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Abstract

Equilibrium problems provide a mathematical framework which includes optimization, variational inequalities, fixed-point and saddle point problems, and noncooperative games as particular cases. This general format received an increasing interest in the last decade mainly because many theoretical and algorithmic results developed for one of these models can be often extended to the others through the unifying language provided by this common format. This survey paper aims at covering the main results concerning the existence of equilibria and the solution methods for finding them.

Keywords: Equilibrium problem, monotonicity, coercivity, auxiliary principle, regularization.

1. Introduction

In scientific contexts the term "equilibrium" has been widely used at least in physics, chemistry, engineering and economics within different frameworks, relying on different mathematical models. For instance, it may refer to physical or mechanical structures, chemical processes, the distribution of traffic over computer and telecommunication networks or over public roads (see, for instance, [17, 28, 35, 80, 90, 98, 99]). In economics it often refers to production competition [27] or the dynamics of offer and demand [10], exploiting the mathematical model of noncooperative games and the corresponding equilibrium concept by Nash [84, 85].

This survey paper deals with those equilibrium problems which are relevant in operations research and mathematical programming. Many problems involving equilibria can be modeled in this framework through different mathematical models such as optimization, variational inequalities and noncooperative games among others. In turn, these mathematical models share an underlying common structure which allows to conveniently formulate them in a unique format. Therefore, theoretical developments and algorithms developed for one of these models can be generally modified to cope with the others through the common format in a unifying language. This format reads

$$\text{find } x^* \in C \text{ such that } f(x^*, y) \geq 0 \text{ for all } y \in C, \qquad \text{(EP)}$$

or equivalently

$$\text{find } x^* \in \operatorname{argmin}\{\ f(x^*, y)\ :\ y \in C\ \},$$

where $C \subseteq \mathbb{R}^n$ is a nonempty closed set and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is an equilibrium bifunction, i.e. $f(x, x) = 0$ for all $x \in C$.

This general problem was named "equilibrium problem" by Blum and Oettli [21], who stressed this unifying feature and provided a thorough investigation of its theoretical properties. Until then, this format did not actually receive much attention: Nikaido and Isoda characterized Nash equilibria as the solutions of (EP) for an appropriate auxiliary bifunction [88] but they did not consider the problem itself in an independent fashion; Gwinner introduced it just as a tool to develop a unified treatment of penalization techniques for optimization and variational inequalities [40]; Antipin formulated an inverse optimization problem as a noncooperative game and therefore in the (EP) format via the Nikaido-Isoda bifunction [6] and provided a solution method for the general problem in [7, 8].

Indeed, equilibrium problems in the above format started to gain real interest only after the publication of the seminal paper of Blum and Oettli. Actually, the possibility to exploit results and algorithms developed for one class of problems in another framework was not a novelty at all: this kind of bridge already finds roots in the analytical development of variational inequalities through the connection with optimization via complementarity problems. Anyway, a large number of applications has been described successfully via the concept of equilibrium solution and therefore many researchers devoted their efforts to study (EP). In fact, nowadays there is a good theory for equilibria and a rapidly increasing number of algorithms for finding them.

In this paper we aim at reviewing two core issues up to the state of the art: the existence of equilibria and the solution methods. In order to make the paper as readable as possible, instead of presenting all the technical details of the results, we propose a structured overview with different levels. In particular, the existence results are divided into groups according to

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the required assumptions, while the solution methods are classified depending on the kind of problems which are solved at each iteration. We hope that this paper may serve as a basis for future research and stimulate further interest in equilibrium problems.

1.1. Particular cases of (EP)

In this subsection we briefly show how some of the main mathematical models for equilibria can be formulated in the general format (EP), and we recall just a few of their recent applications.

Optimization problems: finding a global minimum of a function $\psi : \mathbb{R}^n \to \mathbb{R}$ over a closed set $C \subseteq \mathbb{R}^n$ amounts to solving (EP) with

$$f(x,y) = \psi(y) - \psi(x).$$

Pareto optimization problems: given m real-valued functions $\psi_i : \mathbb{R}^n \to \mathbb{R}$, a weak Pareto global minimum of the vector function $\psi = (\psi_1, \ldots, \psi_m)$ over a closed set $C \subseteq \mathbb{R}^n$ is any $x^* \in C$ such that for any $y \in C$ there exists an index i such that $\psi_i(y) - \psi_i(x^*) \geq 0$. Finding a weak Pareto global minimum amounts to solving (EP) with

$$f(x,y) = \max_{i=1,\ldots,m} [\psi_i(y) - \psi_i(x)].$$

Saddle point problems: given two closed sets $C_1 \subseteq \mathbb{R}^{n_1}$ and $C_2 \subseteq \mathbb{R}^{n_2}$, a saddle point of a function $L : C_1 \times C_2 \to \mathbb{R}$ is any $x^* = (x_1^*, x_2^*) \in C_1 \times C_2$ such that

$$L(x_1^*, y_2) \leq L(x_1^*, x_2^*) \leq L(y_1, x_2^*)$$

holds for any $y = (y_1, y_2) \in C_1 \times C_2$. Finding a saddle point of L amounts to solving (EP) with $C = C_1 \times C_2$ and

$$f((x_1, x_2), (y_1, y_2)) = L(y_1, x_2) - L(x_1, y_2).$$

Complementarity problems and systems of equations: given a closed convex cone $C \subseteq \mathbb{R}^n$ and a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, the complementarity problem asks to determine a point $x^* \in C$ such that $\langle F(x^*), v\rangle \geq 0$ for any $v \in C$, i.e., $F(x^*) \in C^*$ where C^* denotes the dual cone of C. The system of equations $F(x) = 0$ is a special complementarity problem with $C = \mathbb{R}^n$. Solving the complementarity problem amounts to solving (EP) with

$$f(x,y) = \langle F(x), y - x\rangle.$$

Variational inequality problems: given a closed set $C \subseteq \mathbb{R}^n$ and a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$, the Stampacchia variational inequality problem asks to determine a point $x^* \in C$ such that

$$\langle F(x^*), y - x^*\rangle \geq 0, \quad \forall\, y \in C. \tag{1}$$

Solving this problem amounts to solving (EP) with

$$f(x,y) = \langle F(x), y - x\rangle.$$

If $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ is a set-valued mapping with compact values, then finding $x^* \in C$ and $u^* \in F(x^*)$ such that

$$\langle u^*, y - x^*\rangle \geq 0, \quad \forall\, y \in C$$

amounts to solving (EP) with

$$f(x,y) = \max_{u \in F(x)} \langle u, y - x\rangle.$$

Fixed point problems: given a closed set $C \subseteq \mathbb{R}^n$, a fixed point of a mapping $F : C \to C$ is any $x^* \in C$ such that $x^* = F(x^*)$. Finding a fixed point amounts to solving (EP) with

$$f(x,y) = \langle x - F(x), y - x\rangle.$$

If $F : C \rightrightarrows C$ is a set-valued mapping with compact values, then finding $x^* \in C$ such that $x^* \in F(x^*)$ amounts to solving (EP) with

