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Existence results for strong vector equilibrium problems and their applications

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Abstract

New existence results for the strong vector equilibrium problem are presented, relying on a well-known separation theorem in infinite dimensional spaces. The main results are applied to strong cone saddle-points and strong vector variational inequalities providing new existence results, and furthermore they allow to recover an earlier result from the literature.

Keywords: equilibrium problems, existence results, Henig proper solutions, strong cone saddle-points, strong vector variational inequalities

AMS Subject Classification: 49K10; 90C47

1 Introduction

In the last years a great attention has been devoted to equilibrium problems. Many authors studied intensively the following *equilibrium problem* (see [7, 8, 9, 10, 12, 28, 30, 31]):

$$\text{find } \bar{a} \in A \text{ such that } \psi(\bar{a}, b) \geq 0 \text{ for all } b \in B, \qquad (EP)$$

where A and B are two nonempty sets and $\psi : A \times B \to \mathbb{R}$. Actually, it has often been assumed that $A = B$ and that the equilibrium bifunction satisfies $\psi(a, a) = 0$ for all $a \in A$.

It is well-known (see for instance [5, 12]) that this problem contains, in particular, optimization problems, variational inequalities, saddle point problems, Nash equilibria, and other problems of interest in many practical applications. If the scalar function ψ is replaced by a vector-valued function, say $\varphi : A \times B \to Z$

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with Z being a topological vector space, at least two different vector equilibrium problems can be considered:

$$\text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \notin -C \setminus \{0\} \text{ for all } b \in B \qquad (VEP)$$

and

$$\text{find } \bar{a} \in A \text{ such that } \varphi(\bar{a}, b) \notin -\operatorname{int} C \text{ for all } b \in B, \qquad (WVEP)$$

where the convex cone $C \subseteq Z$ with $\operatorname{int} C \neq \emptyset$ provides a partial order on Z. We refer to the first problem as to the *strong vector equilibrium problem*, while to the second one as to the *weak vector equilibrium problem*. Vector equilibrium problems are natural extensions of several problems of practical interest like vector optimization problems or vector variational inequality problems. As far as we know, there are many existence results for $(WVEP)$ and its particular cases (see for instance [1, 2, 3, 10, 11, 14, 16, 18, 27, 34, 37]), but not for (VEP).

In Section 2 we recall some notions that are needed in the paper. Exploiting the idea developed by Kassay and Kolumbán [30] for the scalar equilibrium problem (EP) and relying on the Eidelheit's separation theorem in infinite dimensional spaces (see for instance [33]), new existence results for the strong vector equilibrium problem are given in Section 3. Furthermore, a result of Gong [21] about the existence of solutions of (VEP) follows as a particular case of our existence results, using scalarization techniques and considering a parameterized strong vector equilibrium problem.

Motivated by the lack of results for the existence of strong cone saddle points and strong vector variational inequalities, these particular cases of the strong equilibrium problem (VEP) are studied. In particular, the same scalarization techniques are exploited in Section 4 in order to obtain an existence result for strong cone saddle-points under the same assumptions used in [14] for the weak case, while in Section 5 an existence result for a Stampacchia type strong vector variational inequality is finally given.

2 Preliminaries

Given the topological vector space Z, we consider the following partial order relation induced on Z by a convex pointed cone $C \subseteq Z$ with $\operatorname{int} C \neq \emptyset$:

$$z_1 \leq_C z_2 \text{ if and only if } z_2 - z_1 \in C.$$

We recall that a cone is pointed if $C \cap (-C) = \{0\}$. The positive dual of the cone C is the set

$$C^* = \{z^* \in Z^* \mid z^*(c) \geq 0, \text{ for all } c \in C\},$$

while the quasi-interior of the cone C^* is the set

$$C^\sharp = \{z^* \in C^* | z^*(c) > 0, \text{ for all } c \in C \setminus \{0\}\}.$$

A nonempty convex subset V of C is called a base of C if

$$C = \{\lambda v : \lambda \geq 0, v \in V\} \text{ and } 0 \notin \operatorname{cl} V.$$

If C is a nontrivial convex pointed cone of a Hausdorff locally convex space Z, then $C^\sharp \neq \emptyset$ if and only if C has a base. It is worth to recall also that the pointedness of the closure of C guarantees the existence of a base in finite dimensions (see for instance [25]).

We consider some generalized convexity notions for a vector function that are needed in what follows. Let $f : A \to Z$ be a given function. For the next notion we refer to the paper of Ky Fan [17]. We say that f is C-convexlike on A if for all $t \in [0,1]$ and $a_1, a_2 \in A$ there exists $a \in A$ such that

$$f(a) \leq_C tf(a_1) + (1-t)f(a_2).$$

We notice that f is C-convexlike on A if and only if $f(A) + C$ is a convex set. The next notion is due to Jeyakumar [29]. We say that f is C-subconvexlike on A if there exists $c \in \operatorname{int} C$ such that for all $\varepsilon > 0$, $t \in [0,1]$ and $a_1, a_2 \in A$ there exists $\bar{a} \in A$ such that

$$f(\bar{a}) \leq_C tf(a_1) + (1-t)f(a_2) + \epsilon c.$$

The function f is C-subconvexlike on A if and only if $f(A) + \operatorname{int} C$ is a convex set (see [35]).

Lemma 2.1 *[35] Suppose that C is closed. The following conditions are equivalent:*

(i) f is C-subconvexlike on A;

(ii) there exists $c \in C$ such that for all $\varepsilon > 0$, $t \in [0,1]$ and $a_1, a_2 \in A$ there exists $a \in A$ such that

$$f(a) \leq_C tf(a_1) + (1-t)f(a_2) + \epsilon c;$$

(iii) for each $l \in \operatorname{int} C$, $a_1, a_2 \in A$ and $t \in [0,1]$ there exists $a \in A$ such that

$$f(a) \leq_C tf(a_1) + (1-t)f(a_2) + l.$$

Obviously, if f is C-convexlike on A then f is also C-subconvexlike on A.

