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Merawti: a method for the ranking of the alternatives with ties

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Abstract

In this technical report we present a method that can be used by a set of decision makers in order to rank a certain number of alternatives according to a given set of criteria. The method aims at producing a directed multigraph involving all the alternatives (as nodes of the multigraph) so that it is possible for the decision makers to identify the worst alternatives and the best alternatives. The worst alternatives are never selected by the deciders that perform their final selection among the best alternatives.
As usual, reports of errors and inaccuracies are gratefully appreciated.

Contents

List of Figures

1 Introduction

In this technical report we present a method that can be used by a set $\mathscr{D}$ of d decision makers or **deciders** in order to rank the a alternatives of the set $\mathscr{A}$ according to the c criteria of the set $\mathscr{C}$.
The method is grounded on the following basic hypotheses:

- the deciders are in peer-to-peer relations among themselves,
- the criteria have the same weight or importance since they are assumed to be incommensurable.

The proposed method aims at producing a directed multigraph[1] involving all the alternatives (as nodes of the multigraph) so that it is possible for the deciders to identify, at least:

- the worst alternatives as nodes with no outgoing arc but with incoming arcs;
- the best alternatives as nodes with no incoming arc but with outgoing arcs.

The **worst alternatives** are never selected by the deciders that perform their final selection among the **best alternatives** of the set $\hat{\mathscr{A}}$.
The method has been nicknamed **merawti** (an acronym that stands for ***me*thod for the *ra*nking of *a*lternatives *w*ith *ti*es**) and has been presented in the author's PhD thesis ([2]). For the best of our knowledge it is a novel method and is grounded on a new type of order, that we present in section 4.

2 A preliminary remark

At the present time a very wide multitude of multicriteria methods is available and we can say, without any fear of refutation, that almost every day a new method or a variant of an existing method is devised and presented to the world. This technical report, of course, cannot present them in any detail neither individually nor in pairwise or group comparisons. For these purposes we refer to the literature cited, for instance, in [4] and [1].
In this section we essentially aim at forestalling the following objection: if so many multicriteria methods have already been devised why wasting time in devising one more supposedly brand new method? This is a sound objection to which I can give only an indirect answer.
During the writing of my PhD thesis ([2]) I had the need to use a method where

[1] A directed multigraph is a graph where between any pair of nodes we can have more than one directed arc. A directed arc is an oriented link from a source node to a destination node. Within our framework (and for reasons that the technical report aims at making clear) we do not collapse a pair of arcs with opposite directions between the same nodes in an undirected arc.

a set of d deciders rank the a alternatives of a given set according to the c criteria of a given set and so a multideciders multicriteria method.
The method I devised was based on the following high level steps:

(1) every decider, independently from the others, defines his own partial order of the alternatives;

(2) the various partial orders are merged together in order to produce a final global partial order;

(3) the final global partial order is used as a decision aiding tool from the deciders in order to perform the final selection.

I chose to adopt a partial order since it has a total (or complete) order as a particular case and since the alternatives are ranked according to a set of independent and incommensurable criteria and so they are not necessarily comparable among themselves. With this we mean hat given two distinct alternatives $a_i, a_j \in \mathscr{A}$ every decider may be unable to state (on the whole set of the criteria) one of the following mutually exclusive conditions:

- a strict preference of a_i over a_j;
- a strict preference of a_j over a_i;
- an indifference between a_i and a_j.

At step (1) I imagined that every decider produces a directed graph and such graphs are merged, at step (2) in order to produce a directed multigraph.
The key step is step (1) where I could use an existing multicriteria method ([1]) or devise a new one. Against any advice of my tutor I decided to devise a new method. According to this method ([2], [3]), every decider:

- performs c total orders of the alternatives, one for each criterion[2];
- merges the c total orders in a single partial order that is represented with a directed graph where the obtained order is, in general, partial.

The need to define the individual merging step made me devise a new binary relation that is, in general, not transitive but satisfies asymmetry and that defines a partial order that I termed a **particular order**.
The collective merging of step (2) can be easily accomplished since the various directed graphs that correspond to each particular order have the same set of nodes whereas the final selection of step (3) is guided by the nodes in the final multigraph that have no outgoing arc and that, therefore, represent the best alternatives.

[2]We note that it should be self-evident that if we rank a alternatives according to a single criterion we always get a total order, though possibly with some ties, among the alternatives.

3 The basic ingredients

In section 1 we have already informally introduced some of the basic ingredients of the **merawti** method. In this section we both complete and formalize their presentation.
Our main aim with this technical report is to present a multideciders multicriteria method where the d deciders of the set $\mathscr{D}$ rank the a alternatives of the set $\mathscr{A}$ according to the c criteria of the set $\mathscr{C}$ in order to identify the following sets:

- the set $\hat{\mathscr{A}}$ of the best alternatives;
- the set $\check{\mathscr{A}}$ of the worst alternatives.

