2 - RND)$ can be formulated as follows: given a directed network G, a set of origin-destination pairs K, a bounded non empty polyhedron $\mathcal{D}$, and a positive value C, are there a capacity installation x, and two routing templates, say y^1 and y^2, such that each demand $d \in \mathcal{D}$ can be served either by y^1 or by y^2 (or possibly by both), by respecting the capacity constraints defined by x, at a cost at most C? Specifically, we shall address the special $(2 - RND)$ case where some components of y^1 and y^2 are given, and where only a subset of the demands in the polyhedron $\mathcal{D}$ needs to be routed.

Theorem 1.1 $(2 - RND)$*, in its decisional version, is NP-complete.*

Proof: Given an instance of the Partition problem, define an instance of $(2-RND)$ as follows. Construct the layered network G_q depicted in Figure 1. The instance of $(2 - RND)$ defined on this layered network has $q + 2$ origin-destination pairs: (s_i, t_i), $i = 1, \ldots, q$, plus the pairs (s, t) and (o, g). Consider the demand polyhedron $\mathcal{D}$ defined as the convex envelope of the following $q + 2$ demand vectors: $d^1 = (a_1, 0, 0, ...0, 0)$, $d^2 = (0, a_2, 0, ...0, 0)$, ..., $d^q = (0, 0, 0, ...a_q, 0, 0)$, $d^{q+1} = (0, 0, 0, ... \frac{\sum_{i=1}^q a_i}{2}, 0)$ and $d^{q+2} = (0, 0, 0, ...0, \frac{\sum_{i=1}^q a_i}{2})$.

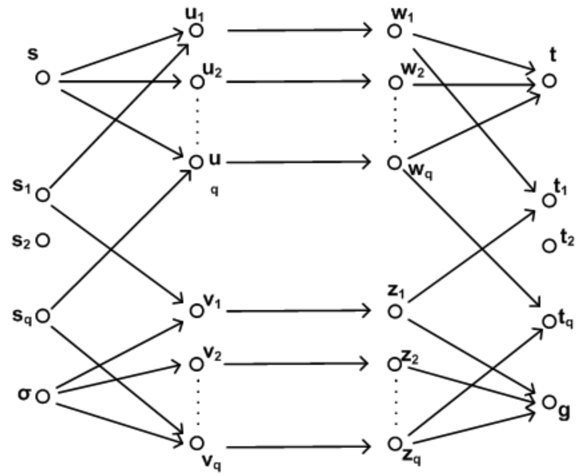

Figure 1: Layered network G_q

Assume that the alternative routing templates are given for the first q origin-destination pairs. Specifically, the pair (s_i, t_i) may use either the routing template (s_i, u_i, w_i, t_i) (this path defines the routing template $y^{1s_it_i}$) or (s_i, v_i, z_i, t_i) (this path defines the routing template $y^{2s_it_i}$), $i = 1, \ldots, q$. On the other hand, the routing templates related to the pairs (s, t) and (o, g) have to be determined.

Let the capacity installation cost be 1 for each arc (u_i, w_i) and for each arc (v_i, z_i), $i = 1, \ldots, q$, while it is 0 for the other arcs, and assume that only the extreme points of $\mathcal{D}$, i.e. d^1, d^2, ..., d^{q+2}, have to be routed along the layered network (as previously indicated, a special $(2-RND)$ instance is so defined). We shall show that there exists a feasible solution for the constructed instance of $(2-RND)$, having cost $\leq \sum_{i=1}^q a_i$, if and only if we can answer YES to the given instance of the Partition problem. Since this problem is NP-complete, we will able to conclude that $(2-RND)$, in its decisional version, is NP-complete. Observe that, in order to route the extreme points of the given polyhedron, the cost of installation is necessarily $\geq \sum_{i=1}^q a_i$, due to the demands d^{q+1} and d^{q+2} and due to the topology of the network.