Now, we recall a generalized convexity notions for a vector bifunction.

Definition 2.1 *[14] A bifunction $\varphi : A \times B \to Z$ is said to be*

(i) C-subconcavelike in its first variable if for all $l \in \operatorname{int} C$, $x_1, x_2 \in A$, $t \in [0,1]$ there exists $x \in A$ such that

$$\varphi(x,y) \geq_C t\varphi(x_1,y) + (1-t)\varphi(x_2,y) - l \textit{ for all } y \in B;$$

(ii) *C-subconvexlike in its second variable if for all $l \in \operatorname{int} C$, $y_1, y_2 \in B$ and $t \in [0,1]$, there exists $y \in B$ such that*

$$\varphi(x,y) \leq_C t\varphi(x,y_1) + (1-t)\varphi(x,y_2) + l \text{ for all } x \in A.$$

(iii) *C-subconcavelike-subconvexlike on $A \times B$ if it is C-subconcavelike in its first variable and C-subconvexlike in its second variable.*

In case $Z = \mathbb{R}$ and $C = \mathbb{R}_+$ we use the terms *subconcavelike, subconvexlike* omitting the $\mathbb{R}_+$ symbol.

3 Sufficient conditions for strong solutions

The main results of this section provide sufficient conditions for the existence of solutions of the strong vector equilibrium problem (VEP) under different sets of assumptions.

Theorem 3.1 *Suppose φ satisfies the following assumptions:*

(i) *if the system $\{U_b \mid b \in B\}$ covers A, then it contains a finite subcover, where*

$$U_b = \{a \in A \mid \varphi(a,b) \in -C \backslash \{0\}\};$$

(ii) *for each $a_1, \ldots, a_m \in A$, $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + ... + \lambda_m = 1$, $b_1, \ldots, b_n \in B$ there exists $u^* \in C^\sharp$ such that*

$$\min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i u^*\big(\varphi(a_i, b_j)\big) \leq \sup_{a \in A} \min_{1 \leq j \leq n} u^*\big(\varphi(a, b_j)\big);$$

(iii) *for each $b_1, \ldots, b_n \in B$, $z_1^*, \ldots, z_n^* \in C^*$ not all zero, it holds*

$$\sup_{a \in A} \sum_{j=1}^{n} z_j^*\big(\varphi(a, b_j)\big) > 0.$$

Then, the vector equilibrium problem (VEP) admits a solution.

Proof. Suppose by contradiction that (VEP) admits no solution, i.e., for each $a \in A$ there exists $b(a) \in B$ such that

$$\varphi(a, b(a)) \in -C \setminus \{0\}.$$

Since the family $\{U_{b(a)}\}_{a \in A}$, where

$$U_{b(a)} := \{a' \in A \mid \varphi(a', b(a)) \in -C \setminus \{0\}\}, \tag{1}$$

covers the set A, then assumption (i) guarantees that there exist $b_1, \dots, b_n \in B$ such that

$$A \subseteq \bigcup_{j=1}^{n} U_{b_j}. \tag{2}$$

We define the vector-valued function $F : A \to Z^n$ by

$$F(a) := \big(\varphi(a, b_1), \dots, \varphi(a, b_n)\big),$$

and we have

$$\operatorname{co} F(A) \cap (C \setminus \{0\})^n = \emptyset, \tag{3}$$

where $\operatorname{co} F(A)$ denotes the convex hull of the set $F(A)$. To prove it, we suppose by contradiction there exist $a_1, \dots, a_m \in A$ and $\lambda_1, \dots, \lambda_m \geq 0$ with $\lambda_1 + ... + \lambda_m = 1$ such that

$$\sum_{i=1}^{m} \lambda_i F(a_i) \in (C \setminus \{0\})^n.$$

This is equivalent to

$$\sum_{i=1}^{m} \lambda_i \varphi(a_i, b_j) \in C \setminus \{0\} \text{ for each } j \in \{1, \dots, n\}. \tag{4}$$

Let $u^* \in C^\sharp$ be a functional for which assumption (ii) holds. Applying u^* to the above relation and taking the minimum over all $j \in \{1, \dots, n\}$, we get

$$\min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i u^*\big(\varphi(a_i, b_j)\big) > 0. \tag{5}$$

Thus, assumption (ii) and (5) imply

$$\sup_{a \in A} \min_{1 \leq j \leq n} u^*\big(\varphi(a, b_j)\big) > 0. \tag{6}$$

Relation (2) guarantees that for each $a \in A$ there exists $j_0 = j_0(a) \in \{1, \dots, n\}$ such that $a \in U_{b_{j_0}}$, i.e. $\varphi(a, b_{j_0}) \in -C \setminus \{0\}$ for each $a \in A$. Applying $u^* \in C^\sharp$, we get

$$u^*\big(\varphi(a, b_{j_0})\big) < 0 \text{ for all } a \in A.$$

Taking the minimum over $j \in \{1, \dots, n\}$ and then the supremum over $a \in A$ in the previous relation, we obtain

$$\sup_{a \in A} \min_{1 \leq j \leq n} u^*\big(\varphi(a, b_j)\big) \leq 0, \tag{7}$$

which is a contradiction to (6). Hence, condition (3) holds.