It is obvious that we have (see also section 8):

$$\mathscr{A} \supseteq \hat{\mathscr{A}} \cup \check{\mathscr{A}} \tag{1}$$

and:

$$\hat{\mathscr{A}} \cap \check{\mathscr{A}} = \emptyset \tag{2}$$

The method is composed of three steps.
At the **first step** every decider $d_i \in \mathscr{D}$ defines a partial order among the a alternatives according to the c criteria. To such order it corresponds a directed non complete graph[3]:

$$G_i = (N, A_i) \tag{3}$$

where N is the set of the nodes, the same for all the deciders, one node for each of the alternatives, whereas the set A_i is the set of the directed arcs (i, j) with $i, j \in N$. Each directed arc (i, j) (where i is the tail and j the head) is associated to a preference relation $\succ$ (see section 4) between the alternatives $a_i, a_j \in \mathscr{A}$ so that it translates the binary relation:

$$a_i \succ a_j \tag{4}$$

At the **second step** the deciders merge the d directed graphs in a single directed multigraph MG where:

- we label each arc (i, j) with its multiplicity;
- we neither merge a pair of opposite arcs between the same nodes in an undirected arc nor cancel them out.

Once the second step has been carried out the deciders have defined MG so they can switch to the **third step** where they identify the sets $\hat{\mathscr{A}}$ and $\check{\mathscr{A}}$ and use the former to select the best alternatives (see also section 8 for some difficulties

[3] A graph ([6]) is complete if we have an arc between any pair of nodes.

and some ways to deal with them).
More formally we can define[4]:

$$MG = (N, A) \tag{5}$$

where N is the same set as in the case of the G_i whereas A is a set of pairs defined as:

$$A = \{(v_j, m_j) \mid \exists G_i \; with \; v_j \in A_i\} \tag{6}$$

where:

> $v_j = (h, k)$ is the directed arc from h to k with $h, k \in N$,
>
> m_j is the multiplicity of that connection or the number of directed arcs v_j from h to k over the various G_i.

The multiplicity assigned to a directed arc v_j defines the number of graphs G_i that have a directed arc between the same nodes and so counts the number of the deciders that share that preference relation between the corresponding pair of alternatives a_h, a_k. From this we derive that:

$$d \geq max_j\{m_j\} \tag{7}$$

In Figure 1 we give an example of a directed multigraph that involves four alternatives and (at least) three deciders.

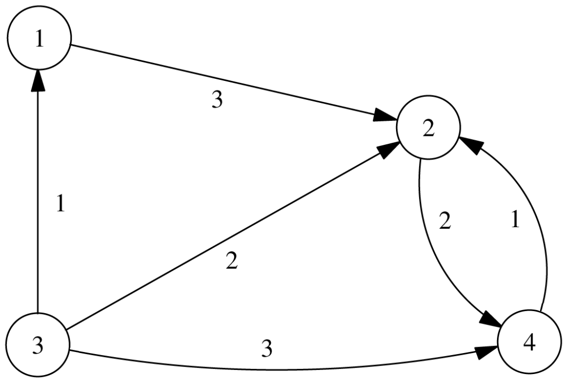

Figure 1: *An example of directed multigraph*

For the MG of Figure 1 it is easy to see that we have:

> $\hat{\mathscr{A}} = \{3\}$

[4]We underline the fact that every G_i may have at the most $a(a-1)/2$ directed arcs so that MG may have at the most $da(a-1)/2$ directed arcs.

$\check{\mathscr{A}} = \emptyset$

from the definitions of such sets.
It is easy to see how we have no guarantee to have $\hat{\mathscr{A}} \neq \emptyset$ and $\check{\mathscr{A}} \neq \emptyset$. An example where both these conditions are falsified is given in Figure 2. We refer to section 8 for an analysis of such cases.

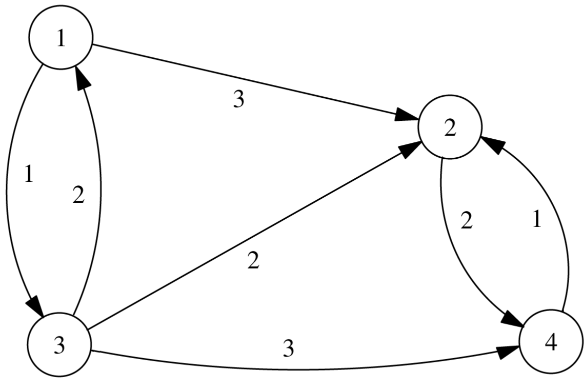

Figure 2: *Neither best nor worst alternatives*

4 The mathematical core

At the core of the proposed method we have the binary relation $\succ$. Each of the d deciders uses such relation in order to produce the graph G_i from the alternatives of the set $\mathscr{A}$. As we are going to show shortly, for the generic graph G_i we may assume to have[5]:

$\hat{\mathscr{A}_i} \neq \emptyset$

$\check{\mathscr{A}_i} \neq \emptyset$

The binary relation $\succ$ is not primitive and is defined on the basis of two other primitive binary relations or:

- an indifference relation $\sim_i$,
- a strict preference relation $\succ_i$.

Such relations are defined on the set $\mathscr{A}$ of the alternatives for each of the the criterion $c_i \in \mathscr{C}$ and are endowed with classical properties. With this we mean that[6]:

[5]We refer to section 8 for a discussion of these assumptions. At this level we note that such assumptions are falsified only if G_i contains only isolated nodes since the binary relation is asymmetric so that we cannot have cycles.

[6]For the basic definitions of such properties we refer, for instance, to [5].