First assume that the given instance of the Partition problem is such that there exists a partition of the coefficients into two subsets such that the sum of the coefficients in each subset is $\frac{\sum_{i=1}^q a_i}{2}$. Let P_1 and P_2 denote the two subsets. Then, if a_i belongs to P_1, route the demand d^i using the routing template y^1 (observe that this involves the only origin-destination pair (s_i, t_i)); otherwise, route d^i using the routing template y^2. In addition, route the demands d^{q+1} and d^{q+2} in such a way as to exploit the capacity $\frac{\sum_{i=1}^q a_i}{2}$ which has been installed on the two bipartite subgraphs of G_q induced by the nodes (u_i, w_i), $i = 1, \ldots, q$, and (v_i, z_i), $i = 1, \ldots, q$. This choice guarantees an installation cost equal to $\sum_{i=1}^q a_i$.

Now suppose that there exists a capacity vector x for the constructed layered network which is feasible for routing d^1, d^2, ..., d^{q+2}, and whose cost is exactly $\sum_{i=1}^q a_i$ (as observed before, the cost can not be $< \sum_{i=1}^q a_i$). This implies the existence of a partition of the subset of demands $\{d^1, d^2, ..., d^q\}$, say D_1 and D_2, such that the demands in D_1 are routed by means of routing templates of type (s_i, u_i, w_i, t_i), the ones in D_2 are routed by means of routing templates of type (s_i, v_i, z_i, t_i), and such that the overall capacity requirement for the demands in each subset of the partition is exactly $\frac{\sum_{i=1}^q a_i}{2}$. Otherwise, in fact, a capacity greater than $\frac{\sum_{i=1}^q a_i}{2}$ should be necessarily allocated either on the bipartite subgraph of G_q induced by the nodes (u_i, w_i), $i = 1, \ldots, q$, or on the bipartite subgraph induced by the nodes (v_i, z_i), $i = 1, \ldots, q$, in order to route some demand subset, say D_1. In addition, a capacity equal to $\frac{\sum_{i=1}^q a_i}{2}$ should be necessarily allocated on the arcs of the subgraph used to route the demands in the latter subset, that is D_2, in order to route either the demand of (s, t) or the demand of (o, g). But that would require an overall capacity (and so, an overall cost) strictly greater than $\sum_{i=1}^q a_i$. Therefore, if we denote by P_1 the subset of the coefficients a_i such that d^i belongs to D_1, and by P_2 the subset of the coefficients a_i such that d^i belongs to D_2, then the partition (P_1, P_2) is such that the sum of the coefficients in each subset is $\frac{\sum_{i=1}^q a_i}{2}$. The thesis follows.

Corollary 1.1 *$(h - RND)$, that is the generalization of $(2 - RND)$ where h alternative routing templates can be used to satisfy the demands in $\mathcal{D}$, is NP-Hard for $h \geq 2$.*

Corollary 1.2 *$(2 - RND)$ is NP-Hard also when some routing templates are given, and when only a subset of the demands of the polyhedron $\mathcal{D}$ has to be routed.*

2. Conclusions and future research

We have investigated the time complexity of $(2 - RND)$, i.e. the robust network design problem where the demands of a given polyhedron $\mathcal{D}$ can be routed through two alternative routing templates. We have shown that the problem, in its decisional version, is NP-complete by reduction from the Partition problem. This result applies also to the generalizations of $(2 - RND)$ where a constant number (greater than or equal to 2) of alternative routing templates can be used to satisfy the demands in $\mathcal{D}$. In addition, the problem has been proved to be hard also when some routing templates are given as input data, and when only some demands in $\mathcal{D}$ are required to be routed.

It is therefore a challenging topic of research to devise tractable approaches for special $(2-RND)$ cases, along the lines discussed in [6]. Moreover, we plan to perform a computational study aimed at discovering cases where the routing solution provided by (special cases of) $(2 - RND)$ is less expensive than the optimal oblivious routing solution. This will be the subject of further investigation.

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