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Particular order

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Abstract

This technical report presents a new type of binary relation that satisfies asymmetry but not transitivity and that can be used by a decision maker in order to derive a partial order on a set of alternatives according to a given set of criteria. The order is partial since the various criteria (and the corresponding orders) are not always comparable among themselves. The technical report defines the binary relation, presents its features as well as some toy examples to show it at work together with a discussion of its potential weaknesses.
As usual, reports of errors and inaccuracies are gratefully appreciated.

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1 Introduction

In this technical report we present a method that can be used by a decision maker or **decider** in order to rank the a alternatives of a set $\mathscr{A}$ according to the c criteria of a set $\mathscr{C}$. The method aims at producing a directed graph involving all the alternatives so that it is possible for the decider to identify, at least:

- the worst alternatives;
- the best alternatives.

The **worst alternatives** are never selected by the decider that performs his final selection among the **best alternatives** (see section 6).
The order has been nicknamed **particular order** (for reasons that will be made clear shortly from its definition) and has been presented in the author's PhD thesis ([3]). For the best of our knowledge it is a novel order and is endowed with some interesting features.

2 A preliminary remark

At the present time a very wide multitude of orders is available (see, for instance, [7] for a detailed presentation of many of them).
This technical report, of course, cannot present them in a detailed way neither individually nor in pairwise or group comparisons. For these purposes we refer to the literature cited, for instance, in [7] , [5] and [1].
In this section we essentially aim at forestalling the following objection: if so many orders have already been devised why wasting time in devising one more brand new order? This is a sound objection to which I can give only an indirect answer.
During the writing of my PhD thesis ([3]) I had the need to devise a method where a set of d deciders rank the a alternatives of a given set according to the c criteria of another given set and so I needed a multideciders multicriteria method.
The method I devised was based on the following high level steps:

(1) every decider, independently from the others, defines his own partial order of the alternatives;

(2) the various partial orders are merged together in order to produce a final global partial order;

(3) the final global partial order is used as a decision aiding tool from the deciders in order to perform the final selection.

I chose to adopt a partial order since it has a total (or complete) order as a particular case and since the alternatives are ranked according to a set of independent criteria and so they are not necessarily comparable among themselves.

With this we mean hat given two distinct alternatives[1] $a_i, a_j \in \mathscr{A}$ the decider may be unable to state (on the whole set of the criteria) one of the following mutually exclusive conditions:

- a strict preference of a_i over a_j;
- a strict preference of a_j over a_i;
- an indifference between a_i and a_j.

At step (1) I imagined that every decider produces a directed graph and such graphs are merged, at step (2) in order to produce a directed multigraph[2].
The key step is step (1) where I could use an existing multicriteria method ([1]) or devise a new one. Against any advice of my tutor I decided to devise a new method. According to this method, to be described in detail in the present technical report[3] (but see also my PhD thesis [3]), every decider:

- performs c total orders of the alternatives, one for each criterion[4];
- merges the c total orders in a single partial order that is represented with a directed graph[5].

The need to define the individual merging step made me devise a new binary relation that is, in general, not transitive and that defines a partial order that I termed a **particular order**.
The collective merging of step (2) can be easily accomplished since the various directed graphs that correspond to each particular order have, in general, the same set of nodes whereas the final selection of step (3) is guided by the nodes in the final multigraph that have no incoming arc and that, therefore, represent the best alternatives.

[1] The set $\mathscr{A}$ is assumed to have been pruned of both equivalent and dominated alternatives. An alternative $a_i \in \mathscr{A}$ is termed **equivalent** to another alternative $a_j \in \mathscr{A}$ if for every criterion $c_h \in \mathscr{C}$ we have $a_i \sim_h a_j$ where $\sim_h$ is a classical indifference relation. In this case we can discard any of the two alternatives with a caveat. If we discard a_i we must record this fact since, if at the end of the procedure we get $a_j \in \hat{\mathscr{A}}$ (or if a_j is one of the best alternatives), we must include also a_i in the set $\hat{\mathscr{A}}$ owing to this equivalence.
On the other hand, an alternative a_j is said to be **dominated** by an alternative a_i if:

- for a proper subset of the criteria we have $a_i \sim_h a_j$,
- for the remaining disjoint non empty subset of the criteria we have $a_i \succ_h a_j$ where $\succ_h$ is a classical strict preference relation.