$$f(x,y) = \max_{u \in F(x)} \langle x - u, y - x\rangle.$$

Nash equilibrium problems: in a noncooperative game with p players, each player i has a set of possible strategies $K_i \subseteq \mathbb{R}^{n_i}$ and aims at minimizing a loss function $f_i : K \to \mathbb{R}$ with $K = K_1 \times \cdots \times K_p$. A Nash equilibrium is any $x^* \in K$ such that no player can reduce its loss by unilateraly changing its strategy, in formulas any $x^* \in K$ such that

$$f_i(x^*) \leq f_i(x^*(y_i))$$

holds for any $y_i \in K_i$ and any $i = 1, \ldots, p$, with $x^*(y_i)$ denoting the vector obtained from x^* by replacing x_i^* with y_i. Finding a Nash equilibrium amounts to solving (EP) with the so-called Nikaido-Isoda bifunction [88], i.e.,

$$f(x,y) = \sum_{i=1}^{p} \left[f_i(x(y_i)) - f_i(x) \right]. \tag{2}$$

On the contrary, the problem of finding a Nash equilibrium in the case of jointly convex strategies (see [31]) cannot be formulated in the (EP) format. Anyway, the solution set of (EP) with the corresponding Nikaido-Isoda bifunction coincides with the subclass of the so-called normalized Nash equilibria.

Inverse optimization problems: given a closed set $C \subseteq \mathbb{R}^n$, m functions $f_i : \mathbb{R}^n \to \mathbb{R}$ and p functions $g_j : \mathbb{R}^n \to \mathbb{R}$, this problem asks to determine a parameter $\lambda^* \in \mathbb{R}^m_+$ such that at least one optimal solution x^* of the minimization problem

$$\min \Big\{ \sum_{i=1}^{m} \lambda_i^* f_i(x) \ :\ x \in C \Big\}$$

satisfies the constraints $g_j(x^*) \leq 0$ for all $j = 1, \ldots, p$. Actually, it is equivalent to the Nash equilibrium problem with three players in which the first player controls the x variables and aims at solving

$$\min \Big\{ \sum_{i=1}^{m} \lambda_i f_i(x) \ :\ x \in C \Big\},$$

the second player controls the auxiliary variables y and aims at solving

$$\max\Big\{ \sum_{j=1}^{p} g_j(x)y_j \ : \ y \geq 0 \Big\}$$

while the third player simply chooses a vector $\lambda \in \mathbb{R}^m_+$ or equivalently minimizes a constant objective function over $\mathbb{R}^m_+$. Therefore, also this inverse optimization problem can be formulated in the (EP) format via the Nikaido-Isoda bifunction.

As showed above, many problems have been proposed and studied which belong to the class of equilibrium problems. Due to the huge number of applications, it is not possible to cite all the corresponding references. We provide some references to books or surveys on such topics (see [2, 12, 31, 32, 54] and references therein) together with some recent papers about economic problems [29, 42, 55, 56, 57, 68, 72, 74, 81, 82, 83], environmental problems [30, 38, 66], and problems arising from Information and Communication Technologies [2, 3, 9, 48, 89, 97, 105].

1.2. Organization of this paper

The paper is divided into two parts: Section 2 is devoted to existence results, while Section 3 is devoted to solution methods.

Section 2 first recalls the well-known Knaster-Kuratowski-Mazurkiewicz Theorem, which is a basic tool to prove the existence of solutions of an equilibrium problem. Next, the most significant known existence results are divided into three groups based on the assumptions they require. In Section 2.1 we state a basic existence theorem and analyze the topological and algebraic ingredients which are needed to prove it. Then, we describe how the basic existence theorem can be extended by weakening some of its assumptions. Section 2.2 deals with the existence results based on generalized monotonicity assumptions on the bifunction f and their connections with the above mentioned basic theorem. Section 2.3 describes some results which do not need convexity assumptions on the set C or on the bifunction f.

Section 3 describes the most significant known algorithms for finding equilibria and analyzes the assumptions which allow to obtain convergence results. The section is divided into two subsections. Subsection 3.1 is devoted to methods based on successive convex optimization. The section includes methods based on a fixed-point reformulation of the equilibrium problem, the so-called extragradient methods, and descent methods based on gap or D-gap functions. Subsection 3.2 concerns regularization methods based on successive approximations of (EP). The section covers the proximal point and the so-called Tikhonov-Browder methods.

2. Existence results

Since the solution set S of an equilibrium problem can be given as the intersection of a family of values of set–valued maps, the so–called Three Polish Theorem [49] by Knaster, Kuratowski and Mazurkiewicz provides a powerful tool to achieve existence results for the solutions of equilibrium problems.

KKM-Theorem. *Let C be a subset of $\mathbb{R}^n$ and $T : C \rightrightarrows \mathbb{R}^n$ be a set–valued map satisfying the following conditions:*

(KKM1) T is a KKM–map, that is any point of the convex hull of any finite set $\{x_1, x_2, \dots, x_p\} \subseteq C$ belongs at least to $T(x_i)$ for some i, i.e.,

$$\mathrm{conv}\,\{x_1, x_2, \dots, x_p\} \subseteq \bigcup_{i=1}^{p} T(x_i),$$

(KKM2) $T(x)$ is closed for each $x \in C$,

(KKM3) there exists $x \in C$ such that $T(x)$ is compact,

then

$$\bigcap_{x \in C} T(x) \neq \emptyset.$$

This result was originally stated in $\mathbb{R}^n$ and later it was extended to the case of an infinite–dimensional topological vector space by Ky Fan [33].

2.1. The classical results

The KKM-Theorem is fundamental for ensuring a sufficient condition for the existence of solutions of (EP). Indeed, let us consider the set–valued map

$$T(y) = \{x \in C \ : \ f(x, y) \geq 0\}.$$

Since S is clearly the intersection of the above sets, i.e.,

$$S = \bigcap_{y \in C} T(y),$$

we exam which assumptions on f and C are enough to satisfy the assumptions of the KKM-Theorem. If C is bounded and $f(\cdot, y)$ continuous for every $y \in C$, then $T(y)$ is compact for every fixed $y \in C$ and therefore Assumptions KKM2 and KKM3 hold. Moreover, the convexity of C and of the functions $f(x, \cdot)$ for every fixed $x \in C$ implies that T is a KKM-map (Assumption KKM1). Therefore, we get following basic existence theorem for (EP).

Basic existence theorem. *Suppose that*

(A1) C is convex,

(T1) C is bounded,

(A2) $f(x, \cdot)$ is convex for each $x \in C$,

(T2) $f(\cdot, y)$ is continuous for each $y \in C$,

then S is nonempty.

As underlined above, the main ingredients of the previous result are four, two of a topological character (the boundedness of

C and continuity of the functions $f(\cdot,y)$) and two of an algebraic character (the convexity of C and of the functions $f(x,\cdot)$).

Starting from this result, we show how it is possible to weaken these assumptions. To our knowledge, the history of existence theorems for equilibrium problems can be traced back to 1972 when Ky Fan [34] proposed a famous minimax result in a real Hausdorff topological vector space, which implies a stronger result than the above basic existence theorem. His result is based on two simple statements of fact. Since the upper semicontinuity of a function is equivalent to the closedness of each its superlevel set, then Assumption T2 can be weakened to the upper semicontinuity of $f(\cdot,y)$. Instead, Assumption A2 of convexity of $f(x,\cdot)$ can be weakened to the quasiconvexity of the functions $f(x,\cdot)$ which is equivalent to the convexity of their sublevel sets, and it implies that T is a KKM–map.