(1) the binary relation $\succ_i$ satisfies transitivity and asymmetry;

(2) the binary relation $\sim_i$ satisfies reflexivity, symmetry and transitivity.

Since both relations satisfy transitivity we have that also their "compositions" satisfy that property so that from situations such as:

$$a_i \sim_h a_j \succ_h a_k \tag{8}$$

we derive $a_i \succ_h a_k$. The same occurs also in other similar situations of easy interpretation.
Within our context over the elements of the set $\mathscr{A}$ we use both relations $\succ_h$ and $\sim_h$ (with $h \in I$) so we can define for any pair $(a_i, a_j) \in \mathscr{A}$ if we have $a_i \succ_h a_j$ or $a_j \succ_h a_i$ or $a_i \sim_h a_j$.
With this we mean that for every $c_h \in \mathscr{C}$ a decider can define a total order with possible ties of the alternatives so to end this step with c total orders and the need to get them merged in a final order that has no guarantee to be total.
At this point we define the binary relation $\succ$. For this purpose we introduce the following quantities, for any pair $(a_i, a_j) \in \mathscr{A}$:

x as the number of times where we have $a_i \succ_h a_j$ over all the criteria $h \in I$;

y as the number of times where we have $a_j \succ_h a_i$ over all the criteria $h \in I$;

z as the number of times where we have $a_j \sim_h a_i$ over all the criteria $h \in I$.

At this point we can have the following cases:

(c_1) $x > y$ so we can state that $a_i \succ a_j$;

(c_2) $x < y$ so we can state that $a_j \succ a_i$;

(c_3) $x = y$ so we cannot state any relation between a_i and a_j.

The occurrence of case (c_3) qualifies the order that we obtain through the relation $\succ$ as a partial order. Since, moreover, such a binary relation, in general, fails transitivity (that is a basic property of many orders as we find them defined in the literature, see for instance [5]) but satisfies asymmetry we called such an order a **particular order**.
Some more comments are in order.
First of all, at the basis of the three foregoing cases there is the fact that the criteria are assumed to have the same weight or importance so that:

- if we are in the cases (c_1) and (c_2) we can identify a preferred alternative from a given pair;

- if we are in case (c_3) we are in an particular condition.

The last particular condition can be specified as either of undecidability (for any pair of alternatives i, j joined by a directed path but not by a directed arc) or of incomparability (for any pair of alternatives i, j without any directed connection between them).
In the (c_3) case the number of the criteria that favor the alternative a_i against the alternative a_j is equal to the number of the criteria that favor the alternative a_j against the alternative a_i so that we cannot identify a preferred alternative nor we can state an indifference condition since the criteria are, in general, not comparable among themselves.
We underline how, for each pair of alternatives, the constraint is $x + y + z = c$ and that, independently from z, any combination of the foregoing three cases is possible.
Next we have to see why relation $\succ$ fails, in general, transitivity.
If we consider the three alternatives a_i, a_j, a_h we can have:

- for the pair a_i, a_j $x > y$ and so $a_i \succ a_j$,

- for the pair a_j, a_h $x > y$ and so $a_j \succ a_h$,

but for the pair a_i, a_h we can have any of the three cases (o_1) (when transitivity is satisfied), (o_2) and (o_3), when transitivity is not satisfied. All this depends from the fact that the alternatives are pairwise compared according to independent criteria ([7], [8]). In this way there is no relation between the evaluations of the three pairs so that transitivity is not necessarily satisfied, as it will be clear also from the toy example that we are going to provide in section 6.
On the other hand the binary relation $\succ$ satisfies **asymmetry**. Let us verify this. If we consider the alternatives $a_i, a_j \in \mathscr{A}$ and we have $x > y$ then we have $a_i \succ a_j$. From $x > y$ we have $\neg(y > x)$ or $\neg(a_j \succ a_i)$ and so the asymmetry.

5 The method

As we have outlined in section 3, the **merawti** method is based on the following steps:

(1) every decider individually ranks the given alternatives according to the common criteria and produces a directed graph G_i;

(2) the d directed graphs are merged in a single collective directed multigraph MG;

(3) the deciders use MG to select the best alternative.

At step (1) every decider[7] considers the common alternatives of the set $\mathscr{A}$ and evaluates them according to each criterion from the common set of criteria $\mathscr{C}$. In this way he is able to produce c total orders (with possible ties) of the alternatives and merge them to produce a single directed graph G_i. This individual merging is made as follows:

[7] We refer to section 7 for some possible variations to this uniform scheme.

- the decider draws the nodes of his graph G_i;
- he considers a pair of nodes $i, j \in N$ and performs the comparisons we have shown in section 4 so to draw an arc (i, j) if $i \succ j$, an arc (j, i) if $j \succ i$ and no arc in the last possible case;
- he repeats the previous step for all the possible pairwise comparisons of the nodes.

At step (2) the deciders collectively merge the d graphs G_i in a single multigraph MG through a simple mechanical procedure.
They firstly draw the nodes of the set N and then, for each pair of distinct nodes $i, j \in N$, count the number of arcs between such nodes as both (i, j) and (j, i). If at least one of the counts is greater than zero they draw the corresponding arc and assign that count as the multiplicity of the arc. As we show in Figure 2, for a given pair of nodes $i, j \in N$ we may have the arcs (i, j) and (j, i) each with its own multiplicity.
We devote section 9 to the treatment of the step (3).