Dominated alternatives can be harmlessly removed form the set $\mathscr{A}$.

[2] A directed graph is a graph where the links between the nodes are directed arcs each with a tail and with an head. A multigraph is a graph where we can have more than one arc between any two nodes. In this case we can state for any pair of alternatives $a_i, a_j \in \mathscr{A}$ one of the conditions we have listed in the main text.

[3] Steps (2) and (3) will be presented in greater detail in a forthcoming technical report.

[4] We note that it should be self-evident that if we rank a alternatives according to a single criterion we always get a total order though possibly with some ties among the alternatives.

[5] The rest of this technical report is devoted to the illustration of the steps through which this merging is performed and to show how the obtained order is, in general, partial.

3 The mathematical background

We now recall some properties (some of them will be referred to in section 4) of a binary relation R on a set A.
If for any $a \in A$ we have ([7], [8]):

aRa then R satisfies **reflexivity**.

If for any $a, b \in A$ we have:

$aRb \Rightarrow bRa$ then R satisfies **symmetry**;

aRb and bRa if and only if $a = b$ then R satisfies **antisymmetry**;

$aRb \Rightarrow \neg bRa$ then R satisfies **asymmetry**.

If for any $a, b, c \in A$ we have:

aRb and $bRc \Rightarrow aRc$ then R satisfies **transitivity**.

If a binary relation on a set A satisfies reflexivity, symmetry and transitivity it is termed an **equivalence relation** so that it partitions the set A in a certain number of disjoint subsets also called equivalence classes. Given an element $a \in A$ its equivalence class $[a]$ is defined as:

$$[a] = \{y \in A \mid yRa\} \tag{1}$$

If a binary relation on a set A satisfies reflexivity, antisymmetry and transitivity it is termed a **partial order** but we do not exclude that the set A is totally or completely ordered[6].
If a binary relation on a set A satisfies asymmetry and transitivity it is termed a **strict simple order** ([7]) that is **complete** (so that for any $a, b \in A$ with $a \neq b$ we have either aRb or bRa) but not **strongly complete** ([7]). In case of strong completeness we have that for any $a, b \in A$ we have either aRb or bRa so we allow also $a = b$.

4 The basic ingredients

In sections 1 and 2 we have already informally introduced some of the basic ingredients of our **particular order**. In this section we both complete and formalize their presentation.
We start with the set of the alternatives:

$$\mathscr{A} = \{a_1, \ldots, a_a\} \tag{2}$$

[6]We recall that the terms total and complete to characterize an order are fully synonyms and the use of one term rather than the other depends on the cultural background of the writer. In this technical report we are going use the terms complete and partial, the former to be defined more formally shortly in the text.

to which we can associate an index set:

$$J = \{1, \dots, a\} \tag{3}$$

so that we can identify a pair of alternatives either as (a_i, a_j) or simply as (i, j) and a single alternative a_i through its index $i \in J$.
Next we have the set of the criteria:

$$\mathscr{C} = \{c_1, \dots, c_c\} \tag{4}$$

to which we can associate an index set:

$$I = \{1, \dots, c\} \tag{5}$$

so that a criterion $c_h \in \mathscr{C}$ may be identified by its index $h \in I$.
Over the elements of the set $\mathscr{A}$ we[7] define two basic binary relations:

$\succ_i$

$\sim_i$

for each $c_i \in \mathscr{C}$ and use them to define the binary relation $\succ$ that characterizes the particular order.
Such basic binary relations are endowed with classical properties ([7]) so that (see section 3):

(1) the binary relation $\succ_i$ satisfies transitivity and asymmetry;

(2) the binary relation $\sim_i$ satisfies reflexivity, symmetry and transitivity.

