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An iterative procedure for the Paretian ranking of competing projects

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Abstract

In this Technical Report we present an iterative procedure for the evaluation, from a set of decision makers or **deciders**, of a set of projects according to a given set of criteria and the possible selection of the best or most preferred project from that set.
The procedure uses Pareto principles and a Borda classical voting method and aims at attaining fair allocations whenever this is possible.
Comments, observations and reports of errors are very welcome.

Contents

1 Introduction

In this technical report we present a procedure that can be used by a set E of **evaluators** (or decision makers also termed deciders) for the evaluation of a given set P of competing projects and the selection of one project from the set P.
The evaluation turns in the ranking of the projects according to a given set C of criteria and is carried out with the use of the concepts of **Pareto dominance** and **Pareto equivalence** or **indifference**.
The problem of project evaluation and selection is very remarkable and has received attention even from European Union authorities (see, for instance, [17]).
In this technical report we propose an approach that sees the projects of the set P as both feasible and eligible ([17]) and the elements of the set C as having the same importance or weight[1].
The proposed approach is based on a **static setting** since both the set P and the set C are seen as exogenously given by the selection procedure ([6, 8, 9]). If, during the execution of the procedure, the evaluators could vary dynamically the sets P and C by adding or removing elements we would be in a dynamic setting ([9]). In the present technical report we do not deal with these possibilities so that we restrict our treatment to the static setting.

2 The motivations

In order to understand the rationale of the proposed procedure we must consider a set E of the deciders that act within a well defined normative context that provides them with the set C of the evaluation criteria.
In this context the deciders have a common problem as a problem that can be solved, for either economic or logistic reasons or both, only with the contribution of all the deciders at various degrees of involvement.
In order to solve such a problem every decider, however, is interested in the implementation of one or more preferred projects p_j.
In this way the deciders of the set E collect the set P of the projects as the union of the sets P_k of the projects that each decider $e_k \in E$ would be willing to implement in order to solve the common problem.
Once they have collected the projects they must evaluate such projects in order to select one of them to have it implemented. We assume that all the presented projects are included in the set P but for the projects that are fully equivalent according to the opinion of all the deciders themselves. With this we mean that if one of the deciders opposes to the claim of equivalence for a project that project must be inserted in the set P.
In order to give some concreteness to our reasoning we note how this way of proceeding can be applied to solve problems of solid wastes disposal, special

[1] In section 11 we show the case of different weights associated to the elements of C.

and hazardous wastes disposal, local transports, water distribution management, energy production and dispatchment, localization of either commercial or industrial areas and all the similar problems that affect, with both positive and negative externalities, more or less wide areas that fall under the jurisdiction of a certain number of deciders.
As it will be clear from the rest of this technical report, in solving this type of problems we cannot use approaches such that proposed in [13] (where the authors present procedures for the fair division of indivisible goods) or such those proposed in [2] (where the focus is on the sharing of one or more either divisible or indivisible goods) since we have more complex aims. We, indeed, aim at providing tools (firstly) for the selection of an element from a set and (secondly) for the sharing of the divisible features associated to such an element among a set of deciders with the possibility, for the deciders, to iteratively examine all the elements of that set also with the aid of either recovery or reevaluation phases.

3 The basic ingredients

The procedure that we are going to introduce in section 4 and analyze in the following sections is based on a certain number of sets that have already been introduced but that we formally define and characterize in the present section. Such basic sets are:

- the set E of the deciders;
- the set C of the criteria;
- the set P of the projects.