The next step is to replace Assumption T1 with weaker conditions, which clearly have to involve some suitable form of coercivity on f. The first result was presented in [22]. Under the same assumptions of the Ky Fan result but Assumption T1, the authors replaced the boundedness of C with the following coercivity condition

$$\begin{array}{l}\exists\, r>0,\ \exists\, y\in C \ \text{ with } \ \|y\|\le r \ \text{ s.t.}\\ f(x,y)<0,\ \forall\, x\in C \ \text{ with } \ \|x\|>r,\end{array} \tag{3}$$

where $\|\cdot\|$ denotes the Euclidean norm, proving that S is nonempty and bounded. Actually, the result was given in a real Hausdorff topological vector space, which required to state the coercivity condition in a more complex form.

2.2. Results under generalized monotonicity

The existence of a point $y\in C$ such that $f(x,y)\to-\infty$ as $\|x\|\to\infty$ with $x\in C$ guarantees the coercivity assumption (3). In particular, this condition holds if $f(x,\cdot)$ is convex and f is *strongly monotone*, i.e., there exists $\gamma>0$ such that

$$f(x,y)+f(y,x)\le-\gamma\|x-y\|^2,\quad \forall x,y\in\mathbb{R}^n. \tag{4}$$

Indeed, fixed any $y\in C$, there exists $\alpha\in\mathbb{R}$ such that

$$f(y,x)\ge\alpha\|y-x\|,\quad \forall x\in\mathbb{R}^n,$$

since $f(y,\cdot)$ is convex and $f(y,y)=0$. Hence, (4) implies

$$f(x,y)\le-\gamma\|x-y\|^2-\alpha\|x-y\|\to-\infty$$

as $\|x\|\to\infty$ with $x\in C$. Moreover, the strong monotonicity of f ensures that the equilibrium problem has exactly one solution within this framework. Unfortunately, strong monotonicity is a sharp assumption. For instance, if f describes an optimization or a saddle point problem, it is never satisfied. Anyway, it paves the way towards a different approach for obtaining weaker coercivity conditions.

One of the most common approaches to weaken the coercivity condition (3) is based on strengthening the assumption of quasiconvexity of $f(x,\cdot)$ while introducing suitable conditions on f weaker than strong monotonicity.

The basic concept of monotonicity for bifunctions is an adaptation of the well–known definition of monotonicity for variational inequalities. A bifunction f is said to be *monotone* on C if

$$f(x,y)+f(y,x)\le 0,\quad \forall\, x,y\in C.$$

This condition is satisfied by a large number of equilibrium problems, for instance optimization and saddle point problems. If f describes the variational inequality (1), then it is monotone if and only if the operator F is monotone, i.e.,

$$\langle F(x)-F(y),x-y\rangle\ge 0,\quad \forall\, x,y\in C.$$

The corresponding concepts of strict monotonicity are defined analogously just requiring strict inequalities to hold.

The crucial point for the analysis of coercivity conditions weaker than (3) is based on the relationships between S and the solution set of the *Minty equilibrium problem*, which reads

$$\text{find } y^*\in C \text{ such that } f(x,y^*)\le 0 \text{ for all } x\in C. \tag{MEP}$$

The Minty equilibrium problem was initially introduced for variational inequality [73] and its relevance to applications was pointed out in [39]. A well-known result, formulated by Minty in [73], states the equivalence of the Minty and Stampacchia variational inequalities under continuity and monotonicity assumptions of the involved operator.

An analogous relationship holds between the solution set S of (EP) and the solution set M of (MEP). Indeed, the inclusion $S\subseteq M$ is an immediate consequence of the monotonicity of f: if $x^*\in S$, then it belongs also to M since $f(y,x^*)\le-f(x^*,y)$ for every $y\in C$. Actually, the inclusion holds whenever $f(x^*,y)\ge 0$ implies $f(y,x^*)\le 0$ for every $y\in C$. This leads to a weaker concept of monotonicity introduced in [16]. A bifunction f is said to be *pseudomonotone* on C if the implication

$$f(x,y)\ge 0 \quad\Longrightarrow\quad f(y,x)\le 0$$

holds for every $x,y\in C$. Clearly, every monotone bifunction is also pseudomonotone and pseudomonotonicity is sufficient to guarantee the inclusion.

The converse inclusion $M\subseteq S$ holds if $f(\cdot,y)$ is upper semicontinuous for any $y\in C$ and $f(x,\cdot)$ is *explicitly quasiconvex* for any $x\in C$, i.e., it is quasiconvex and the inequality

$$f(x,ty_1+(1-t)y_2)<\max\{f(x,y_1),f(x,y_2)\}.$$

holds for any $y_1,y_2\in C$ with $f(x,y_1)\ne f(x,y_2)$ and any $t\in(0,1)$. In the recent years, the assumption of upper semicontinuity has been deeply weakened in [15] introducing the concept of *upper sign continuity*. This concept is an adaption of a similar one introduced in [41] for set–valued mappings. Actually, in this case the inclusion holds just between the two sets of local solutions.

In short, pseudomonotonicity, explicit quasiconvexity and upper semicontinuity imply $S=M$.

The equivalence between (EP) and (MEP) is the key tool for estabilishing weaker coercivity conditions and the assumption of pseudomonotonicity is often required for this reason.

Roughly speaking, the skeleton of the proofs of existence is usually the following: first the nonemptiness of M is established applying KKM-Theorem to the set–valued map

$$T(x) = \{y \in C \ : \ f(x, y) \leq 0\}.$$

Pseudomonotonicity and quasiconvexity are fundamental to show that T is a KKM–map. Afterwards the inclusion $M \subseteq S$ implies the nonemptiness of S. Moreover, the quasiconvexity of $f(x, \cdot)$ implies that $T(x)$ is a convex set, and therefore the reformulation of (MEP) as the intersection of a family of convex sets, i.e.,

$$M = \bigcap_{x \in C} T(x),$$

allows to reduce the Minty equilibrium problem to a so–called convex feasibility problem [11].

One of the first and most used coercivity conditions, which has been introduced for pseudomonotone equilibrium problems, is the following:

$$\begin{array}{c} \exists\, r > 0 \text{ s.t. } \forall\, x \in C \text{ with } \|x\| > r, \\ \exists\, y \in C \text{ with } \|y\| \leq r \text{ s.t. } f(x, y) < 0. \end{array} \tag{5}$$

To our knowledge, it was originally introduced in [47] for complementarity problems and later adapted to (EP) [15]. This condition is weaker than the coercivity condition (3). Following the scheme previously described, it is possible to prove that if f is pseudomonotone and C is convex, condition (5) together with

(a) the upper semicontinuity of $f(\cdot, y)$ (or also the upper sign continuity),

(b) the lower semicontinuity of $f(x, \cdot)$,

(c) the explicit quasiconvexity of $f(x, \cdot)$

imply the nonemptiness and boundedness of S. We observe that if $f(x, \cdot)$ is convex, conditions (b) and (c) are automatically satisfied. Moreover it is possible to show that the coercivity condition (5) is, in a certain sense, equivalent to the nonemptiness and the boundedness of S. More precisely, if (a) and (c) hold and S is nonempty and bounded, then (5) holds [15].