Therefore, the Eidelheit's separation theorem implies that there exists a nonzero functional $z^* \in (Z^n)^*$ such that

$$z^*(u) \leq 0, \text{ for all } u \in \operatorname{co} F(A) \tag{8}$$

and

$$z^*(k) \geq 0, \text{ for all } k \in (C \setminus \{0\})^n. \tag{9}$$

Using the representation $z^* = (z_1^*, ..., z_n^*)$, we deduce $z_j^* \in C^*$ for all $j \in \{1, \dots, n\}$ by a standard argument.

By (8) we have $z^*(F(a)) \leq 0$ for all $a \in A$, or equivalently,

$$\sum_{j=1}^{n} z_j^*(\varphi(a, b_j)) \leq 0.$$

Because the above inequality holds for each $a \in A$ we obtain

$$\sup_{a \in A} \sum_{j=1}^{n} z_j^* \big(\varphi(a, b_j)\big) \leq 0,$$

which is a contradiction to assumption (*iii*). □

In order to guarantee that the assumptions of the theorem are satisfied, some continuity and generalized convexity properties are useful. In particular, the next two continuity type concepts are generalizations of the upper semicontinuity of real-valued functions (see [32, 38] for other generalizations).

Definition 3.1 *A vector-valued function $f : A \to Z$ is said to be*

- *(i) [32, 37] C-upper semicontinuous on A (C-usc in short) if for each $x \in A$ and any $c \in \operatorname{int} C$, there exists an open neighbourhood $U \subset A$ of x such that $f(u) \in f(x) + c - \operatorname{int} C$ for all $u \in U$.*
- *(ii) properly C-upper semicontinuous on A (properly C-usc in short) if for each $x \in A$ and any $c \in C \setminus \{0\}$, there exists an open neighbourhood $U \subset A$ of x such that $f(u) \in f(x) + c - C \setminus \{0\}$ for all $u \in U$.*

The function f is called (properly) C-lower semicontinuous if $-f$ is (properly) C-upper semicontinuous.

Notice that every properly C-upper semicontinuous function is C-upper semicontinuous, but not the vice versa. For example, taking $A = Z$ and any cone $C \subseteq Z$ such that $C \setminus \{0\}$ is not an open set, the identity function is not properly C-usc, as the following characterization of this new concept of continuity shows.

Proposition 3.1 *Let $f : A \to Z$. The following conditions are equivalent:*

- *(i) f is properly C-upper semicontinuous on A;*
- *(ii) the set $f^{-1}(z - C\setminus\{0\})$ is open in A for each $z \in Z$.*

Proof. Let $z \in Z$. If $f^{-1}(z - C\setminus\{0\}) = \emptyset$, then (ii) holds. Assume that there exists $x_0 \in f^{-1}(z - C \setminus \{0\})$. Thus, we have $c := z - f(x_0) \in C \setminus \{0\}$. By the definition of properly C-usc, there exists an open neighbourhood U of x_0 such that it holds

$$f(x) \in f(x_0) + c - C \setminus \{0\} = z - C \setminus \{0\}$$

for all $x \in U$. So, $f^{-1}(z - C \setminus \{0\})$ is an open subset of the space A.

For the reverse implication let $x_0 \in A$ and $c \in C \setminus \{0\}$. Since $x_0 \in f^{-1}(f(x_0) + c - C \setminus \{0\})$, which is an open set by condition (ii), there exists an open neighbourhood U of x_0 such that

$$x \in f^{-1}(f(x_0) + c - C \setminus \{0\}) \text{ for all } x \in U,$$

and therefore f is properly C-usc at x_0. Since x_0 was arbitrarily taken, we deduce that f is properly C-usc on A. □

Proposition 3.2 *Suppose that A is a compact topological space and the function $\varphi(\cdot, b) : A \to Z$ is properly C-usc on A for each $b \in B$. Then, the assumption (i) of Theorem 3.1 is satisfied.*

Proof. Let $U_b := \{a \in A \mid \varphi(a, b) \in -C \setminus \{0\}\}$, for any $b \in B$. In what follows we show that the family of these sets is an open covering of A.

Take $a_0 \in U_b$ and consider $c' := -\varphi(a_0, b) \in C \setminus \{0\}$. Since the function $\varphi(\cdot, b)$ is properly C-usc at $a_0 \in A$, there exists a neighbourhood $U \subset A$ of a_0 such that

$$\begin{aligned} \varphi(u, b) &\in \varphi(a_0, b) + c' - C \setminus \{0\} \\ &= \varphi(a_0, b) - \varphi(a_0, b) - C \setminus \{0\} \\ &= -C \setminus \{0\} \end{aligned}$$

for all $u \in U$. Hence, we get $\varphi(u, b) \in -C \setminus \{0\}$ for all $u \in U$, which implies that U_b is an open set. Therefore, assumption (i) of Theorem 3.1 follows from the compacteness of A. □

Proposition 3.2 and the C-subconcavelikeness of a bifunction allow to achieve the following existence result as a corollary of Theorem 3.1.

Corollary 3.1 *Suppose A is a compact topological space, C a closed convex cone with a nonempty interior such that $C^\sharp \neq \emptyset$ and the bifunction φ satisfies the conditions:*

(i) *$\varphi(\cdot, b)$ is properly C-usc for all $b \in B$ and φ is C-subconcavelike in its first variable;*

(ii) *for each $b_1, \ldots, b_n \in B$, $z_1^*, \ldots, z_n^* \in C^*$ not all zero it holds*

$$\sup_{a \in A} \sum_{j=1}^{n} z_j^*\big(\varphi(a, b_j)\big) > 0.$$

Then, the vector equilibrium problem (VEP) *admits a solution.*

Proof. It is enough to show that the C-subconcavelikeness of the function φ in its first variable implies condition (ii) of Theorem 3.1. Take $a_1, \dots, a_m \in A$, $b_1, \dots, b_n \in B$, $\lambda_1, \dots, \lambda_m \geq 0$ with $\lambda_1 + ... + \lambda_m = 1$ and $u^* \in C^\sharp$.