The set E (of[2] $e = |E|$ elements) identifies the set of the deciders that, during the execution of our iterative procedure, act also as evaluators of every project of the set P.
In the roles of proposers, evaluators and selectors the members of the set E act as **peers** without any subordination relation and without the possibility to exert an absolute veto power[3] over any of the projects during the whole process.
The set C (of $c = |C|$ elements) derives from the normative context about the problem that the deciders are trying to solve and contains a certain number of criteria that are seen as exogenously fixed from the deciders.
Last but not least, the set P (of $p = |P|$ elements) contains the projects to be evaluated and selected. In order to perform such operations the deciders use the elements of the set C.
Every criterion $c_j \in C$ is assumed to be characterized by the following features:

- the various c_j have the same weight or importance but may be either independent (so that the corresponding rankings can be determined one

[2]Given a set A with $a = |A|$ we denote its cardinality or the number of its elements. An empty set has, rather obviously, a null cardinality.
[3]With this power we denote the power to oppose to any of the projects so to have it abandoned without any possibility of being either examined or discussed.

independently from the others) or interdependent (so that the corresponding rankings reflect these links);

- every c_j is associated to a numerical scale of some type, either ordinal or cardinal ([19, 20]), that allows the deciders to rank the projects $p_i \in P$ according to each criterion;

- every c_j allows the definition of a total ordering with possible ties among the elements of the set P.

We note how the presence of interdependencies among the criteria turns into the need of either compensations or transfers and that the absence or the inadequacy of such compensations or transfers may give rise to the lack of unanimous approval from the deciders for the project currently under examination (see sections 4 and 7).
We underline, moreover, how the numerical scale associated to each criterion is used by the deciders to map their preferences on a total ordering with possible ties among the projects for every criterion. For these purposes every criterion is considered as an independent dimension so that, in many cases, such scales can be, at least in part, arbitrarily defined by the deciders (but under the constraints posed by the normative in force for the current problem).
From these features we have that every $p_i \in P$ is associated to a point

$$x_i = (x_{i1}, \dots, x_{ic}) \in \mathbb{R}^c \tag{1}$$

with $x_{ij} \in \mathbb{R}$ for $j \in [1, c]$. By using these points we can characterize the projects through Pareto criteria as either **Pareto dominated** or **Pareto equivalent** (see section 5): those of the former type can be safely discarded (since for each of them there is surely a project with better or not worse properties) whereas those of the latter type are considered incomparable so that can be seen as equivalent.

4 The proposed procedure

The iterative procedure that represents the core of the present technical report consists of the following high level steps (whose details will be provided in the next sections of this technical report):

(1) evaluation of the projects of the set P;

(2) identification of the set P_d of the Pareto dominated and of the set P_e of the Pareto equivalent projects;

(3) selection and removal of a project p_i from the set P_e;

(4) sharing of the project p_i's shareable features among the deciders of the set E;

(5) request for approval from the deciders;

(6) if all the deciders approve the sharing then the procedure is over with a success and p_i is the selected project;

(7) if at least one decider opposes to the sharing then the project p_i must be, at least temporarily, shelved;

(8) if the set P_e is not empty the procedure recovers from step (3) otherwise it would end with a failure;

(9) in order to avoid this failure the deciders can either perform a recovery of at least one of the temporarily shelved projects or repeat the evaluation of the projects;

(10) in the former case the procedure enters in a **recovery step** that will be described in detail in section 9;

(11) in the latter case the procedure is repeated from step (1).

As it should be evident from its description the procedure is iterative over the set P of the projects. The iteration points are represented by:

- step (8) that can give rise to at the most $|P_e|$ iterations;
- step (11) that is meaningful only if P_e is varied (also considering the subset relation) with respect to the preceding realizations of the same set so that, since $|P|$ is finite, the procedure can be repeated in this way a finite number of times.

The key points of the procedure are steps (5) and (10) since the presence of the latter step as a common knowledge among the deciders can cause the adoption of strategic behaviors at the former step. With this we mean that a decider can oppose to a project, even if he could be in favor to its selection, simply because he knowns that that project can be recovered at step (10) with a simple request and since he wants to examine also the other projects of the set P_e to verify if there may be a better sharing for him.

5 The structure of the rest of the technical report

Now that the basic elements have been introduced the technical report continues with a discussion of the Pareto criteria and with a detailed description of the key steps of the proposed procedure. Then it describes in greater detail two of such steps (or the sharing and recovery steps) before presenting a discussion of the main properties that are satisfied by the proposed approach. The technical report ends with a section devoted to the analysis of the possible use of different weights for the criteria of the set C followed by a short section devoted to a conclusive discussion and to the presentation of future activities within this research activity.