Since both relations satisfy transitivity we have that also their "compositions" satisfy that property so that from situations such as:

$$a_i \sim_h a_j \succ_h a_k \tag{6}$$

we derive $a_i \succ_h a_k$. The same occurs also in other similar situations of easy interpretation.
Within our context, over the elements of the set $\mathscr{A}$ we use both relations $\succ_h$ and $\sim_h$ (with $h \in I$) so we can define, for any $(a_i, a_j) \in \mathscr{A}$, if we have $a_i \succ_h a_j$ or $a_j \succ_h a_i$ or $a_i \sim_h a_j$.
With this we mean that for every $c_h \in \mathscr{C}$ a decider can define a total order of the alternatives so to end this step with c total orders and the need to get them merged in a final order that has no guarantee to be total. We devote section 5 to the definition of this merging procedure.

[7] In many cases we use the term "we" as a shorthand for "the decider".

5 The definition of the order

After all the premises we made in sections 3 and 4 it is time to define the order that we can obtain on the set $\mathscr{A}$ through the binary relation $\succ$.
In order to define the $\succ$ binary relation we introduce the following quantities, for any $(a_i, a_j) \in \mathscr{A}$:

x as the number of times where we have $a_i \succ_h a_j$ over all the criteria $h \in I$;

y as the number of times where we have $a_j \succ_h a_i$ over all the criteria $h \in I$;

z as the number of times where we have $a_j \sim_h a_i$ over all the criteria $h \in I$.

At this point we can have the following cases:

(o_1) $x > y$ so we can state that $a_i \succ a_j$;

(o_2) $x < y$ so we can state that $a_j \succ a_i$;

(o_3) $x = y$ so we cannot state any relation between a_i and a_j.

The occurrence of case (o_3) qualifies the order we obtain through the relation $\succ$ as a partial order. Since, moreover, such a binary relation, in general, fails transitivity (that is a basic property of many orders as we find them defined in the literature, see for instance [7]) but satisfies asymmetry[8] we called such an order a **particular order**.
Some more comments are in order.
First of all, at the basis of the three foregoing cases there is the fact that the criteria are assumed to have the same weight or importance so that:

- if we are in the cases (o_1) and (o_2) we can identify a preferred alternative from a given pair;
- if we are in case (o_3) we are in an particular condition.

In the last particular condition (that can be specified as either of undecidability or of incomparability, see further on) the number of the criteria that favor the alternative a_i against the alternative a_j is equal to the number of the criteria that favor the alternative a_j against the alternative a_i so that we cannot identify a preferred alternative nor we can state an indifference condition since the criteria are, in general, not comparable among themselves.
We underline how, for each pair of alternatives, the constraint is $x + y + z = c$ and that, independently from z, any combination of the foregoing three cases is possible.
Next we have to see why relation $\succ$ fails, in general, transitivity.
If we consider the three alternatives $a_i, a_j, a_h \in \mathscr{A}$ we can have:

[8] This property can be easily derived from the definition of the $\succ$ binary relation and will be presented more formally in section 7.

- for the pair a_i, a_j $x > y$ and so $a_i \succ a_j$,

- for the pair a_j, a_h $x > y$ and so $a_j \succ a_h$,

but for the pair a_i, a_h we can have any of the three cases (o_1) (where transitivity is satisfied), (o_2) and (o_3), where transitivity is not satisfied. All this depends from the fact that the alternatives are pairwise compared according to independent criteria ([9], [10]). In this way there is no relation between the evaluations of the three pairs so that transitivity is not necessarily satisfied, as it will be clear also from the toy examples that we are going to provide in section 6.

6 Two toy examples

We now give two toy examples from [3] in order to show what we have described in section 5 at work.
We examine firstly the case of four criteria and four alternatives so that we have:

$I = \{1, 2, 3, 4\}$ to index the criteria,

$J = \{1, 2, 3, 4\}$ to index the alternatives.