6 A toy example

We devote this section to a toy example where three deciders must rank four alternatives according to four criteria. Each of them produces his own directed graph G_i and then such graphs are merged in a single directed multigraph MG. We consider three deciders d_1, d_2 and d_3.
Decider d_1 may have the following preferences on the four alternatives according to the four criteria:

$1 \sim_1 2 \succ_1 3 \succ_1 4$

$2 \sim_2 3 \succ_2 4 \succ_2 1$

$3 \sim_3 4 \succ_3 1 \succ_3 2$

$1 \succ_4 3 \succ_4 2 \succ_4 4$

Each of such relations is translated in the corresponding graph of Figure 3. In this and similar graphs we use an undirected arc to translate an indifference relation and a directed arc to translate a strict preference relation.
If we preform the six pairwise comparisons among the four alternatives we get:

(1,2) $1 \sim_1 2$, $2 \succ_2 1$, $1 \succ_3 2$, $1 \succ_4 2$ and so an arc from 1 to 2;

(1,3) $1 \succ_1 3$, $3 \succ_2 1$, $3 \succ_3 1$, $1 \succ_4 3$ and so no arc between 1 and 3;

(1,4) $1 \succ_1 4$, $4 \succ_2 1$, $4 \succ_3 1$, $1 \succ_4 4$ and so no arc between 1 and 4;

(2,3) $2 \succ_3 3$, $2 \sim_2 3$, $3 \succ_3 2$, $3 \succ_4 2$ and so an arc from 3 to 2;

(2,4) $2 \succ_1 4$, $2 \succ_2 4$, $4 \succ_3 2$, $2 \succ_4 4$ and so an arc from 2 to 4;

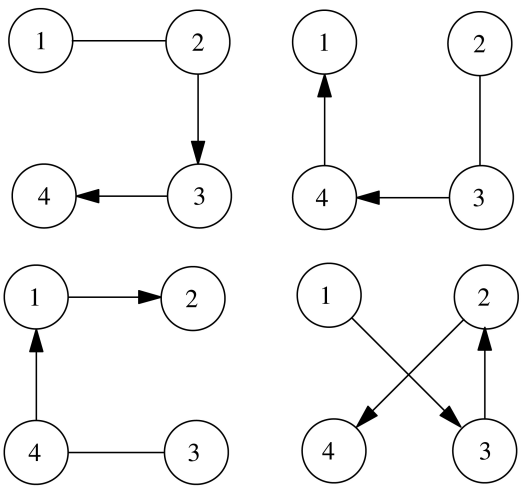

Figure 3: *The preferences of* d_1

(3,4) $3 \succ_1 4$, $3 \succ_2 4$, $3 \sim_3 4$, $3 \succ_4 4$ and so an arc from 3 to 4.

The result of such pairwise comparisons is represented in Figure 4 as the directed graph G_1.

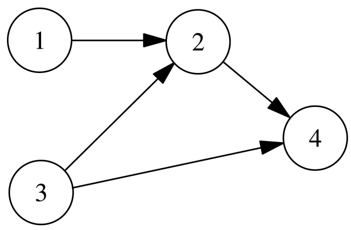

Figure 4: G_1

Decider d_2 may have the following preferences on the four alternatives according to the four criteria:

$1 \succ_1 2 \succ_1 3 \sim_1 4$

$2 \sim_2 3 \succ_2 4 \succ_2 1$

$3 \succ_3 4 \sim_3 1 \succ_3 2$

$2 \sim_3 3 \succ_4 4 \sim_4 1$

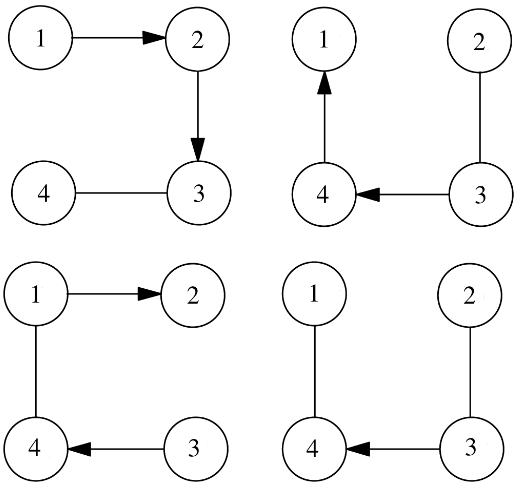

Figure 5: *The preferences of* d_2

Each of such relations is translated in the corresponding graph of Figure 5. If we preform the six pairwise comparisons among the four alternatives we get:

(1,2) $1 \succ_1 2$, $2 \succ_2 1$, $1 \succ_3 2$, $2 \succ_4 1$ and so no arc between 1 and 2;

(1,3) $1 \succ_1 3$, $3 \succ_2 1$, $3 \succ_3 1$, $3 \succ_4 1$ and so an arc from 3 to 1;

(1,4) $1 \succ_1 4$, $4 \succ_2 1$, $4 \sim_3 1$, $4 \sim_4 1$ and so no arc between 1 and 4;

(2,3) $2 \succ_3 3$, $2 \sim_2 3$, $3 \succ_3 2$, $2 \sim_3 3$ and so no arc between 3 and 2;

(2,4) $2 \succ_1 4$, $2 \succ_2 4$, $4 \succ_3 2$, $2 \succ_4 4$ and so an arc from 2 to 4;

(3,4) $3 \sim_1 4$, $3 \succ_2 4$, $3 \succ_3 4$, $3 \succ_4 4$ and so an arc from 3 to 4.