6 About Pareto criteria

The key point is the definition of the sets P_d and P_e respectively of the **Pareto dominated** and **Pareto equivalent** projects ([19, 3, 4]) according to the criteria of the set C.
For our purposes a project $p_i \in P$ can be either Pareto dominated or Pareto equivalent to some other project so that, if we are able to determine if a project is not Pareto dominated, we can assign it to the set P_e.
Given a project $p_i \in P$ we say that it is **Pareto dominated** if and only if there exists at least one project $p_h \in P$ such that:

$$p_h \succ_j p_i \ \forall\, c_j \in C \tag{2}$$

In relation (2) with $\succ_j$ we denote a strict preference binary relation endowed with classical properties ([20]). In this way we speak of a **strict dominance**.
We can relax such definition and define a **weak dominance** binary relation so that p_h weakly dominates p_i if and only if we have, using a classical weak preference binary relation ([20]):

$$p_h \succeq_j p_i \tag{3}$$

for all the $c_j \in C$ but at least for one criterion where we have a strict preference.
In both cases an either strictly or weakly dominated project belongs to the set P_d of the Pareto dominated projects.
It is easily seen how:

- if $p_i \in P_d$ then there is at least one dominating project;
- there may be projects that are neither dominated nor dominating and therefore belong to the set P_e.

The set P_e can be iteratively derived from the set P through a simple iterative procedure that runs in $p = |P|$ steps and has the following structure:

(0) we define $P_e = \emptyset$ and set an index $i = 1$;

(1) we extract p_i from P (so[4] $P = P \setminus \{p_i\}$), insert it in P_e (so $P_e = P_e \cup \{p_i\}$) and update i as $i = i + 1$;

(2) we extract $p_i \in P$ so that we have the following possible cases:

  (2a) p_i is dominated by at least one of the projects that have already been inserted in P_e so we can discard it,

  (2b) if p_i is not dominated we insert it in P_e (so $P_e = P_e \cup \{p_i\}$) and remove from such a set all the projects that are dominated by the current p_i (so $P_e = P_e \setminus P_{d_i}$ if with P_{d_i} we denote the set of the projects to be removed from P_e owing to their being dominated);

[4] We note how expressions like $a = a \oplus b$, where $\oplus$ denotes a binary operation, must be read as assignment operations meaning that a assumes the old value modified in some way with the current value of b.

(3) we update i as $i = i + 1$ so that if $i \leq p$ the procedure is repeated from step (2) otherwise it is over so that it produces, as its output, the set P_e of the Pareto equivalent projects.

We note how the foregoing procedure assumes that the projects of the set P have been numbered in some way from p_1 to p_p, an easy task, indeed. It is easy to see why step (1) is always feasible from step (0).
At this point we have to fully characterize the binary relations $\succ_j$ and $\succeq_j$ ([20]) for every $c_j \in C$. In order to make this characterization we recall how every c_j allows the assignment of a real value x_{ij} to every project p_i so that we have the following cases:

$p_i \succ_j p_h$ if and only if $x_{ij} > x_{hj}$ so that c_j represents a **benefit** or a property where the higher is the better;

$p_i \succ_j p_h$ if and only if $x_{ij} < x_{hj}$ so that c_j represents a **cost** or a property where the lower is the better.

This for the strict dominance. We can easily characterize also the weak dominance if we properly replace $\succ_j$ with $\succeq_j$, $>$ with $\geq$ and $<$ with $\leq$.
In order to make some more comments and give concreteness to our arguments we consider a toy example where we have:

$C = \{c_1, c_2\}$

$P = \{p_1, p_2, p_3, p_4\}$

According to the proposed procedure we start with considering p_1 to which we associate the vector $x_1 = (x_{11}, x_{12})$ so that we may have the following cases:

- if both c_1 and c_2 represent benefits we have that the projects p_j such that $x_{j1} < x_{11}$ and $x_{j2} < x_{12}$ are in P_d;
- if both c_1 and c_2 represent costs we have that the projects such that $x_{j1} > x_{11}$ and $x_{j2} > x_{12}$ are in P_d;
- if c_1 represents a benefit and c_2 represents a cost we have that the projects p_j such that $x_{j1} < x_{11}$ and $x_{j2} > x_{12}$ are in P_d;
- dual considerations hold if c_1 is a cost and c_2 is a benefit.