A decider may have the following rankings of the four alternatives according to the four criteria:

$1 \sim_1 2 \succ_1 3 \succ_1 4$

$2 \sim_2 3 \succ_2 4 \succ_2 1$

$3 \sim_3 4 \succ_3 1 \succ_3 2$

$1 \succ_4 3 \succ_4 2 \succ_4 4$

In Figure 1 we have represented the four corresponding linear graphs[9], in the order from left to right from top to bottom. In such graphs we represent the relations $\sim_i$ as undirected arcs and the relations $\succ_i$ as directed arcs. From such graphs it is possible to derive the following six pairwise comparisons[10,11]:

(1,2) $1 \sim_1 2$, $2 \succ_2 1$, $1 \succ_3 2$, $1 \succ_4 2$ from where we get $1 \succ 2$ and so a directed arc from 1 to 2;

(1,3) $1 \succ_1 3$, $3 \succ_2 1$, $3 \succ_3 1$, $1 \succ_4 3$ from where we get a particular condition over the alternatives 1 and 3 and so no directed arc between them;

[9] A graph is said to be linear if it can be drawn along a line so that we have a starting node, an ending node and any node but the ending one has only one connection with another node.

[10] We note that for a alternatives we have $a(a-1)/2$ possible pairwise comparisons and so at the most $a(a-1)/2$ possible directed connections.

[11] We recall that the relations $\sim_i$ and $\succ_i$ are endowed with classical properties so they are assumed to be transitive so that, for instance, from $1 \sim_i 2 \succ_i 3$ we derive $1 \succ_i 3$.

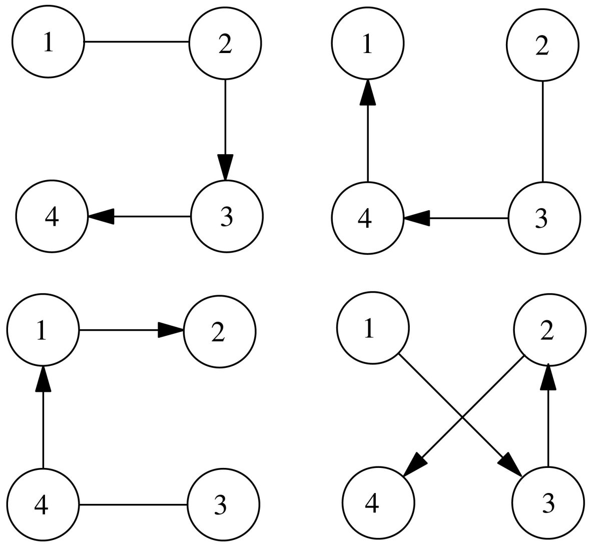

Figure 1: *The four linear graphs of the four rankings*

(1,4) $1 \succ_1 4$, $4 \succ_2 1$, $4 \succ_3 1$, $1 \succ_4 4$ from where we get a particular condition over the alternatives 1 and 4 and so no directed arc between them;

(2,3) $2 \succ_3 3$, $2 \sim_2 3$, $3 \succ_3 2$, $3 \succ_4 2$ from where we get $3 \succ 2$ and so a directed arc from 3 to 2;

(2,4) $2 \succ_1 4$, $2 \succ_2 4$, $4 \succ_3 2$, $2 \succ_4 4$ from where we get $2 \succ 4$ and so a directed arc from 2 to 4;

(3,4) $3 \succ_1 4$, $3 \succ_2 4$, $3 \sim_3 4$, $3 \succ_4 4$ from where we get $3 \succ 4$ and so a directed arc from 3 to 4.

All these relations (or lack of relations) are summarized in the graph of Figure 2 that, therefore, represent the particular order of the four alternatives according to the four criteria. In that graph we use a directed arc between the alternatives i and j with the proper orientation in order to represent the relation between such alternatives according to relation $\succ$.
From the graph of Figure 2 we can see how the decider:

- considers the alternatives 1 and 3 as the best alternatives,
- considers the alternative 4 as the worst alternative,
- ranks the alternatives 1 and 3 as incomparable (see section 7),
- ranks the alternatives 1 and 4 as undecidable (see section 7).