The result of such pairwise comparisons is represented in Figure 6 as the directed graph G_2.

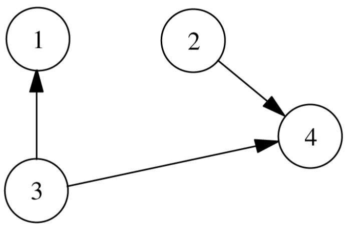

Figure 6: G_2

Decider d_3 may have the following preferences on the four alternatives according to the four criteria:

$1 \succ_1 2 \succ_1 3 \sim_1 4$

$1 \succ_2 3 \succ_2 4 \succ_2 2$

$3 \succ_3 2 \sim_3 1 \succ_3 4$

$3 \succ_3 1 \succ_4 2 \succ_4 4$

Each of such relations is translated in the corresponding graph of Figure 7.

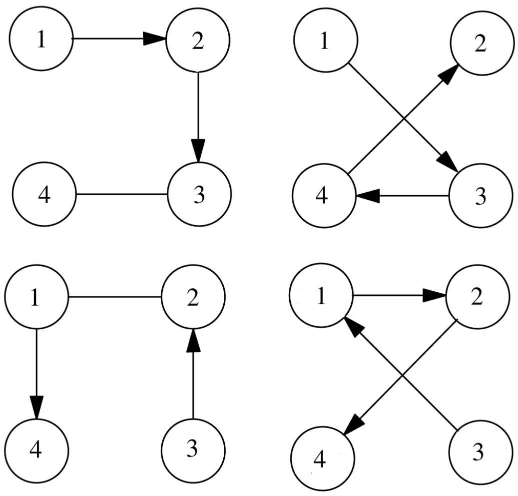

Figure 7: *The preferences of d_3*

If we preform the six pairwise comparisons among the four alternatives we get:

(1,2) $1 \succ_1 2$, $1 \succ_2 2$, $1 \sim_3 2$, $1 \succ_4 2$ and so an arc from 1 to 2;

(1,3) $1 \succ_1 3$, $1 \succ_2 3$, $3 \succ_3 1$, $3 \succ_4 1$ and so no arc between 1 and 3;

(1,4) $1 \succ_1 4$, $1 \succ_2 4$, $1 \succ_3 4$, $1 \succ_4 4$ and so an arc from 1 to 4;

(2,3) $2 \succ_3 3$, $3 \succ_2 2$, $3 \succ_3 2$, $3 \succ_3 2$ and so an arc from 3 to 2;

(2,4) $2 \succ_1 4$, $4 \succ_2 2$, $2 \succ_3 4$, $2 \succ_4 4$ and so an arc from 2 to 4;

(3,4) $3 \sim_1 4$, $3 \succ_2 4$, $3 \succ_3 4$, $3 \succ_4 4$ and so an arc from 3 to 4.

The result of such pairwise comparisons is represented in Figure 8 as the directed graph G_3.
If the three deciders merge the graphs G_1, G_2 and G_3 they easily get the multigraph MG of Figure 9.
It is easy to see how we have:

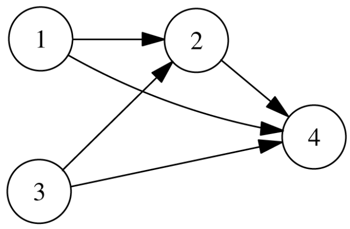

Figure 8: G_3

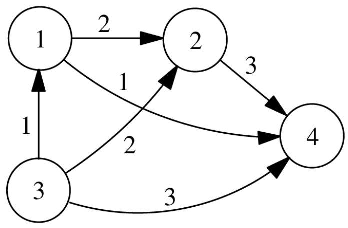

Figure 9: MG

- $\check{\mathscr{A}} = \{4\}$
- $\hat{\mathscr{A}} = \{3\}$

so that in this toy example the final selection is really easy. At this level we only note that:

- a_3 is preferred to alternatives a_2 and a_4 from either a majority or the unanimity of the deciders as voters;
- the unanimity of the deciders as voters agrees to consider the alternative a_4 as an alternative worse than alternatives a_2 and a_3;
- a majority of the deciders as voters agrees to consider the alternative a_2 as an alternative worse than alternatives a_1 and a_3;
- only one decider strictly prefers alternative a_3 to alternative a_1;
- only one (possibly different) decider strictly prefers alternative a_1 to alternative a_4.

We refer to section 9 for a few more comments.