According to our procedure we put p_1 in P_e and consider its relations of dominance with the other three projects of the set P. With this we mean that we compare p_2 with the projects that are present in P_e and update this set accordingly then we go on with p_3 and end with p_4.
When we are over we have defined the set P_e of the Pareto equivalent projects for this case. With this we mean that, for instance:

(a) if p_1 dominates all the other projects we then have $P_e = \{p_1\}$,

(b) if p_1 is dominated by p_2 that dominates also p_3 but not p_4 we have $P_e = \{p_2, p_4\}$.

We note how in case (a) we discard all the other three projects since they are dominated by p_1 whereas in case (b) we start with $P_e = \{p_1\}$ then we switch to $P_e = \{p_2\}$ to end with $P_e = \{p_2, p_4\}$. Similar considerations hold also if we properly consider the weak relations $\geq$ and $\leq$.

7 The procedure in detail

In this section we describe in detail some of the steps that we only listed in section 4 within the general procedure whereas we devote section 8 to a discussion of the **sharing** step and section 9 to a discussion of the **recovery** step.

We start with the **evaluation** step (or step (1)) where each project $p_i \in P$ is associated to a vector $x_i \in \mathbb{R}^c$. This step must be necessarily carried out by the deciders before they can identify the set P_e of the Pareto equivalent projects from the set P of the competing projects.

In order to accomplish this task we have that:

- every decider $e_k \in E$, privately and independently from the others, produces an evaluation matrix X^k of p rows and c columns such that

$$X^k = \{x_{ij}\}_{i \in [1,p], j \in [1,c]} \tag{4}$$

  contains, for a given decider, his evaluations x_{ij} of every project p_i according to every criterion c_j;

- the e matrices X^k are merged, through a mechanical procedure, into a single global matrix X;

- this matrix is used by the deciders to define the set P_e as we have seen in section 6.

As it is easily seen the rows of the matrix X are the vectors x_i that we have already introduced in section 3 whereas its columns define how each criterion ranks the different projects.

The deciders define their matrices X^k using the set C that is common knowledge among them ([14, 16]) and possibly after that also the main features of the projects of the set P have become common knowledge.

The merging of the matrices X^k into X is performed as follows:

- if c_j is a cost we have $x_{ij} = min_{k \in [1,e]} x^k_{ij}$ for $i \in [1,p]$ with $j \in [1,c]$;

- if c_j is a benefit we have $x_{ij} = max_{k \in [1,e]} x^k_{ij}$ for $i \in [1,p]$ with $j \in [1,c]$.

These rules are an ex-ante common knowledge among the deciders. In this way we encourage the revelation from every decider of his true evaluations of the benefits and the costs associated to every project. Such rules, indeed, tend to discourage the adoption of **strategic evaluations** that, anyway, could be filtered by the deciders in the sharing and unanimous approval steps. We say that an evaluation is strategic if:

(s_1) it assigns high costs and low benefits to a project in order to discredit it;

(s_2) it assigns low costs and high benefits to a project in order to favor it.

We note that the assignment of either low costs and benefits or of high costs and benefits, although it may be insincere, is not classified as strategic since it is both more realistic and less damaging for the selection procedure as a whole. As it is evident evaluations of the (s_1) type tend to be filtered out by our merging rule that, on the other hand, tends to amplify the effect of evaluations of the (s_2) type. On the ground of this we have that if some of the deciders adopt strategic evaluations they contribute to the definition of an unrealistic global matrix X so that when one of the affected projects is evaluated in the sharing and request for unanimous approval steps it may be rejected since, depending on the case:

- some deciders feel that they suffer too high costs as compared to the benefits they get;
- some deciders are thought to get too high benefits and to suffer too low costs.