It turns out that the coercivity condition (5) is quite strong, since it entails also the boundedness of the solution set S. For this reason in [15] the authors adapted to equilibrium problems the following coercivity condition which was introduced for variational inequalities in [13]:

$$\begin{array}{c} \exists\, r > 0 \text{ s.t. } \forall\, x \in C \text{ with } \|x\| > r, \\ \exists y \in C \text{ with } \|y\| < \|x\| \text{ s.t. } f(x, y) \leq 0. \end{array} \tag{6}$$

The coercivity condition (6) is weaker than (5) and it is enough to achieve the nonemptiness of S under the same assumptions (a), (b) and (c). Besides, condition (6) was compared with other coercivity conditions introduced in literature (see [37, 104]) and it was proved that it is really the weakest [15]. Very recently [60] a coercivity condition weaker than (6) has been proposed for existence of solutions of (EP). This condition works without any monotonicity assumption on f but it requires the convexity of $f(x, \cdot)$ instead of just the explicit quasiconvexity.

An analogous result of equivalence between the nonemptiness of S and a coercivity condition was proved in [43] for bifunctions such that $f(x, \cdot)$ is pseudoconvex, which is a stronger property than the quasiconvexity but it is not comparable with the explicit quasiconvexity. Nevertheless it is possible to prove [43] that if $f(\cdot, y)$ is upper semicontinuous, $f(x, \cdot)$ is pseudoconvex, and f is pseudomonotone then S is nonempty if and only if the following condition holds

$$\forall \{x_k\} \subseteq C \text{ with } \|x_k\| \to \infty, \exists y \in C \ : \ f(x_k, y) \leq 0, \text{ definitively.}$$

In the last years quasimonotone equilibrium problems have been considered [15]. We recall that the bifunction f is said to be *quasimonotone* on C if the implication

$$f(x, y) > 0 \quad \Longrightarrow \quad f(y, x) \leq 0.$$

holds for every $x, y \in C$. Cleary, every pseudomonotone bifunction is also quasimonotone. Unfortunately, quasimonotonicity is not enough to ensure the equivalence between (EP) and (MEP). Nevertheless the quasimonotonicity of f paired with the coercivity condition (6) ensures the nonemptiness of S provided that upper semicontinuity and quasiconvexity assumptions are strengthened [15].

2.3. *Results without convexity*

In order to avoid any assumption of convexity both for the constraint set C and for the bifunction f, some authors (see for instance [1, 14, 67] and references therein) proposed a different approach in which the existence of solutions for (EP) is obtained assuming that the bifunction f satisfies the following triangular inequality

$$f(x, y) \leq f(x, z) + f(z, y), \qquad \forall x, y, z \in C. \tag{7}$$

The first results are a consequence of a generalization of the famous Ekeland's variational principle to the equilibrium problem. For instance, if f satisfies (7) and C is bounded, the existence of a solution for (EP) has been proved in [14] along with the following further topological assumptions on f:

- lower semicontinuity and lower boundedness of $f(x, \cdot)$ for any $x \in C$,
- upper semicontinuity of $f(\cdot, y)$ for any $y \in C$.

Moreover the same result holds if the boundedness of C is replaced by the coercivity condition (6).

Following a different approach, the same results are achieved in [24], for instance removing the assumptions of lower boundedness and upper semicontinuity.

3. Solution methods

Throughout all the section we suppose that $C \subseteq \mathbb{R}^n$ is a nonempty closed convex set and the equilibrium bifunction f is continuously differentiable and $f(x, \cdot)$ is convex for all $x \in C$. In this way we can describe solution methods for (EP) in a unified common framework. Actually, some methods simply require that f is continuous (or even satisfies weaker continuity assumptions) and exploit the subgradients of $f(x,\cdot)$ instead of the gradient $\nabla_y f$ of f with respect to the second argument y.

It is worth remarking that this framework includes all the assumptions of the basic existence theorem of Section 2.1 except for the boundedness of C. Indeed, all the algorithims require it or alternatively additional assumptions implying one of the coercivity conditions of the previous section, so that the existence of a solution is always guaranteed.

In this framework (EP) is equivalent to the variational inequality

$$\begin{array}{c}\text{find } x^* \in C \text{ such that}\\ \langle \nabla_y f(x^*, x^*), y - x^* \rangle \geq 0 \text{ for all } y \in C,\end{array} \tag{8}$$

in the sense that the soluton sets of the two problems actually coincide [23]. Therefore, any method for variational inequalities could be applied to solve (EP) through (8). For instance, projection methods for (8) have been developed in [95, 96]. Clearly, all the assumptions required by algorithms for the operator of the variational inequality must be satisfied by the gradient mapping $G(x) = \nabla_y f(x, x)$.

Whenever f is pseudomonotone, (EP) reduces through the equivalent Minty equilibrium problem to finding a point in the intersection of a family of convex sets. Therefore, any method for the so-called convex feasibility problem (see [11]) could be applied to solve pseudomonotone equilibrium problems. For instance, methods based on (gradient) projections [45] and analytic center cutting plane techniques [94] have been developed.

All the other solution methods we review share a common feature: the bifunction f is modified by adding some (parametric) term, generally in order to improve the properties of the bifunction which is actually managed by the algorithms. We classify the solution methods into two large families according to the nature of the additional term: the first family is based on optimization techniques as each iteration of the algorithms requires the solution of at least one optimization problem, while the second is based on successive approximations of (EP) with equilibrium problems with better properties.

3.1. Methods based on successive optimization

One possible approach to solve (EP) through the solution of a sequence of optimization problems is due to its reformulation as a fixed point problem. Indeed, since $f(x, x) = 0$, x^* solves (EP) if and only if it is a fixed point of the multivalued map

$$\Phi(x) = \operatorname{argmin}\{\, f(x,y) \ : \ y \in C \,\},$$

i.e., $x^* \in \Phi(x^*)$. Therefore, the fixed point iterative scheme $x^{k+1} \in \Phi(x^k)$ could be exploited, solving one convex optimization problem per iteration. Actually, there are a few reasons why this approach is not effective: unless C is bounded, Φ might not be defined everywhere; even where it is defined, it is generally multivalued and thus rules to choose among different solutions would be needed; finally, even if it is single-valued, it is not necessarily continuous.

If Φ is defined on C, then the value function

$$\varphi(x) = -f(x, \Phi(x)) = -\min\{\, f(x,y) \ : \ y \in C \,\}$$

allows to reformulate (EP) as a constrained optimization problem. In fact, φ is a *gap function*, i.e., it is non negative on C and $x^* \in C$ solves (EP) if and only if $\varphi(x^*) = 0$ [71]. Therefore, (EP) is equivalent to finding the global minima of φ over C. Anyway, φ obviously inherits all the troubles of Φ.