Thanks to the C-subconcavelikeness of φ in its first variable, for each $l \in \operatorname{int} C$ there exists $\bar{a} \in A$ such that

$$\sum_{i=1}^{m} \lambda_i \varphi(a_i, b_j) \leq_C \varphi(\bar{a}, b_j) + l \text{ for each } j \in \{1, \dots, n\}. \tag{10}$$

Applying u^* to (10), we obtain

$$\sum_{i=1}^{m} \lambda_i u^* \varphi(a_i, b_j) \leq u^*\big(\varphi(\bar{a}, b_j)\big) + u^*(l) \text{ for each } j \in \{1, \dots, n\}, \tag{11}$$

and taking the minimum over j we get

$$\begin{aligned} \min_{1 \leq j \leq n} \sum_{i=1}^{m} \lambda_i u^*\big(\varphi(a_i, b_j)\big) &\leq \min_{1 \leq j \leq n} u^*\big(\varphi(\bar{a}, b_j)\big) + u^*(l) \\ &\leq \sup_{a \in A} \min_{1 \leq j \leq n} u^*\big(\varphi(a, b_j)\big) + u^*(l). \end{aligned}$$

Since this inequality holds for each $l \in \operatorname{int} C$, we obtain the assumption (ii) of Theorem 3.1 just taking $l \to 0$. □

In [4] the existence result of [34] for $(WVEP)$ is extended to an abstract setting involving multifunctions and more general ordering structures. Another existence result for (VEP) can be achieved from Theorem 2 in [4] if the concrete setting of strong vector equilibirum problems is considered. Anyaway, such a result is not related to the above ones: while proper C-upper semicontinuity guarantees condition (ii) in Theorem 2 of [4] (see Proposition 3.2 above), the other assumptions are independent of each other.

In the special case $Z = \mathbb{R}$, Corollary 3.1 collapses to the following result.

Corollary 3.2 *Suppose A is a compact topological space and the bifunction φ satisfies the conditions:*

(i) $\varphi(\cdot, b)$ is usc for all $b \in B$ and φ subconcavelike in its first variable;

(ii) for each $b_1, \dots, b_n \in B$, $\mu_1, \dots, \mu_n \geq 0$ it holds

$$\sup_{a \in A} \sum_{j=1}^{n} \mu_j \varphi(a, b_j) > 0.$$

Then, the scalar equilibrium problem (EP) *admits a solution.*

The next result follows from Theorem 3.1 via the above corollary.

Theorem 3.2 *Let A be a nonempty compact subset of a metrizable topological vector space E, C a closed convex cone with a nonempty interior and $e^* \in C^\sharp$. Suppose that $\varphi : A \times B \to Z$ is C-subconcavelike-subconvexlike and the function $a \mapsto e^*(\varphi(a,b))$ is upper semicontinuous on A for each fixed $b \in B$. Furthermore assume $\sup_{a\in A} e^*(\varphi(a,b)) \geq 0$ for all $b \in B$. Then, the strong vector equilibrium problem (VEP) admits a solution.*

Proof. We prove the theorem in two steps.

Step 1. Let $\tau \in C \setminus \{0\}$ and define the function $\psi : A \times B \to Z$ by $\psi(a,b) = \varphi(a,b) + \tau$ for all $a \in A$ and $b \in B$. For the given $e^* \in C^\sharp$ we consider the real-valued function $e^* \circ \psi : A \times B \to \mathbb{R}$, which is defined as $(e^* \circ \psi)(a,b) = e^*(\psi(a,b))$ for all $a \in A$ and $b \in B$. We show that this function satisfies the assumptions of Corollary 3.2. Given any $\epsilon > 0$, there exists $l \in \operatorname{int} C$ such that $e^*(l) = \epsilon$. Since φ is C-subconcavelike in its first variable, for each a_1, $a_2 \in A$ and $t \in [0,1]$ there exists $\bar{a} \in A$ such that

$$\psi(\bar{a}, b) \geq_C t\psi(a_1, b) + (1-t)\psi(a_2, b) - l \text{ for all } b \in B.$$

Applying e^* to this inequality we obtain

$$e^*(\psi(\bar{a}, b)) \geq te^*(\psi(a_1, b)) + (1-t)e^*(\psi(a_2, b)) - \epsilon \text{ for all } b \in B.$$

Thus, the function $e^* \circ \psi$ is subconcavelike in its first variable.

Take $b_1, \ldots, b_n \in B$, $\mu_1, \ldots, \mu_n \geq 0$ with $\mu_1 + \ldots + \mu_n = 1$. Since φ is C-subconvexlike in its second variable, we have that for each $l' \in \operatorname{int} C$ there exists an element $\bar{b} \in B$ such that

$$\psi(a, \bar{b}) \leq_C \sum_{j=1}^{n} \mu_j \psi(a, b_j) + l' \text{ for all } a \in A. \tag{12}$$

Applying the nonzero functional e^* to relation (12) we get

$$e^*(\psi(a, \bar{b})) \leq \sum_{j=1}^{n} \mu_j e^*(\psi(a, b_j)) + e^*(l') \text{ for all } a \in A. \tag{13}$$

By the assumptions and inequality (13), we deduce that

$$e^*(\tau) \leq \sup_{a\in A} e^*(\varphi(a, \bar{b})) + e^*(\tau) \leq \sup_{a\in A} \sum_{j=1}^{n} \mu_j e^*(\psi(a, b_j)) + e^*(l').$$