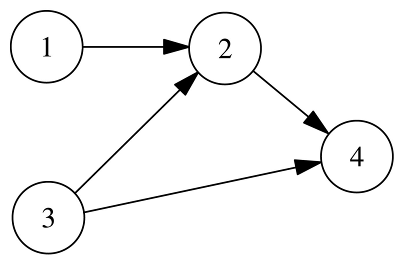

Figure 2: *The resulting graph*

The final selection is made by the decider from the set $\hat{\mathscr{A}} = \{a_1, a_3\}$ so that he can choose the alternative a_3 that is strictly better than alternatives a_2 ad a_4 and is incomparable with alternative a_1. Anyway the full treatment of this final selection step is out of the scope of the present technical report.
We now give the second toy example, in this case with three criteria and four alternatives so that we have:

$$I = \{1, 2, 3\},$$

$$J = \{1, 2, 3, 4\}.$$

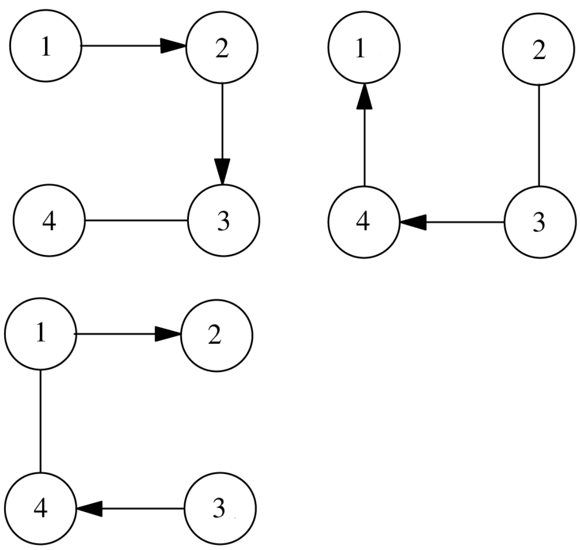

Figure 3: *The three linear graphs of the three rankings*

In this case the decider may have the following rankings of the four alternatives according to the three criteria:

$1 \succ_1 2 \succ_1 3 \sim_1 4$

$2 \sim_2 3 \succ_2 4 \succ_2 1$

$3 \succ_3 4 \sim_3 1 \succ_3 2$

To such total orders there correspond the linear graphs of Figure 3 (in the order from left to right and from top to bottom) so that, following the same steps that we have followed in the previous example, we get the following rankings according the $\succ$ binary relation:

$1 \succ 2$

$3 \succ 1$

$2 \succ 4$

$3 \succ 4$

Such relations can be represented through the directed graph of Figure 4.

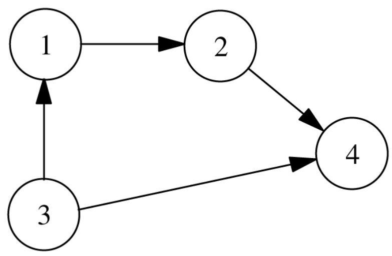

Figure 4: *The second resulting graph*

We note that the pairs of alternatives $2, 3$ and $1, 4$ are characterized by null differences between the corresponding x and y values (see section 5) so that no arc is drawn between their representative nodes.
In this case the decider can argue that the alternative 3, though not directly comparable with alternative 2, is the only good candidate for being the best alternative from the set $J = \{1, 2, 3, 4\}$ according to the criteria of the set $I = \{1, 2, 3\}$. Also in this seemingly simple and clearcut case, however, the full treatment of this final selection step is out of the scope of the present technical report.
On the ground of the foregoing examples we comment a little on the fact that the preference relation $\succ$, in general, is not transitive.
Such relation is defined through pairwise comparisons of possibly tied alternatives through, as primitives, relations that we assume transitive such as $\sim_h$ and

$\succ_h$ for each criterion h. This means that relation $\succ$ is derived from such primitive relations and from the rules we have established for its definition.
From such rules we may have both $i \succ j$ and $j \succ k$ but not $i \succ k$. Examples of this fact, caused also by the presence of the relation $\sim_h$ over pairs of distinct alternatives, can be found, for instance, in the second example where we have:

- $1 \succ 2$ and $2 \succ 4$ though not $1 \succ 4$,
- $3 \succ 1$ and $1 \succ 2$ though not $3 \succ 2$.