The auxiliary principle provides the adequate technique to overcome all the above drawbacks. Given any $\alpha > 0$, consider the equilibrium problem

$$\begin{array}{c}\text{find } x^* \in C \text{ such that}\\ f(x^*, y) + \alpha\|y - x^*\|^2/2 \geq 0 \text{ for all } y \in C.\end{array} \tag{EP_α}$$

Indeed, it is equivalent to the original equilibrium problem as the solution sets of (EP_α) and (EP) coincide [70]. Therefore, algorithms for (EP) can exploit all the properties that the additional term $\|y - x\|^2$ provides to the bifunction of (EP_α). In fact, the corresponding argmin function

$$\Phi_\alpha(x) = \operatorname{argmin}\{\, f(x,y) + \alpha\|y - x\|^2/2 \ : \ y \in C \,\}$$

is defined everywhere and single-valued for any $x \in \mathbb{R}^n$ since the objective function of the inner optimization problem is strongly convex. Moreover, the corresponding value function

$$\varphi_\alpha(x) = -f(x, \Phi_\alpha(x)) = -\min\{\, f(x,y) + \alpha\|y - x\|^2/2 \ : \ y \in C \,\}$$

is a continuously differentiable gap function for (EP_α) and thus also for (EP).

Most algorithms actually solve directly (EP_α) for some fixed $\alpha > 0$ or at most consider a sequence of parameters α_k bounded above and away from 0, while just a few solve (EP) exploiting a whole range of parameters for $\alpha_k \downarrow 0$ or $\alpha_k \uparrow +\infty$.

Finally, it is worth remarking that the quadratic regularization term $\|x - y\|^2$ can be replaced by any Bregman distance, i.e., any

$$D(x,y) = g(y) - g(x) - \langle \nabla g(x), y - x \rangle$$

where $g : \mathbb{R}^n \to \mathbb{R}^n$ is a differentiable strongly convex function (other than the squared norm), preserving the good features of the auxiliary problem which have been descrived above.

3.1.1. Fixed point methods

The fixed point approach can be effectively exploited to solve (EP), applying the iterative scheme to (EP_α) for some fixed $\alpha > 0$. In fact, the fixed point iteration $x^{k+1} = \Phi_\alpha(x^k)$ is well defined and it amounts to find the unique optimal solution x^{k+1} of the convex optimization problem

$$\min\{\, f(x^k, y) + \alpha\|y - x^k\|^2/2 \ : \ y \in C \,\}. \tag{9}$$

The whole sequence $\{x^k\}$ converges to the unique solution of (EP) provided that f is strongly monotone with constant γ and there exists $c_1, c_2 > 0$ such that the triangular inequality

$$f(x,y) + f(y,z) \geq f(x,z) - c_1\|x - y\|^2 - c_2\|y - z\|^2 \tag{10}$$

holds for all $x, y, z \in C$, while the above parameters satisfy the inequalities $\gamma > c_1$ and $2c_2 \leq \alpha$ [70]. The rate of convergence of the algorithm is linear [79], in fact

$$\|x^{k+1} - x^*\| \leq \sqrt{K}\, \|x^k - x^*\|$$

holds with $K = [1-2(\gamma-c_1)/\alpha]$, where x^* is the unique solution of (EP). The rate of contraction $\sqrt{K}$ can be improved under a triangular condition stronger than (10), solving two strongly convex problems at each iteration instead of a single one [93]: the objective function of the first depends upon the points obtained in all the previous iterations while the second is the usual fixed point iteration (9).

It is worth noting that the uniqueness of the solution follows from the strong monotonicity assumption, which is rather restrictive. Actually, convergence can be achieved also if f is pseudomonotone whenever (10) holds with $c_2 = 0$, $f(x, \cdot)$ is Lipschitz continuous on C uniformly in x, i.e., there exists $L > 0$ such that

$$|f(x, y) - f(x, z)| \leq L\|y - z\|, \quad \forall\ x, y, z \in C,$$

and α is replaced by a sequence $\{\alpha_k\}$ such that the series whose terms are $1/\alpha_k^2$ is convergent [86]. Actually, both the above sets of conditions on f are particular cases of a more general set of conditions ensuring convergence [86], which moreover are satisfied if f is the variational inequality (1) and the operator F is co-coercive on C, i.e., there exists $\gamma > 0$ such that

$$\langle F(y) - F(x), y - x\rangle \geq \gamma\|F(x) - F(y)\|^2, \quad \forall\ x, y \in C.$$

Furthermore, this fixed point approach can exploit also bundling techniques by replacing the convex function $f(x^k, \cdot)$ in (9) with suitable linear approximations in such a way that the fixed point iteration asks for the solution of an easier optimization problem [86].

Another family of methods for solving (EP) can be developed applying the fixed point approach to the variational inequality (8). Actually, in order to exploit the basic iterative scheme rather restrictive assumptions such as monotonicity and Lipschitz properties of the gradient mapping G are needed. Anyway, they can be dropped and convergence can be achieved under the (pseudo)monotonicity of f if the fixed point iteration

$$y^k = \text{argmin}\{\ \langle \nabla_y f(x^k, x^k), y - x^k\rangle + \alpha\|y - x^k\|^2/2\ :\ y \in C\ \},$$

or equivalently $y^k = \pi_C(x^k - \alpha^{-1}\nabla_y f(x^k, x^k))$, i.e., y^k is the projection of $x^k - \alpha^{-1}\nabla_y f(x^k, x^k)$ onto C, is combined with a suitable line search along the direction $y^k - x^k$ and appropriate further projections [50, 65, 100]. Chosen the parameters $\theta, \beta, \tau \in (0, 1)$, the line search identifies a point $z^k = x^k + \theta\beta^m(y^k - x^k)$ where m is the smallest non-negative integer such that

$$f(z^k, x^k) \geq \tau\theta\beta^m\langle \nabla_y f(x^k, x^k), x^k - y^k\rangle. \tag{11}$$

In turn, z^k provides the hyperplane

$$H^k = \{\ x \in \mathbb{R}^n\ :\ \langle \nabla_y f(z^k, x^k), x^k - x\rangle = f(z^k, x^k)\ \}$$

which strictly separates x^k from the solution set of (EP). Therefore, the next iterate is obtained projecting onto C the projection of x^k onto H^k, i.e.,

$$x^{k+1} = \pi_C(x^k - \eta f(z^k, x^k)\nabla_y f(z^k, x^k)/\|\nabla_y f(z^k, x^k)\|^2) \tag{12}$$

with $\eta = 1$. Actually, the method converges also if the projection onto H^k is somehow relaxed taking any $\eta \in (0, 2)$. Moreover, x^{k+1} can be alternatively chosen as a convex combination of the projection in (12) and x^k. It is worth noting that if the functions $f(\cdot, y)$ were concave, and therefore the gap function φ were convex, and $z^k \in \Phi(x^k)$, then (12) would be a step of a relaxation subgradient method for the minimization of φ.