Taking the limit as $l' \to 0$, we obtain

$$0 < e^*(\tau) \leq \sup_{a\in A} \sum_{j=1}^{n} \mu_j e^*(\psi(a, b_j)).$$

Hence, the assumptions of Corollary 3.2 are satisfied. Therefore, there exists a solution $\tilde{a} \in A$ of (EP), i.e.,

$$e^*(\varphi(\tilde{a}, b)) + e^*(\tau) \geq 0 \text{ for all } b \in B.$$

Step 2. Applying Step 1 with e^*/n, we get that for all $n \in \mathbb{N}$ there exists a point $\tilde{a}_n \in A$ such that

$$e^*(\varphi(\tilde{a}_n, b)) + \frac{1}{n}e^*(\tau) \geq 0 \text{ for all } b \in B \text{ and } n \in \mathbb{N}. \tag{14}$$

In this way we achieve a sequence $\{\tilde{a}_n\}$ of points of the compact set A. Since E is metrizable, compactness guarantees sequential compactness: thus, there exists a convergent subsequence of $\{\tilde{a}_n\}$ (also denoted by $\{\tilde{a}_n\}$ for the sake of simplicity), i.e., there is $\tilde{a} \in A$ such that $\tilde{a}_n \to \tilde{a}$ when $n \to \infty$. We show that $\tilde{a}$ solves (VEP).

Since $a \mapsto e^*(\varphi(a, b))$ is upper semicontinuous on A for any point $b \in B$, we have that it is upper semicontinuous at $\tilde{a}$, i.e, for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that

$$e^*(\varphi(\tilde{a}_n, b)) < e^*(\varphi(\tilde{a}, b)) + \epsilon \text{ for all } n \geq n_0.$$

Thanks to condition (14), we deduce

$$0 \leq e^*(\varphi(\tilde{a}_n, b)) + (1/n)e^*(\tau) < e^*(\varphi(\tilde{a}, b)) + (1/n)e^*(\tau) + \epsilon$$

for all $n \geq n_0$. Taking $n \to \infty$, we have

$$0 \leq e^*(\varphi(\tilde{a}, b)) + \epsilon.$$

Since this inequality holds for any ϵ, we conclude that

$$0 \leq e^*(\varphi(\tilde{a}, b)).$$

Since this inequality holds for any $b \in B$, then $\tilde{a}$ is a solution of (VEP). $\square$

When the space E is normed, weak compactness is equivalent to weak sequential compactness: therefore, the following stronger result can be achieved for the case of normed spaces, just arguing as in the previous proof.

Theorem 3.3 *Let A be a nonempty weakly compact subset of a normed space E, C a closed convex cone with a nonempty interior and $e^* \in C^\sharp$. Suppose that $\varphi : A \times B \to Z$ is C-subconcavelike-subconvexlike and the function $a \mapsto e^*(\varphi(a, b))$ is weakly upper semicontinuous on A for each fixed $b \in B$. Furthermore assume $\sup_{a \in A} e^*(\varphi(a, b)) \geq 0$ for all $b \in B$. Then, the strong vector equilibrium problem (VEP) admits a solution.*

Theorem 3.3 allows to get the following slight generalization of Theorem 3.2 of [21], in which convexlikeness is replaced by the weaker subconvexlikeness.

Corollary 3.3 *Let A be a nonempty weakly compact subset of a normed space E, C a closed convex cone with a nonempty interior and $e^* \in C^\sharp$. Suppose $\varphi : A \times A \to Z$ is C-subconcavelike-subconvexlike and the function $a \mapsto e^*(\varphi(a,b))$ is weakly upper semicontinuous on A for each fixed $b \in A$. Furthermore assume $\varphi(a,a) \in C$ for all $a \in A$. Then, the strong vector equilibrium problem (VEP) admits a solution.*

Proof. The thesis follows immediately from Theorem 3.3, just taking $A = B$ and noticing that $\varphi(a,a) \in C$ for all $a \in A$ implies

$$\sup_{a \in A} e^*(\varphi(a,b)) \geq e^*(\varphi(b,b)) \geq 0 \text{forall} b \in A.$$

□

Theorem 3.3 extends Theorem 3.2. of [21] also in two other ways: two different sets A and B are considered and the equilibrium condition $\varphi(a,a) \in C$ is replaced by a weaker assumption involving appropriate suprema over A.

4 Strong vector saddle points

Many of the existence results in literature are devoted to approximate saddle-points due to their numerical applications (see [24, 26] and the reference therein). Existence results for weak C-saddle points have also been developed (see [14, 37]). Gong [23] has recently given existence results for ideal strong C-saddle points for vector-valued functions. It is well-known that whenever it is not empty, the set of ideal minima of a set coincide with the set of Pareto minima, and it is a singleton when the ordering cone is pointed (see [32]). Hence, Gong stated results for the existence and uniqueness of strong C-saddle points.

Given $S \subseteq Z$, we denote the set of minima of S with respect to the cone C by $\operatorname{Min} S$, i.e., $z_0 \in \operatorname{Min} S$ means $z_0 \in S$ and

$$(S - z_0) \cap (-C) = \{0\},$$

while the set $\operatorname{Max} S$ denotes the set of maxima of S with respect to the cone C, i.e., $z_0 \in \operatorname{Max} S$ means $-z_0 \in \operatorname{Min}(-S)$. In a similar fashion, $\operatorname{Min}_w S$ denotes the set of weak minima of S with respect to the cone C, i.e., $z_0 \in \operatorname{Min}_w S$ means $z_0 \in S$ and

$$(S - z_0) \cap (-\operatorname{int} C) = \emptyset,$$

while $\operatorname{Max}_w S$ denotes the set of weak maxima of S with respect to the cone C, i.e., $z_0 \in \operatorname{Max}_w S$ means $-z_0 \in \operatorname{Min}_w(-S)$.