To explain this feature we may note that:

· for the pair $(1, 2)$ the alternative 1 is preferred to the alternative 2 for the first and the third criterion,

· for the pair $(2, 4)$ the alternative 2 is preferred to the alternative 4 for the first and the second criterion.

This fact prevents the satisfaction of the transitivity since the alternative 1 is preferred to the alternative 4 on the first criterion (where we have concordance) but no agreement is possible on the other two criteria where we have either discordance or indifference (whereas our definition of the relation $\succ$ imposes $1 \succ 4$ if there is a majority of criteria that establish such preference through the relations $\succ_i$).
Another instance can be found in the first example where we have $1 \succ 2$ and $2 \succ 4$ but, applying the rules that define the relation $\succ$, we do not have $1 \succ 4$.

7 Some features of the particular order

At this point, after the two toy examples that we have given in section 6, we say something more about the **particular order** based on the binary relation $\succ$ that we defined in section 5.
First of all the defined order is not complete since we can have:

- **decidable alternatives** for any pair of alternatives i, j joined by a directed arc so that we can say if it is either $i \succ j$ or $j \succ i$;
- **undecidable alternatives** for any pair of alternatives i, j joined by a directed path but not by a directed arc;
- **incomparable alternatives** for any pair of alternatives i, j without any directed connection between them.

In some particular cases we could have an alternative a_h that is preferred neither to nor by any other alternative a_k (so that we have $x = y\ \forall a_k$) and that corresponds to an **isolated node**. In order for this to occur we must have an alternative a_h for which we have $x = y$ for any other alternative a_k. We deal with this feature in section 8.
In our toy examples we have no isolated node. In the first toy example indeed we have:

- the alternatives $1, 3$ are incomparable whereas the alternatives $1, 4$ are undecidable;
- all the other alternatives are decidable.

In the second toy example we have:

- the alternatives $1, 4$ and $2, 3$ are undecidable;
- all the other alternatives are decidable.

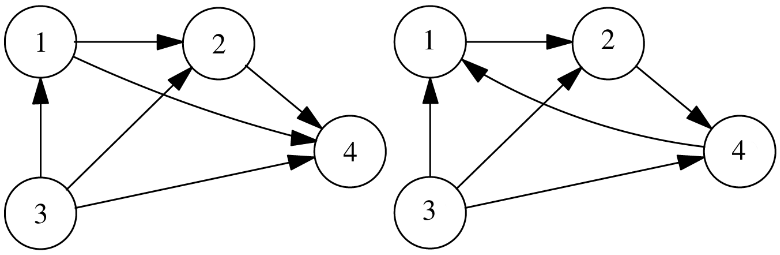

Figure 5: *Conditions for transitivity*

As we have already seen in sections 5 and 6, relation $\succ$ dos not satisfies transitivity (owing to the presence of incomparable and undecidable alternatives).
A necessary condition for transitivity is that among the a alternatives we have (see footnote 10):

$$\frac{a(a-1)}{2} \tag{7}$$

directed connections. Such condition is not sufficient as we show with the examples of Figure 5 where we have:

- on the left a transitive relation among the alternatives;
- on the right a non transitive relation among the same alternatives since we have $1 \succ 2$ and $2 \succ 4$ but $4 \succ 1$ against the definition of transitivity.

From all these considerations we have that transitivity is, in general, not satisfied by the $\succ$ binary relation and must be verified case by case as an ex-post property.
On the other hand the binary relation $\succ$ satisfies **asymmetry**. Let us verify this. If we consider the alternatives $a_i, a_j \in \mathscr{A}$ and we have (see section 5) $x > y$ we have $a_i \succ a_j$. From $x > y$ we have $\neg(y > x)$ or $\neg(a_j \succ a_i)$ and so the asymmetry.
Since relation $\succ$ is asymmetric we cannot have cycles like the one shown in Figure 6 where we would have $1 \succ 4$ and $4 \succ 1$. This situation can never occur since, from the asymmetry of the $\succ$ binary relation, the former relation prevents the latter from occurring and vice versa.