3.1.2. Extragradient methods

In order to weaken strong monotonicity to pseudomonotonicity, another approach relies on a double step procedure: the fixed point iteration

$$\bar{x}^k = \text{argmin}\{\ f(x^k, y) + \alpha\|y - x^k\|^2/2\ :\ y \in C\ \}$$

is taken as a predictive intermediate step followed by the correction step

$$x^{k+1} = \text{argmin}\{\ f(\bar{x}^k, y) + \alpha\|y - x^k\|^2/2\ :\ y \in C\ \}.$$

The whole sequence $\{x^k\}$ converges to a solution of (EP) provided that C is bounded, f is pseudomonotone and there exists $\Lambda > 0$ with $\alpha \geq \Lambda$ such that

$$|f(v, w) - f(x, w) - f(v, y) + f(x, y)| \leq 2\Lambda\, \|v - x\|\, \|w - y\| \tag{13}$$

holds for all $v, w, x, y \in C$ [36]. Notice that (13) means that the functions $f(v, \cdot) - f(x, \cdot)$ are Lipschitz with constant $2\Lambda\|v - x\|$ and the functions $f(\cdot, w) - f(\cdot, y)$ are Lipschitz with constant $2\Lambda\|w-y\|$. Since C is bounded, these conditions surely hold if f is twice continuously differentiable. The boundedness assumption on C can be removed provided that f satisfies the triangular condition (10), instead of (13), and $\alpha \leq \min\{1/2c_1, 1/2c_2\}$ [92]. Furthermore, when C is a polyhedron, the regularization term $\|x - y\|^2$ can be replaced by an interior-quadratic term to perform unconstrained minimization [87]: this term is actually composed of two parts, one plays the role of a barrier function to keep the iterates x^k in the interior of C while the other is a quadratic convex regularization function which exploits the linear description of C. In case $C = \mathbb{R}^n_+$ the usual regularization term can be used along with the barrier part [4].

In order to drop the triangular condition (10), the correction step can be replaced by a suitable line search along the direction $\bar{x}^k - x^k$ and a double projection [87, 92]. Chosen $\beta \in (0, 1)$ and $\eta \in (0, 2)$, the line search identifies a point $z^k = x^k + \beta^m(\bar{x}^k - x^k)$ where m is the smallest non-negative integer such that

$$f(z^k, x^k) - f(z^k, \bar{x}^k) \geq \eta\alpha\|x^k - \bar{x}^k\|^2/2, \tag{14}$$

and the next iterate is obtained projecting onto C the projection of x^k onto the separating hyperplane H^k just like in the combined relaxation method described at the end of the previous subsection. It is worth noting that the line searches (11) and (14) are different as they relate to different predictive steps but they both aim at separating strictly the current iterate from the solution set of (EP).

Alternatively, if C has a nonempty interior, it is possible to replace the predictive step with the following projection

$$\bar{x}^k = \pi_C\left(\bar{x} + \alpha^{-1}\sum_{j=1}^{k-1}\nabla_y f(x^j, x^j)\right)$$

where $\bar{x}$ belongs to the interior of C, and replace also x^k by $\bar{x}^k$ in the correction step [91]. In this way the centroid of the correction iterates, i.e., $(x^1 + \cdots + x^k)/k$, converges to a solution of (EP) provided that f is monotone and the gradient mappings $\nabla_y f(x, \cdot)$ are Lipschitz continuous uniformly in x for some constant $L \leq \alpha$.

3.1.3. Descent methods

As already explained at the beginning of Section 3.1, the gap function φ_α allows to reformulate (EP_α), and thus (EP), as a constrained optimization problem, whose global minima are indeed the solutions of the two equilibrium problems. Though φ_α is continuously differentiable, in general it is not convex and therefore finding its global minima is not an easy task.

One possibility to overcome this difficulty is to consider assumptions which guarantee that any stationary point of the minimization problem

$$\min\{\ \varphi_\alpha(x)\ :\ x \in C\ \} \tag{15}$$

is actually a global minimum, and thus solves (EP). This "stationarity property" holds if f is strictly ∇–monotone on C (see [18, 71]), i.e.,

$$\langle \nabla_x f(x, y) + \nabla_y f(x, y), y - x \rangle > 0, \quad \forall\ x, y \in C \text{ with } x \neq y, \tag{16}$$

while it does not hold if f is ∇–monotone on C, i.e., the left-hand side of (16) is just greater or equal to 0. Actually, no relationship holds between these conditions and the monotonicity assumptions which have been exploited in the previous sections. Anyway, they can be considered some kind of monotonicity too: in fact, f is surely (strictly) ∇–monotone on C if the mappings $\nabla_x f(x, \cdot) + \nabla_y f(x, \cdot)$ are (strictly) monotone on C for any $x \in C$.

Strict ∇-monotonicity guarantees also that $\Phi_\alpha(x) - x$ is a descent direction for φ_α at any non-stationary point x (see [18, 71]). Therefore, a basic descent method can be devised moving away from a non-stationary point x^k along the direction $d^k = \Phi_\alpha(x^k) - x^k$ with a suitable stepsize $t_k \in (0, 1]$ to obtain the new iterate $x^{k+1} = x^k + t_k d^k$. Notice that, due to the choice of the stepsize, x^{k+1} is a convex combination of x^k and $\Phi_\alpha(x^k)$, and hence belongs to C.

If f is strictly ∇–monotone and C is bounded, then this descent method with the exact line search

$$t_k \in \arg\min\{\ \varphi_\alpha(x^k + td^k)\ :\ t \in [0, 1]\ \}$$

generates a bounded sequence $\{x^k\}$ such that any of its limit points solves (EP) [71]. An Armijo-type inexact line search can be considered choosing the stepsize $t_k = \beta^m$ with $\beta \in (0, 1)$ and m being the smallest nonnegative integer such that

$$\varphi_\alpha(x^k + \beta^m d^k) \leq \varphi_\alpha(x^k) - \theta\beta^m \|d^k\|^2.$$

Convergence is guaranteed provided that C is bounded, f is strongly ∇–monotone with constant $\gamma > 0$, i.e.,

$$\langle \nabla_x f(x, y) + \nabla_y f(x, y), y - x \rangle \geq \gamma \|y - x\|^2, \qquad \forall\ x, y \in C,$$

and $\theta < \gamma$ [25, 64, 71].

If C is not bounded, some additional assumptions on f are needed to obtain the boundedness of the sequence $\{x^k\}$, which is a key property to achieve convergence. Whenever the value of the gap function φ_α provides an error bound for the unique solution x^* of (EP), i.e., there exists $\sigma > 0$ such that

$$\varphi_\alpha(x) \geq \sigma \|x - x^*\|^2, \quad \forall\ x \in C, \tag{17}$$

the sequence $\{x^k\}$ is bounded since the sequence of values $\{\varphi_\alpha(x^k)\}$ is decreasing. The error bound (17) holds if f is strongly monotone [71] or if $\nabla_x f$ and $\nabla_y f$ are Lipschitz continuous and $G(x) = \nabla_y f(x, x)$ is strongly monotone [25], i.e., there exists $\gamma > 0$ such that

$$\langle G(x) - G(y), x - y \rangle \geq \gamma \|x - y\|^2,$$

or if the mappings $\nabla_y f(\cdot, y)$ are strongly monotone with the same constant γ for all $y \in C$ [61, 64]. Therefore, the descent method (with exact or inexact line search) converges also in case C is not bounded if any of the three above additional assumptions holds too.