The outcome of step (1) is the matrix X that, at step (2), allows the deciders to identify, as we have seen in section 6, the set P_e from which they select (and remove) one project at a time (step (3)).
Once the sharing step (step (4), see section 8) has been executed the procedure goes on with a **request for approval** (step (5)) for the project currently under examination.
At this step there is a call for unanimity or for an **unanimous approval** so that every decider can act as a temporary veto player ([14]) so to cause the shelving of a project since the outcome of the sharing step is believed to be unfair by at least one decider.
The other meaningful steps are step (10) and step (11): at the former we have a **recovery** (see section 9) whereas at the latter we have a **reevaluation**. We note how the recovery step is a way to avoid boycotting from the temporary veto players.
A **recovery** is possible if at least one decider asks for it otherwise the procedure can go on with a reevaluation of the projects and so can be repeated from step (1).
If, however, after a new execution of the steps (1) and (2), the set P_e is unchanged (by comparing it, also as a subset, with any of the previously determined P_e sets) the procedure must be aborted with a failure. This interruption is a consequence of the fact that no decider requested a recovery (and so none of them thought a project from P_e as worth being reconsidered) and that a new definition of the matrix X produced a set P_e of Pareto equivalent projects that coincides with or is a subset of one of the sets that have already been examined without success in one of the preceding iterations. We underline how the execution of a recovery step prevents the deciders from performing any reevaluation of the projects.

From these facts and from the fact that the set P has a finite cardinality we derive, as we already noted in section 4, that the procedure cannot last forever.

8 The sharing step

Every project $p_i \in P_e$ is associated to its vector x_i (as defined by relation (1)). The elements x_{ij} (with $j \in [1, c]$) of this vector belongs either to shareable features or to non shareable features.
In the former category we have, for instance, implementation costs, functioning costs, maintenance costs, economic benefits, labor related benefits, savings related benefits, all measured according to cardinal scales ([20]).
In the latter category we have, for instance, the degree of compliance to some norms or the measure of the environmental impact through either positive or negative externalities, both possibly measured over ordinal scales ([20]).
From this we derive that we can decompose the vector x_i in the two sub vectors of the shareable and the non shareable features as follows:

$$x_i = (x_{i_s}, x_{i_{ns}}) \tag{5}$$

The sharing step involves, therefore, only the elements of the sub-vector x_{i_s} and is executed through a method called **last modifier** ([9, 10]).
We note that the elements of the sub-vector $x_{i_{ns}}$ do not enter the the sharing step but can be considered by the deciders:

- when they have either to approve or to reject the outcome of a sharing step;

- when they have to ask for a recovery step so to recover the preferred projects according to such values.

This method has been inspired by a method called **last diminisher** ([3]) that, in case of a divisible good, works as follows. Given a good to be shared among a set of partners $A, B, C \ldots$ we have that A cuts a portion of the good for himself and the others can (but are not obliged to) reduce such portion (if they judge it is too big). If none of the others reduces the portion then A gets it and exits the game. If some of them reduces it then the last one who reduces the portion must take it and exit the game. The procedure is repeated until all the partners receive their own share of the good (with the last two partners that can use a simple divide and choose procedure as it is shown in [3]).
Although simple, elegant and attracting this procedure only guarantees **proportionality**[5] but fails **envy-freeness**[6] since every partner but the last two

[5] We recall that an allocation procedure satisfies proportionality ([3, 4]) if each of n players thinks to have received a portion that has size or value of at least $1/n$ of the total size or value.