Let X and Y be nonempty subsets of two metrizable topological vector spaces and $f : X \times Y \to Z$. Considering the sets

$$f(x,Y) := \{f(x,y) |\, y \in Y\} \text{ and } f(X,y) := \{f(x,y) |\, x \in X\}$$

for any $x \in X$ and $y \in Y$, the definition of weak and strong C-saddle point can be given in the following way.

Definition 4.1 *A point* $(x_0, y_0) \in X \times Y$ *is said to be*

(i) a weak C-saddle point of f on $X \times Y$ *if*

$$f(x_0, y_0) \in \mathrm{Max}_w\, f(X, y_0) \cap \mathrm{Min}_w\, f(x_0, Y);$$

(ii) a strong C-saddle point of f on $X \times Y$ *if*

$$f(x_0, y_0) \in \mathrm{Max}\, f(X, y_0) \cap \mathrm{Min}\, f(x_0, Y).$$

Obviously, each weak C-saddle point is a strong C-saddle point, but the vice versa is not true as shown by the following simple example.

Example 4.1 Let $f : [-1, 1] \times [-1, 1] \to \mathbb{R}^2$ be given by $f(x) = x$ and let $C = \mathbb{R}^2_+$. It is easy to check that $(0, 0)$ is a weak C-saddle point, but it is not a strong C-saddle point.

Strong C-saddle points can be obtained as particular cases of strong equilibria as the next statement shows.

Proposition 4.1 *Let* $A = B = X \times Y$ *and* $\varphi : A \times A \to \mathbb{R}$ *given by* $\varphi(a, b) = f(x, v) - f(u, y)$*, where* $a = (x, y)$*,* $b = (u, v)$*,* $x, u \in X$ *and* $y, v \in Y$*. If* $\bar{a} \in A$ *is a solution of* (VEP)*, then* $\bar{a}$ *is a strong C-saddle point of f.*

Proof. Let $\bar{a} = (\bar{x}, \bar{y})$ be a solution of the problem (VEP). Then

$$\varphi(\bar{a}, b) \notin -C \setminus \{0\} \text{ for all } b \in A.$$

This implies that $f(\bar{x}, v) - f(u, \bar{y}) \notin -C \setminus \{0\}$ for all $(u, v) \in X \times Y$. If we take $u := \bar{x}$ and $v := y$ we obtain

$$f(\bar{x}, y) - f(\bar{x}, \bar{y}) \notin -C \setminus \{0\} \text{ for all } y \in Y, \tag{15}$$

which leads to $f(\bar{x}, \bar{y}) \in \mathrm{Min}\, f(\bar{x}, Y)$.

Let $u := x$ and $v := \bar{y}$. Then, we have

$$f(x, \bar{y}) - f(\bar{x}, \bar{y}) \notin C \setminus \{0\} \text{ for all } x \in X, \tag{16}$$

that is $f(\bar{x}, \bar{y}) \in \mathrm{Max}\, f(X, \bar{y})$. □

Unless $Z = \mathbb{R}$ (see [12]), the vice versa implication of the above proposition does not necessarily hold, as the next example shows.

Example 4.2 Let $X = Y = [-1, 0]$, $Z = \mathbb{R}^2$, $C = \mathbb{R}^2_+$ and f be defined by

$$f(x, y) = \begin{cases} (0, 0), & \textit{if } x = 0,\ y = -1 \\ (-1/2, 1/2), & \textit{if } x = 0 \textit{ and } y \neq -1 \\ (-1/4, 1), & \textit{if } x \neq 0 \textit{ and } y = -1 \\ (x, y + 1), & \textit{otherwise.} \end{cases}$$

It is easy to check that $(0,-1)$ is a strong C-saddle point of the function f. On the other hand, for $\bar{a} = (0,-1)$ and $b = (u,v) \in [-1,0) \times (-1,0]$, we get

$$\varphi(a,b) = (-1/2, 1/2) - (-1/4, 1) = (-1/4, -1/2) \in -\mathbb{R}^2_+ \setminus \{0\}.$$

Hence, $(0,-1)$ is not a solution of (VEP) although it is a strong C-saddle point of the function f.

In [14] the authors stated existence results for weak C-saddle points under the same assumptions of Theorem 4.3 of [15]. Next, we show that under a further, not very demanding assumption (namely, that $C^\sharp \neq \emptyset$ or equivalently that C has a base), the existence of strong C-saddle points is granted.

Theorem 4.1 *Suppose $C^\sharp \neq \emptyset$, X and Y are compact sets and f satisfies the following conditions:*

(i) f is C-usc with respect to its first variable and C-lsc with respect to its second variable;

(ii) f is C-subconcavelike-subconvexlike on $X \times Y$.

Then, f admits a strong C-saddle point.

Proof. Let $A = X \times Y$, and define the function $\varphi : A \times A \to Z$ by

$$\varphi(a,b) = f(x,v) - f(u,y) \mathrm{for all} a = (x,y),\ b = (u,v) \in A.$$

For any fixed $e^* \in C^\sharp$ and $\tau \in C \setminus \{0\}$ we consider the function $g : A \times A \to \mathbb{R}$ defined by $g(a,b) = e^*(\varphi(a,b) + \tau)$ for all $a,\ b \in A$ and we show that the assumptions of Corollary 3.2 are satisfied by this function g.

Let $b \in A$ and $\epsilon > 0$ be arbitrary. Thus, there exists $l \in \operatorname{int} C$ such that $e^*(l) = \epsilon$. Because f is C-usc in its first variable and C-lsc in its second variable we have that there exists a neighbourhood $U := U_{x_0} \times U_{y_0}$ of a_0 such that

$$f(x,v) \in f(x_0,v) + \frac{1}{2}l - C \setminus \{0\} \text{ for all } x \in U_{x_0}$$

and

$$f(u,y) \in f(u,y_0) - \frac{1}{2}l + C \setminus \{0\} \text{ for all } y \in U_{y_0}.$$

Therefore, the upper semicontinuity of $g(\cdot, b)$ at $a_0 = (x_0, y_0)$ follows.