A descent method which does not require the strict ∇–monotonicity of f can be devised relying on the concavity-type condition

$$f(x, y) + \langle \nabla_x f(x, y), y - x \rangle \geq 0, \qquad \forall\ x, y \in C, \tag{18}$$

which is indeed satisfied if $f(\cdot, y)$ is concave for all $y \in C$. Moreover, it implies ∇–monotonicity but it is neither stronger nor weaker than strict ∇–monotonicity, and it does not guarantee the stationarity property for (15). Anyway, this concavity-type assumption paired with the boundedness of C guarantees that $\Phi_\alpha(x) - x$ is a descent direction for φ_α at any $x \in C$ which does not solve (EP), provided that α is small enough [18]. This property provides the key idea of the method: if $\Phi_\alpha(x^k) - x^k$ is a descent direction for φ_α at the current iterate x^k, then a (inexact) line search is performed, while otherwise the value of α is decreased, for instance according to some contraction factor. Convergence is achieved under (18) and the boundedness of C [18].

The evaluation of φ_α at a given point $x \in \mathbb{R}^n$ could be computationally expensive if the description of C involves nonlinear convex constraints, i.e.,

$$C = \{\ y \in D\ :\ c_i(y) \leq 0, \quad i = 1, \ldots, m\ \}$$

where D is a polyhedron and $c_i : \mathbb{R}^n \to \mathbb{R}$ are nonlinear convex functions. If the c_i's are (continuously) differentiable, linearizing these constraints around x means replacing C by the outer polyhedral approximation

$$P(x) = \{\ y \in D\ :\ c_i(x) + \langle \nabla c_i(x), y - x \rangle \leq 0, \quad i = 1, \ldots, m\ \}.$$

In fact, $C \subseteq P(x) \subseteq D$. Modifying the inner optimization problem defining $\varphi_\alpha(x)$ in this way leads to the function

$$\phi_\alpha(x) = -\min\{\ f(x, y) + \alpha\|y - x\|^2/2\ :\ y \in P(x)\ \},$$

which is indeed a new gap function for (EP) [19] and moreover the computation of its values amounts to minimize a strongly convex function subject to linear constraints only. Anyway, there is no longer any guarantee that the unique minimizer of $\phi_\alpha(x)$ belongs to C and hence the descent procedure may fail to produce a new iterate belonging to C. Therefore, the introduction of a penalization term and procedures to control the penalization parameter are needed. Whenever D is bounded, convergence is achieved provided that f is strictly ∇-monotone [20] or satisfies the concavity-type condition (18) [19].

Gap functions can be exploited also to reformulate (EP) as an unconstrained optimization problem. The approach is based on a pair of gap functions which provide the so-called D-gap function

$$\varphi_{\alpha\beta}(x) = \varphi_\alpha(x) - \varphi_\beta(x),$$

where $0 < \alpha < \beta$. Indeed, the global minima of the unconstrained minimization problem

$$\min\{\, \varphi_{\alpha\beta}(x) \ : \ x \in \mathbb{R}^n \,\} \tag{19}$$

coincide with the solutions of (EP) (see [62, 101]). Clearly, $\varphi_{\alpha\beta}$ inherits the differentiability properties of φ_α and φ_β but in general it is not convex (just like φ_α and φ_β), thus it could be difficult to find a global minimum.

However, if the mappings $\nabla_x f(x,\cdot) + \nabla_y f(x,\cdot)$ are strictly monotone on $\mathbb{R}^n$ for any $x \in \mathbb{R}^n$, then each stationary point of $\varphi_{\alpha\beta}$ is actually a global minimum of (19), and thus solves (EP) [101]. Actually, descent methods for $\varphi_{\alpha\beta}$ require stronger assumptions: if the mappings $\nabla_x f(x,\cdot)$ are strongly monotone with the same constant $\gamma > 0$ for all $x \in \mathbb{R}^n$ and uniformly Lipschitz continuous, then there exists $\rho > 0$ such that $\varphi_{\alpha\beta}$ and the corresponding argmin function $\Phi_{\alpha\beta} = \Phi_\alpha - \Phi_\beta$ satisfy

$$\langle \nabla\varphi_{\alpha\beta}(x), \Phi_{\alpha\beta}(x) + \rho s(x)\rangle \le -\gamma(\|\Phi_{\alpha\beta}(x)\| + \rho\|s(x)\|)^2/2 \tag{20}$$

for all $x \in \mathbb{R}^n$, where $s(x) = \alpha[x - \Phi_\alpha(x)] - \beta[x - \Phi_\beta(x)]$ (see [26, 62]). The stationarity of a given point x^k is equivalent to $\Phi_{\alpha\beta}(x^k) = 0$ and $s(x^k) = 0$. If this is not the case, then (20) guarantees that $d^k = \Phi_{\alpha\beta}(x^k) + \rho s(x^k)$ is a descent direction and an inexact line search can be performed along this direction. The resulting method converges provided that the sequence $\{x^k\}$ is bounded [26, 62]. Since the sequence of values $\{\varphi_{\alpha\beta}(x^k)\}$ is decreasing, it is enough to guarantee that the sublevel sets of $\varphi_{\alpha\beta}$ are bounded: this is true if $\nabla_y f$ is Lipschitz continuous and $G(x) = \nabla_y f(x,x)$ is strongly monotone [26, 62] or if the mappings $\nabla_y f(\cdot,y)$ are strongly monotone with the same constant for all $y \in C$ [102].

Another descent method based on the D-gap function $\varphi_{\alpha\beta}$ relies on the same direction $d^k = \Phi_\alpha(x^k) - x^k$ which is exploited also by the descent methods for φ_α: if $\varphi_{\alpha\beta}(x^k + d^k) \le \eta\varphi_{\alpha\beta}(x^k)$ holds for some fixed parameter $\eta \in (0,1)$, the new iterate is $x^{k+1} = x^k + d^k$ while otherwise an inexact line search along either d^k or $-\nabla\varphi_{\alpha\beta}(x^k)$ is performed. The method converges to a solution of (EP) provided that the mappings $\nabla_x f(x,\cdot)$ are strictly monotone for any $x \in \mathbb{R}^n$ and $\nabla_y f(\cdot,y)$ are strongly monotone with the same constant for all $y \in C$ [102, 103].

3.2. Regularization methods

Regularization methods for equilibrium problems rely on a well-known solution scheme already developed for nonlinear equations, optimization problems and variational inequalities. The key idea is to solve a sequence of auxiliary equilibrium problems whose solutions converge to a solution of (EP). More precisely, at the k-th iteration any regularization method finds an exact or approximate solution of the auxiliary equilibrium problem

$$\begin{array}{c}\text{find } x^* \in C \text{ such that}\\ f(x^*,y) + \alpha_k\langle x^* - u^k, y - x^*\rangle \ge 0 \text{ for all } y \in C,\end{array} \tag{EP_k}$$

where $\alpha_k > 0$ and $u^k \in \mathbb{R}^n$ are the parameters whose choice determines the different algorithms. The additional term is called regularizing because it allows to strengthen the monotonicity and ∇-monotonicity properties of the original bifunction f. In fact,

$$f_k(x,y) := f(x,y) + \alpha_k\,\langle x - u^k, y - x\rangle$$

is strongly monotone if f is monotone and strongly ∇-monotone if f is ∇-monotone. All the methods described in Section 3.1 can be applied to solve the auxiliary problems. The sequence of the solutions of the auxiliary problems converges to a solution of (EP) under suitable generalized monotonicity or coercivity assumptions on f.