[6] We recall that an allocation procedure satisfies envy-freeness ([3, 4]) if each of n players thinks to have received a portion that is at least tied for the largest or for the most valuable and so does not envy any other player.

partners can envy a portion that another partner receives at a later stage. The procedure neither ex-ante guarantees **Pareto efficiency**[7] so that this property can be verified only ex-post and, if it is failed, can hardly be recovered through bilateral exchanges without violating either proportionality or envy-freeness or both.
The **last modifier** procedure, on the ground of the last diminisher procedure, works as follows:

(1) a decider e_k is selected at random from the set E;

(2) this decider proposes for himself a vector $x_{i_s}^k$ from x_{i_s} so that it contains some fractions of the shareable items;

(3) if no other decider opposes to the proposal then:

- (3a) e_k gets $x_{i_s}^k$ and exits, at least temporarily, the sharing game[8],
- (3b) x_{i_s} is properly updated by subtracting, component by component, the elements of $x_{i_s}^k$,
- (3c) the set E is updated by removing this e_k from it;

(4) if at least one decider opposes the proposal then the opposers, in succession, modify it in some way but only once for each decider that, at that moment, is still in the set E;

(5) let us assume that e_j is the last modifier and that the product of his modifications is the vector $\tilde{x}_{i_s}^k$ so that:

- (5a) if e_k accepts $\tilde{x}_{i_s}^k$ then, with some obvious modifications, the procedure goes on as at the step (3),
- (5b) if e_k refuses $\tilde{x}_{i_s}^k$ then e_j must accept it so that, with some obvious modifications, the procedure goes on as at the step (3);

(6) if the set E contains at least two elements and there are features to be shared in the vector x_{i_s} then this sharing procedure restarts from the step (1);

(7) if the set E contains only one element and there are features to be shared in the vector x_{i_s} then such features are allocated to that decider;

(8) if the set E is empty but there are residual features to be shared in the vector x_{i_s} then this sharing procedure is repeated from scratch on this vector of residuals;

[7] An allocation procedure is Pareto efficient ([3, 4]) if there is no other allocation where a player is strictly better off and none of the other players is worse off.

[8] A player can reenter the sharing game in the following cases: if a recovery is requested, if a reevaluation is executed or in other cases that are going to be discussed shortly. In any case any player is going to reenter the sharing game on a different project if the current project fails to get unanimous approval.

(9) the last case where the set E is not empty but $x_{i_s} = 0$ or there are no more features to be shared among the set of the yet to be involved deciders (the so called leftover deciders of the set E_{lo}) is dealt with in the next paragraphs.

If the case that we mentioned at step (9) occurs the deciders of the set E_{lo} (that have been excluded up to this point from the sharing) can declare to be either satisfied or unsatisfied with the outcome, possibly by using also the elements of the vector $x_{i_{ns}}$ to evaluate the fairness of the current outcome.
If all the leftover deciders of the set E_{lo} declare to be satisfied then the sharing step is over otherwise, if at least one of them declares to be unsatisfied, the procedure is closed with the following steps that implement step (9) above:

(9a) the unsatisfied deciders of the set E_{lo} can try to arrange local agreements with the deciders that got fractions of benefits and costs for the local sharing of such benefits and costs ([18]);

(9b) if such negotiations succeed, so that their outcomes can be made common knowledge among all the deciders, the sharing step is over;

(9c) if such negotiations fail the procedure is repeated from scratch from step (1) on condition that it has not yet been repeated for the same set E_{lo} of deciders since, in this case, it ends with the acceptance of the currently defined allocations.

The presence of step (9c) aims both at encouraging the success of the local negotiations and at discouraging any possible actions of boycotting.
We note how the arising of a dissatisfaction, although it may seem strange, is motivated by the desire from [some of] the deciders of the set E_{lo} either to receive compensations (on the ground of the values of the parameters of the sub vector $x_{i_{ns}}$) or to participate in the sharing of the benefits even at the expense of sharing also some of the costs.
When the **sharing** step (step (4) of the procedure of section 4) is over, every decider $e_k \in E$ has got his vector of shares $x^k_{i_s}$ for the project $p_i \in P$ so that the deciders can go to the **request for unanimous approval** step (or step (5) of the procedure of section 4).
If there is an unanimous approval (step (6) of the procedure of section 4) then the procedure ends with success otherwise the last diminisher procedure may be repeated, if it is possible, on another project of the set P_e, as we have seen in section 4.