7 The robustness of the method

In section 6 we have presented a toy example where three deciders select the best alternative from the set $\mathscr{A}$ of four alternatives by ranking them according

to the four criteria of the set $\mathscr{C}$. They use the method we presented in section 5 in order to devise three graphs G_i and merge them in a single multigraph MG so to use it for the final selection of the best alternative, as we show more formally in section 9.
From what we have seen up to now it may seem that, in order to be applied, the method requires that the deciders share both the set of alternatives and the set of the criteria. In this section we examine the robustness of the method and so how it performs in the following cases:

(1) the deciders share the set $\mathscr{A}$ but each of them has his own set $\mathscr{C}_i$;

(2) the deciders share the set $\mathscr{C}$ but each of them has his own set $\mathscr{A}_i$;

(3) each decider has his own sets $\mathscr{A}_i$ and $\mathscr{C}_i$.

In case (1) the deciders agree on which are the alternatives among which they must perform the selection but disagree on the criteria they can use since each of them thinks that some of the criteria of a global set $\mathscr{C}$ (that contains all the criteria) are either meaningless or inapplicable to the current set of the alternatives.
In this case the graphs G_i have the same nodes and each decider defines the arcs between the various pairs of nodes according to his own criteria.
In this case we do not impose:

$$\cap_i \mathscr{C}_i \neq \emptyset \tag{9}$$

though the fact that such condition is verified (so that the deciders share at least some criteria) surely strengthen the outcome of the selection process.
In case (2) the deciders disagree on which are the alternatives among which they must perform the selection but agree on the criteria they can use because [at least some of] the criteria can be enforced by laws, norms and regulations so that they are imposed to the deciders.
In this case the graphs G_i differ as to their nodes and each decider defines the arcs between his various pairs of nodes according to the criteria of the common set. Every decider, therefore, defines his own G_i but must commit himself at accepting the resulting multigraph MG as the only tool to be used for the final selection.
In this case we must impose:

$$\cap_i \mathscr{A}_i \neq \emptyset \tag{10}$$

If constraint (10) is violated the deciders have no alternative in common so that we can hardly speak of a multideciders method and the selection of one of the alternatives requires that such constraint is re-established in some way.
The key point is represented by the reasons why every decider has his own set $\mathscr{A}_i$.
This is possible only if the alternatives that each decider discards from a set $\mathscr{A}$ (that contains all the currently available alternatives) in order to produce his own set $\mathscr{A}_i$ are, for him, **dominated** alternatives.
An alternative a_h is said to be **dominated** if there exists at least another alternative a_k such that:

- for a proper subset of the criteria we have $a_k \sim_i a_h$,

- for the complementary and non empty subset of the criteria we have $a_k \succ_i a_h$.

In case (3) we face the most general case where each decider has:

- his own set of alternatives $\mathscr{A}_i$;

- his own set of criteria $\mathscr{C}_i$.

On such sets we have the constraints (9) and (10) with the related comments. In these conditions every decider may define his own G_i but must commit himself at accepting the resulting multigraph MG as the only tool to be used for the final selection.
In this way we have shown how the proposed method can be used, with some cautions, even in the last more general and more heterogeneous case.

8 The failure of the method?

As we have seen in section 1 the method aims at the identification of:

- the set $\check{\mathscr{A}}$ of the worst alternatives;

- the set $\hat{\mathscr{A}}$ of the best alternatives.

If we have $\hat{\mathscr{A}} \neq \emptyset$ the deciders may use that set to perform their final selection (see section 9).
In this section we deal with the case $\hat{\mathscr{A}} = \emptyset$ and show how the deciders can behave in such a situation. We note that the condition $\check{\mathscr{A}} = \emptyset$ is of less harm though we will comment a little also about it.
The $\hat{\mathscr{A}} = \emptyset$ condition can occur in the following cases:

(1) canceling preferences,

(2) no node without incoming arcs,

(3) only isolated nodes.

We give an example of case (1) in Figure 10 where for each pair of nodes $i, j \in N$ we have the same number of arcs for the opposite preferences $i \succ j$ and $j \succ i$.
An example of case (2) has been given in Figure 2 where such an equality does not occur.
In order to appreciate the difference between the first two cases we may define the so called **reduced multigraph** $\overline{MG}$.
$\overline{MG}$ can be obtained from MG if we remove from the latter all the pairs of arcs with opposite orientation. In Figure 11 we show, on the left, the reduced multigraph corresponding to the multigraph of Figure 2 and, on the right, the reduced multigraph corresponding to the multigraph of Figure 10. This multigraph is also an example of a multigraph MG made only of isolated nodes.