In the following we classify the regularization methods into two subclasses: the proximal point and the Tikhonov-Browder methods. In the first subclass the parameter u^k depends upon the previous iterate(s) and the parameters α_k are kept fixed (or bounded above and away from zero) while in the second u^k is independent from the previous iterates and $\alpha_k \downarrow 0$.

3.2.1. Proximal Point Methods

The basic version of the proximal point method (shortly PPM) asks to find an exact solution x^k of the auxiliary equilibrium problem (EP_k) with $\alpha_k = \alpha$ for some fixed $\alpha > 0$ while the previous iterate x^{k-1} provides the other parameter, i.e., $u^k = x^{k-1}$. If f is monotone, then each auxiliary problem has a unique solution since f_k is strongly monotone (see Section 2.1), and the sequence $\{x^k\}$ converges to a solution of (EP) [75]. Moreover, if f satisfies a conditioning assumption, then convergence is actually achieved in a finite number of iterations [77].

Since the auxiliary problems cannot be actually solved exactly, inexact versions of PPM are essential in the development of implementable algorithms. One way to consider inexact PPMs is to add an approximation error ε_k to the bifunction f_k so that any solution of the approximated auxiliary problem satisfies

$$f_k(x^k,y) \ge -\varepsilon_k, \quad \forall\, y \in C.$$

With the same choice of parameters of the exact case, the sequence $\{x^k\}$ generated by this inexact PPM converges to a solution of (EP) if f is monotone on C and the series whose terms are ε_k is convergent [75]. Taking u^k as a particular linear combination of the two previous iterates x^{k-2} and x^{k-1}, i.e., $u^k = x^{k-1} + \beta_k\,(x^{k-1} - x^{k-2})$ for some $\beta_k \in [0,1)$, the convergence of the inexact PPM is guaranteed if f is monotone and the parameters α_k, β_k, ε_k satisfy suitable technical conditions [76]. This last method can be further extended considering auxiliary problems which are defined on convex outer approximations of C [5].

Another way to develop inexact PPMs relies on the approximation of a solution v^k of the auxiliary problem (EP_k) with a given accuracy ε_k, i.e., the computation of some $x^k \in C$ such that $\|x^k - v^k\| \leq \varepsilon_k$. Taking $\alpha_k = \alpha$ for some fixed $\alpha > 0$ and $u^k = x^{k-1}$, the sequence $\{x^k\}$ converges to a solution of (EP) provided that f is pseudomonotone, each auxiliary problem admits at least one solution and the series whose terms are ε_k is convergent [51]. In case the Minty equilibrium problem admits at least one solution but f is not necessarily pseudomonotone, the sequence $\{x^k\}$ admits limit points and all of them solve (EP) [51]. Since the accuracy of the approximate solution of the auxiliary problems has to be controlled, some further assumptions on f may provide the required error bound. For instance, if f is weakly monotone, i.e., there exists $\theta > 0$ such that

$$f(x, y) + f(y, x) \leq \theta\|x - y\|^2, \quad \forall\, x, y \in C,$$

then the bifunctions f_k of the auxiliary problems are strongly monotone with constant $\alpha - \theta$ whenever α is chosen greater than θ. In this case each auxiliary problem admits a unique solution and the corresponding gap function allows to estimate the distance from the solution (see also Section 3.1.3).

Considering a regularization term which depends only upon a subset of the variables, a partial version of the latter inexact PPM can be also developed [53].

A further kind of inexact PPMs exploits the auxiliary problem (EP_k) with $u^k = x^{k-1} + e^k$ for some arbitrary error vector e^k whose norm is bounded by a suitable function of the available data at the current iteration. The solution of (EP_k) and the error vector e^k are used to build a hyperplane H^k separating x^{k-1} from the solution set of (EP), and the new iterate x^k is obtained either projecting x^{k-1} onto H^k or making a step from x^{k-1} towards H^k. The convergence of these methods is based on the weak monotonicity and the pseudomonotonicity of f [44]. Notice that these methods allow for constant relative errors in the auxiliary problems unlike the methods previously recalled, which require an increasing accuracy (up to exactness in the limit).

In the case of infinite dimensional spaces, the weak convergence of several methods mentioned above has been proved in Hilbert spaces [46, 75, 76, 77] and Banach spaces [44, 69]. On the other hand, the strong convergence of some methods has been analyzed in [69, 75, 77].

3.2.2. Tikhonov-Browder methods

The basic version of the Tikhonov-Browder method asks to find an exact solution x^k of the auxiliary problem (EP_k) with $u^k = 0$ and it considers a sequence $\alpha_k \downarrow 0$. If f is monotone, then each auxiliary problem admits a unique solution and $\{x^k\}$ converges to the solution of (EP) having minimal norm [78]. More in general, if the parameters u^k are all taken equal to a given vector u, then the sequence $\{x^k\}$ converges to the Euclidean projection of u onto the solution set of (EP).

Since the auxiliary problems can not be solved exactly in practice, an approximate computation of the iterates is required. For instance, if v^k is an exact solution of the auxiliary problem (EP_k) of the basic Tikhonov-Browder method and $\|x^k - v^k\| \leq \varepsilon_k$, then also the sequence $\{x^k\}$ converges to the element of minimal norm of the solution set of (EP), provided that $\varepsilon_k \downarrow 0$. Similarly to the case of PPMs, the error ε_k can be checked using the value of the gap function associated to the auxiliary problem (see Section 3.1.3).

As already mentioned, all the solution methods discussed in Section 3.1 can be exploited to approximate the solution of each auxiliary problem. In particular, descent methods based on gap and D-gap functions have been explicitly considered [59, 63].

In order to provide better approximations of the initial problem, non quadratic regularization terms can be exploited. Considering a suitable strongly monotone bifunction in place of the quadratic term $\langle x - u^k, y - x\rangle$, this modified version of the basic Tikhonov-Browder method is convergent under the monotonicity of f [59, 63]. Considering a regularization term which depends only upon a subset of the variables, a partial version of the method can be also developed [52].

In order to drop any monotonicity assumption on f, the key assumption to guarantee convergence is a coercivity condition more general than (6), which exploits a continuously differentiable and strongly convex function real-valued μ [58]. One method replaces the usual quadratic regularization term with $\langle \nabla\mu(x), y - x\rangle$. The coercivity condition guarantees that (EP) and all the auxiliary problems admit at least one solution. Moreover, the sequence generated by the method has limit points and all of them solve (EP) [58]. Taking $\mu(x) = \|x\|^2$ this result allows to prove the convergence of the basic Tikhonov-Browder under the coercivity condition (6).

Another method uses $\mu(y) - \mu(x)$ as the regularization term and the same convergence result of the previous method holds [58]. However, this new term helps to improve the convexity properties of the auxiliary problems but not the monotonicity properties. In fact, since $f(x, \cdot)$ is convex then $f_k(x, \cdot)$ is strongly convex, while if f is monotone then f_k is also monotone but not necessarily strongly monotone. Even more general coercivity conditions can be exploited [60].

Actually, in order to make these two last methods implementable, some additional monotonicity assumptions on f are needed to control the approximation of the solutions of the auxiliary problems exploiting the methods described in Section 3.1.

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