In order to check the subconcavelikeness of g as a function of its first variable only, let $a_1, \ldots, a_m \in A$, $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + ... + \lambda_m = 1$, and $b \in A$. By the C-subconcavelike-subconvexlikeness of the function f, there exists $\bar{x} \in X$ and $\bar{y} \in Y$ such that

$$f(\bar{x}, v) \geq_C \sum_{i=1}^{m} \lambda_i f(x_i, v) - l/2 \text{ for all } v \in Y,$$

and

$$f(u,\bar{y}) \leq_C \sum_{i=1}^{m} \lambda_i f(u,y_i) + l/2 \text{ for all } u \in X.$$

By the above relations and adding τ, we obtain:

$$\varphi(\bar{a},b) + \tau \geq_C \sum_{i=1}^{m} \lambda_i \varphi(a_i,b) + \tau - l \text{ for all } b \in A,$$

where $\bar{a} := (\bar{x},\bar{y})$. Applying e^* to the above inequality, the first assumption of Corollary 3.2 follows.

Now, let $b_1,\ldots,b_n \in A$, $\mu_1,\ldots,\mu_n \geq$ with $\mu_1 + ... + \mu_n = 1$. Since the C-subconcavelike-subconvexlikeness of the function f implies the subconvexlikeness of g in its second variable, for any $\epsilon > 0$ there exists an element $\bar{b} \in A$ such that

$$\epsilon + \sum_{j=1}^{n} \mu_j g(a,b_j) \geq g(a,\bar{b}) \text{ for all } a \in A.$$

If we take $a = \bar{b}$ in the above inequality and we observe that $g(\bar{b},\bar{b}) = e^*(\tau) > 0$, we get

$$\epsilon + \sup_{a\in A} \sum_{j=1}^{n} \mu_j g(a,b_j) \geq e^*(\tau).$$

Taking $\epsilon \to 0$, the second assumption of Corollary 3.2 follows. Hence, (EP) admits a solution, i.e., there is $\tilde{a} \in A$ such that

$$g(\tilde{a},b) \geq 0 \text{forall} b \in A.$$

Therefore, for each $n \in \mathbb{N}$ there is $\tilde{a}_n \in A$ such that

$$e^*(\varphi(\tilde{a}_n,b) + \frac{\tau}{n}) \geq 0, \text{forall} b \in A. \tag{17}$$

Since the sequence $(\tilde{a}_n)$ is contained in the compact set A, there exists a convergent subsequence of $(\tilde{a}_n)$, also denoted by $(\tilde{a}_n)$, i.e., there exists $\tilde{a} \in A$ such that $\tilde{a}_n \to \tilde{a}$ when $n \to \infty$.

Exploiting assumption (i), we deduce that $e^*(\varphi(\cdot,b))$ is usc for any $b \in A$. In particular, it is usc at $\tilde{a} \in A$, and hence there exists $n_0 \in \mathbb{N}$ such that

$$e^*(\varphi(\tilde{a}_n,b)) < e^*(\varphi(\tilde{a},b)) + \epsilon \text{forall} n \geq n_0.$$

By the above inequality and (17) we have

$$0 < e^*(\varphi(\tilde{a},b)) + \frac{1}{n}e^*(\tau) + \epsilon \text{forall} n \geq n_0.$$

Taking $n \to \infty$, we deduce $0 \leq e^*(\varphi(\tilde{a},b))$.

Thus, this holds for each $b \in A$ and therefore $\tilde{a}$ is a solution of (EP). Since $e^* \in C^\sharp$, we have that $\tilde{a}$ is a solution of (VEP) and therefore it is a strong C-saddle point of f by Proposition 3.1. □

Note that the above result is not related to the existence result for strong C-saddle points given by Gong [23]. While our continuity assumptions are weaker, the convexity assumptions are stronger than those in Theorem 2.1 of [23].

5 Strong vector variational inequalities

The domain of vector variational inequalities received a great attention ever since the paper of Giannessi [20] appeared and the first existence results for vector variational inequalities were published in [15]. In [16] the authors presented some of the most fundamental existence results for vector variational inequalities. Most of the research results in this area deal with a weak version of vector variational inequalities and their generalizations. Hence, the authors of [16] suggested to study the existence of solutions for strong vector variational inequalities. Recently, Fang and Huang [18] obtained some results of this kind.

In this section we give new existence results for strong vector variational inequalities. To this aim, we consider both the Minty and Stampacchia type of strong vector variational inequalities. Given a metrizable topological vector space E, $A \subseteq E$ and $F : E \to LC(E, Z)$, being $LC(E, Z)$ the set of all linear and continuous operators from E to Z, the Minty vector variational inequality reads

$$\text{find } \bar{a} \in A \text{ such that } \langle F(b), b - \bar{a} \rangle \notin -C \setminus \{0\} \text{ for all } b \in A, \qquad (MVI)$$

while the Stampacchia vector variational inequality reads

$$\text{find } \bar{a} \in A \text{ such that } \langle F(\bar{a}), b - \bar{a} \rangle \notin -C \setminus \{0\} \text{ for all } b \in A, \qquad (SVI)$$

where $\langle F(b), b - a \rangle$ denotes the value of $F(b)$ at $b - a$.

Definition 5.1 *We say that $\bar{a} \in A$ is a Henig proper solution of (MVI) if there exists a pointed convex cone K, such that $C \setminus \{0\} \subseteq \operatorname{int} K$ and*

$$\langle F(b), b - \bar{a} \rangle \notin -K \setminus \{0\} \text{ for all } b \in A.$$

See the papers by Benson [6], Borwein [13] and Geoffrion [19] for other notions of proper solutions though given in the context of vector optimization problems.