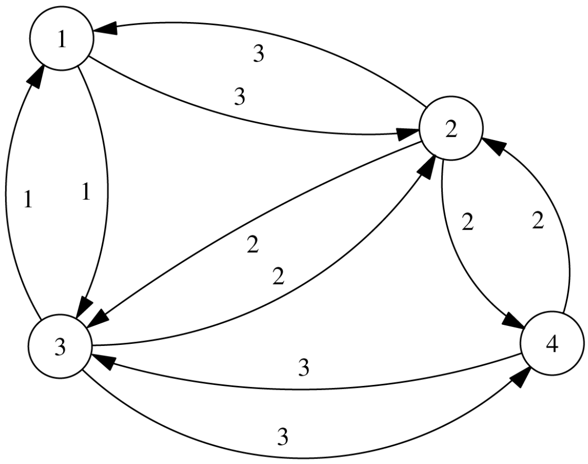

Figure 10: *Canceling preferences*

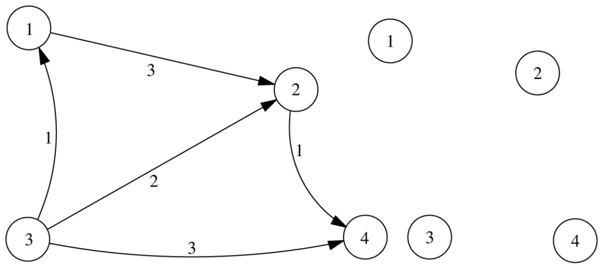

Figure 11: *The reduced multigraphs*

In Figures 12 and 13 we give two examples of the second condition. In both the examples we start with an MG where we have $\hat{\mathscr{A}} = \emptyset$ then we define $\overline{MG}$ where we have $\hat{\mathscr{A}} \neq \emptyset$. In the example of Figure 12 the final selection is an easy task since it is easy to see how in $\overline{MG}$ the best alternative is a_3.
In the example of Figure 13 we have $\hat{\mathscr{A}} = \{1, 3\}$ and, for reasons that will be made clear in section 9, the deciders will select a_1 as the best alternative.
The condition of **canceling preferences** has already been presented in Figure 11 where we show, on the left, the $\overline{MG}$ corresponding to the MG of Figure 2 and, on the right, the $\overline{MG}$ corresponding to the MG of Figure 10.
In the former case we have $\hat{\mathscr{A}} = \{3\}$ so that a_3 is the best alternative in this case.
The latter case where we have only isolated nodes deserves further comments.
Before dealing with it we present another example in Figure 14.

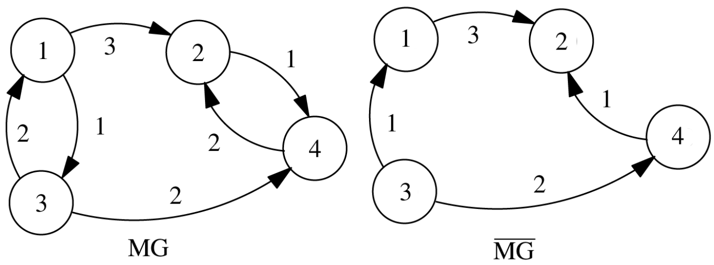

Figure 12: *A first example of no node without incoming arcs*

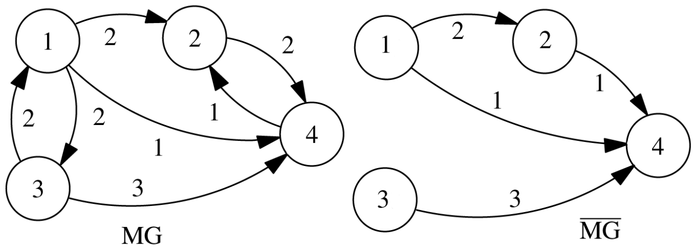

Figure 13: *Another example of no node without incoming arcs*

In that figure we have 3 as an isolated node (so that we can discard a_3 as surely not a best alternative) and:

$$\hat{\mathscr{A}} = \{1\}$$

$$\check{\mathscr{A}} = \{4\}$$

For what concerns the **isolated nodes** (see also footnote 5) we note, indeed, how they can be present:

- already in the multigraph MG,
- only in the reduced multigraph $\overline{MG}$.

In order to be present in the MG, an isolated node must be present in all the composing graphs G_i so that we can discard it together with the corresponding alternative. In this way we might get an empty MG otherwise we may proceed as we have done in the foregoing cases over the MG but without the isolated nodes.
If the condition of an empty MG occurs the method has failed since no decision can be taken by the deciders about the alternatives of the set $\mathscr{A}$ according to the criteria of the set $\mathscr{C}$. The only possibility they have is to enlarge the set $\mathscr{C}$ so to refine the ranking of the alternatives until they get a non empty MG

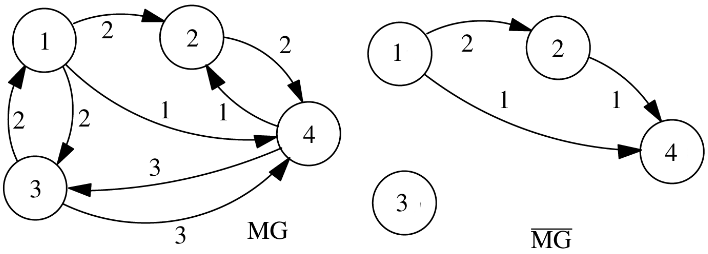

Figure 14: *An example with an isolated node*

possibly without isolated nodes.
If the isolated nodes are present only in the $\overline{MG}$ we can have the following cases:

- only some isolated nodes, as we have seen in Figure 14;
- only isolated nodes, as we have seen in the right side of Figure 11.