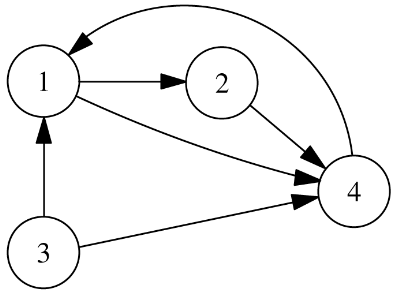

Figure 6: *Possible cycles?*

8 How to deal with isolated nodes

As we have seen in section 7, in the process of producing his directed graph the decider may identify an alternative a_h that is preferred neither to nor by any other alternative a_k and that corresponds to an **isolated node**.
First of all we note that if we have $x = y = 0\ \forall a_k$ with $k \neq h$ we have:

$$a_h \sim_i a_k\ \ \forall c_i \in \mathscr{C} \tag{8}$$

where $\sim_i$ is a classical indifference relation that satisfies transitivity. In this case if we consider the alternatives a_k and a_l (with both $k \neq h$ and $l \neq h$) we have:

$$a_h \sim_i a_k\ \ \forall c_i \in \mathscr{C} \tag{9}$$

and:

$$a_h \sim_i a_l\ \ \forall c_i \in \mathscr{C} \tag{10}$$

and, from the transitivity of $\sim_i$:

$$a_l \sim_i a_k\ \ \forall c_i \in \mathscr{C} \tag{11}$$

so that we have no arc between any pair of alternatives a_k and a_l.
From all this we derive that the alternatives would be equivalent among themselves so that the decider can select one of them at random.
On the other hand if we have $x = y \neq 0$ for some a_k with $k \neq h$ we have that the alternative a_h (to which it corresponds an isolated node) does not belong either to $\hat{\mathscr{A}}$ or to $\check{\mathscr{A}}$ so that it must be discarded from the final selection process.
In the unfortunate case where the final graph contains only isolated nodes we must conclude, therefore, that the method has failed and that the decider is unable to perform his final selection. The only possible solution for the decider is to enlarge the set $\mathscr{C}$ of the classifying criteria and repeat the procedure until he is able to define a directed graph with at least a subset of decidable and undecidable alternatives.

9 Conclusions and future plans

In this technical report we have presented a binary relation $\succ$ that defines a type of order on a set of alternatives $\mathscr{A}$ according to a set of criteria $\mathscr{C}$ and that we nicknamed a **particular order**.
Such binary relation is not complete, satisfies asymmetry but fails transitivity and allows the definition of the set $\hat{\mathscr{A}}$ of the best alternatives among which the decider can perform his final selection.
As we have seen in section 5, relation $\succ$ is based on the execution of pairwise comparisons among the alternatives of the set $\mathscr{A}$ through the use of classical and easy to evaluate binary relations $\succ_i$ and $\sim_i$ for each criterion $c_i \in \mathscr{C}$. Moreover it has a simple definition based on a simple counting and on the evaluation of either a "strictly greater than" relation or an "equal to" relation.
As to the **particular order** we have shown how it can be represented through a directed but in general non complete graph where the set $\hat{\mathscr{A}}$ coincides with the set of the nodes without any incoming arc.
Future plans include the use of these tools in more complex real-world cases and a verification of the claim that the alleged properties are enough to let the decider define, by using the sets $\mathscr{A}$ and $\mathscr{C}$, a well characterized set $\hat{\mathscr{A}}$ from which he can select the best alternative.
If we consider the criteria as voters and the alternatives as candidates another stream of research that could be worth pursuing is a more formal and detailed analysis of the properties of the proposed method as a voting method ([2], [4], [12], [6], [9], [10]) with particular attention to its behavior with regard to some famous impossibility theorems such ah the Arrow's and Sen's theorems ([11], [9], [10]).

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