Definition 5.2 *Let A be a convex set. We say that F is hemicontinuous if for any $a, b \in A$ the function $t \mapsto \langle F(tb + (1-t)a), b - a \rangle$ where $t \in [0, 1]$ is continuous at 0^+.*

Proposition 5.1 *Suppose A is a convex set and F is a hemicontinuous function. If $\bar{a} \in A$ is a Henig proper solution of (MVI), then $\bar{a}$ is a solution of (SVI).*

Proof. Let $\bar{a}$ be a Henig proper solution of (MVI). Then, there exists a pointed convex cone K such that $C \setminus \{0\} \subseteq \operatorname{int} K$ and

$$\langle F(b), b - \bar{a} \rangle \notin -K \setminus \{0\} \text{ for all } b \in A. \tag{18}$$

Given any $b \in A$, define $b(t) := tb + (1-t)\bar{a}$ for all $t \in [0,1]$. Since A is a convex set, then $b(t) \in A$ for each $t \in [0,1]$. By (18) we have

$$t\langle F(b(t)), b - \bar{a} \rangle \notin -K \setminus \{0\} \text{ for all } t \in [0,1].$$

This relation implies that

$$\langle F(b(t)), b - \bar{a} \rangle \notin -\operatorname{int} K \text{ for all } t \in (0,1].$$

Taking the limit as $t \downarrow 0$, the hemicontinuity of the function F implies

$$\langle F(\bar{a}), b - \bar{a} \rangle \notin -\operatorname{int} K,$$

which gives

$$\langle F(\bar{a}), b - \bar{a} \rangle \notin -C \setminus \{0\}.$$

Since this condition holds for each $b \in A$, then $\bar{a}$ solution of (SVI). $\square$

In order to give an existence results for (MVI), we need the next monotonicity assumption.

Definition 5.3 *[16] A mapping $F : E \to LC(E, Z)$ is said to be C-monotone if*

$$\langle F(b) - F(a), b - a \rangle \geq_C 0 \text{ for all } a, b \in E.$$

Exploiting a scalarization technique, we achieve an existence result for (MVI).

Theorem 5.1 *Suppose A is a compact and convex set. If there exists a pointed convex cone K with $K^\sharp \neq \emptyset$ such that $C \setminus \{0\} \subseteq \operatorname{int} K$ and F is K-monotone, then (MVI) admits a Henig proper solution.*

Proof. For a fixed $e^* \in K^\sharp$ and $k \in K \setminus \{0\}$, consider the real valued function $\psi : A \times A \to \mathbb{R}$, defined by $\psi(a,b) = e^*(\langle F(b), b - a \rangle + k)$ for all $a, b \in A$. Since $\psi(\cdot, b)$ is continuous on A for each $b \in A$ and ψ is affine in its first variable, the first assumption of Corollary 3.2 is satisfied.

To check the other assumption of the above corollary, take $b_1, \dots, b_n \in A$ and $\mu_1, \dots, \mu_n \geq 0$ with $\mu_1 + ... + \mu_n = 1$. Since F is K-monotone, we obtain

$$\sum_{j=1}^{n} \mu_j e^*(\langle F(b_j), b_j - a \rangle + k) \geq e^*(\langle F(a), \sum_{j=1}^{n} \mu_j b_j - a \rangle + k) \text{ for all } a \in A.$$

By this, we deduce

$$\sup_{a \in A} \sum_{j=1}^{n} \mu_j \big(\langle F(b_j), b_j - a \rangle \big) > 0.$$

Thus, Corollary 3.2 implies there exists a solution $\tilde{a} \in A$ of (EP), i.e.,

$$e^*(\langle F(b), b - \tilde{a}\rangle + k) \geq 0 \text{ for all } b \in A.$$

For each $n \in \mathbb{N}$, we have that there exits $\tilde{a}_n \in A$ such that

$$e^*(\langle F(b), b - \tilde{a}_n\rangle + k/n) \geq 0 \text{ for all } b \in A.$$

Arguing in the same way as in the previous sections, we obtain the existence of $\tilde{a} \in A$ such that

$$e^*(\langle F(b), b - \tilde{a}\rangle) \geq 0 \text{ for all } b \in A.$$

Since $e^* \in K^\sharp$, we deduce

$$\langle F(b), b - \tilde{a}\rangle \notin -K \setminus \{0\}.$$

Hence, $\tilde{a} \in A$ is a Henig proper solution of (MVI). □

Notice that C-monotonicity implies K-monotonicity. Therefore, we achieve a new existence result for (SVI) as a straightforward consequence of Proposition 5.1 and Theorem 5.1, under assumptions which are different from those of Theorem 2.3 in [18]. Indeed, in Theorem 2.3 of [18] a pseudomonotonicity concept (with respect to the cone C) is assumed, which is not related to C-monotonicity, in the sense that none of them implies the other. Moreover, the existence results in [18] is given for a nonempty bounded closed and convex subset A of a reflexive Banach space, while in the theorem below this set is taken from an arbitrary topological vector space.

We also notice, that the next existence result is different from Corollary 3.23 of [36], which requires the continuity of the operator F.

Theorem 5.2 *Suppose A is a compact convex set and C admits a base. If F is a C-monotone and hemicontinuous operator, then (SVI) admits a solution.*

Proof. Since C admits a base, it follows that there there exists a pointed convex cone K with $K^\sharp \neq \emptyset$ such that $C\setminus\{0\} \subseteq \operatorname{int} K$ (see for instance [22]). Then, the thesis follows from Theorem 5.1 and Proposition 5.1. □

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