In the former case we can discard such nodes as we have seen for the example of Figure 14 whereas in the latter case we can consider the alternatives as undecidable or incomparable among themselves so that the deciders cannot perform any selection but can only proceed as we have seen in the case of an empty MG.
In order to fully appreciate such undecidability or incomparability conditions among the alternatives we can refer to the MG of Figure 10 to which it corresponds the $\overline{MG}$ made only of isolated nodes that we have seen in Figure 11.
In this case we have either pairs of undecidable alternatives (those without any directed link between themselves) or pairs of alternatives over which the deciders have conflicting preferences. From all this we derive that, in this case, the alternatives must be seen as undecidable and this prevents the deciders from performing any selection over such alternatives.
What can we say about the condition $\check{\mathscr{A}} = \emptyset$? Though this condition has no consequence on the final selection (that depends only on the set $\hat{\mathscr{A}}$) it may weaken the strength of such selection since the deciders have not been able to single out any clear cut worst alternative.

9 The final selection

In this section we examine the case where we have $\hat{\mathscr{A}} \neq \emptyset$ though not necessarily we have also $\check{\mathscr{A}} \neq \emptyset$. If we have $\check{\mathscr{A}} \neq \emptyset$ we are sure to have a certain number of worst alternatives otherwise we have no such certainty so that we cannot classify any of the alternatives as absolutely to be discarded (see also section 8).
Under the condition $\hat{\mathscr{A}} \neq \emptyset$ we may have:

$|\hat{\mathscr{A}}| = 1$ so that the selection is easily accomplished and the whole process ends with success;

$|\hat{\mathscr{A}}| > 1$ so we need a further step of selection.

In the latter case we need a method to obtain a ranking of the best alternatives in order to select one of them.
The proposed method is essentially a lexicographic order of the alternatives of the set $\hat{\mathscr{A}}$ according to the number of the outgoing nodes and of the outgoing arcs, counted with their multiplicity, of the corresponding nodes of MG.
More formally, we associate to each $a_i \in \hat{\mathscr{A}}$ a pair of integer values (x_i, y_i) where:

x_i counts, with their multiplicity, the number of the outgoing arcs from the node i of MG;

y_i counts the number of the outgoing nodes of the node i of MG.

We then define the following conditions for any pair of alternatives $a_i, a_j \in \hat{\mathscr{A}}$:

- $a_i \sqsupset a_j$ if and only if we have $y_i > y_j$ or $y_i = y_j$ and $x_i > x_j$,
- $a_i \sim a_j$ if and only if we have $y_i = y_j$ and $x_i = x_j$,
- $a_j \sqsupset a_i$ in all the remaining cases.

In such relations with $\sqsupset$ we denote a classical strict preference relation and with $\sim$ a classical indifference relation.
Once the lexicographic ranking has been performed we can select the best alternative possibly through a random selection in case we get tied best alternatives.

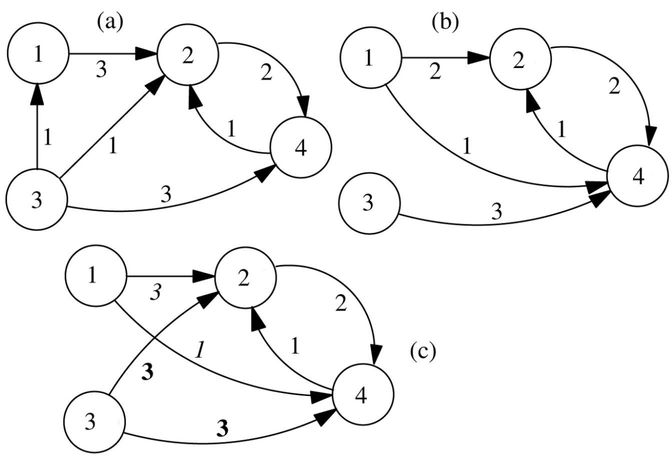

Figure 15: *Examples of MG with $\hat{\mathscr{A}} \neq \emptyset$*

In Figure 15 we give three examples of a final selection under the condition $\hat{\mathscr{A}} \neq \emptyset$.

In the case (a) we have $\hat{\mathscr{A}} = \{3\}$ and $\check{\mathscr{A}} = \emptyset$ so that the deciders elect a_3 as the best alternative but are not able to single out any worst alternative.
In the case (b) we have $\hat{\mathscr{A}} = \{1, 3\}$ and $\check{\mathscr{A}} = \emptyset$. From the definition of our lexicographic ranking we obtain $a_1 \sqsupset a_3$ (since $2 > 1$) so that the deciders elect a_1 as the best alternative. Also in this case they are not able to single out any worst alternative.
In the case (c) we have $\hat{\mathscr{A}} = \{1, 3\}$ and $\check{\mathscr{A}} = \emptyset$. From the definition of our lexicographic ranking we obtain $a_3 \sqsupset a_1$ (since $6 > 4$) so that the deciders elect a_3 as the best alternative. Also in this case they are not able to single out any worst alternative.

10 Conclusions and future plans

In this technical report we have presented a multideciders multicriteria method that can be used by d deciders to select the best alternative from a set of a alternatives according to c criteria.
The method is simple and is composed of an individual phase and a collective phase that aim at producing a single oriented multigraph that the deciders can use to perform the foregoing selection.
In this technical report we have shown the method at work together with some of its strengths and weaknesses.
Future plans include a deeper analysis of the properties of the proposed method together with their formalization. We also aim at characterizing the set of the criteria and at applying the method in more complex and more realistic cases.
Another stream of research that may be worth pursuing is a deeper analysis of the failure cases we have presented in section 8.

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