9 The recovery step

As we have seen in section 7 the **recovery** step is possible if and only if at least one decider asks for it. We note that every decider that is willing to ask for the execution of a recovery step can propose one and only one project p_i from the set P_e as worth being reexamined according to him. By imposing this

limit of one project for each decider we enforce the selection from the deciders of the best project (for each of them) from the set P_e.
In this way the deciders define the set P_e^r of the recovered Pareto equivalent projects. This set is defined without any negotiation through individual decisions that are taken privately and revealed all at the same time. We have two cases:

(1) $|P_e^r| = 1$

(2) $|P_e^r| > 1$

In the (1) case the deciders recovered only one project that undergoes again the sharing step and the request for unanimous approval step. If the project gets an unanimous approval it is selected with the associated sharing of the shareable benefits and costs among the deciders otherwise the whole procedure ends with a failure.
In the (2) case the deciders can order the projects by using the Borda[9] method ([11, 21, 22, 15, 7]).
In this case the Borda method allows the definition of a total ordering with possible ties among the elements of the set $P_e^r \subseteq P_e$. With this we mean that:

- the set P_e^r is partitioned in a certain number $h \leq |P_e^r|$ of disjoint subsets $P_{e_i}^r$ (with $i \in [1, h]$);
- every subset is associated to a Borda score b_i and the Borda scores are assumed to be ordered as

$$b_1 > b_2 > \cdots > b_h \tag{6}$$

The projects are examined according to the foregoing ordering and so starting from those of $P_{e_1}^r$ to end with those of $P_{e_h}^r$.
For every project of each subset the deciders:

- define the sharing of its benefits and costs according to what we have seen in section 8;
- answer to a request for unanimous approval.

If, at any step, a project receives unanimous approval it is selected to be implemented otherwise, if this does not occur for any project, the procedure ends with a failure, this fact being an ex-ante common knowledge among the deciders together with the partitioning and ordering of the projects of the set P_e^r.

[9] We recall how the Borda method with a set of voters and a set of alternatives is based on the following mechanism ([1]): every voter ranks the n alternatives from the best to the worst by assigning n points to the former down to 1 point to the latter so to define a total ordering without ties among the alternatives. Such orderings are merged by simply summing the points assigned to each alternative. In this way the alternative with the highest number of points is the Borda winner. The final ordering can present ties among even all the alternatives and the tied alternatives can be seen as equivalent, at least according to this scoring method.

10 The properties of the proposed procedure

Before presenting the properties of the proposed procedure we note how, at least for some of the deciders or the so called **affected deciders**, the status quo represents a problematic situation that involves some costs that they think justify the selection and the implementation of a solution.
This fact must be taken into consideration when we examine the properties that are satisfied by the procedure that we presented in section 4.
As we have seen from section 7 to section 9 the procedure can terminate either with a failure or with a success.
In the **failure case** the problem remains unsolved so that the affected deciders (those that are directly affected by the problem) are worse off than in presence of any solution whereas the other deciders remain in their current situation and so are surely no worse off. This situation of absence of a solution is decidedly worse than any other situation in which a solution has been selected ad implemented so that the affected deciders are better off whereas the unaffected deciders continue to be no worse off with the possibility to be better off.
From this we derive the strong incentives for the deciders to select a solution among the projects of the set P and implement it.
In the **success case** a project has been selected by unanimous approval of the deciders so that every decider $e_k \in E$ has got his vector of (possibly null) shares $x_{i_s}^k$ associated to the selected project $p_i \in P$.
In this case we have to verify which properties are verified by the selected solution.
In agreement with [3] and [4] we are interested in establishing ways for the identification of **fair** allocations of shareable benefits and costs. In [3] and [4] an allocation is said to be fair if it satisfies the properties of envy-freeness, proportionality, Pareto efficiency and equitability[10] (to be defined and specialized to our context at due time).
In the spirit of [3] and [4] we are primarily interested in **envy-freeness** and **proportionality** (see also footnotes 5 and 7) and to keep satisfied the key relations between these two properties.
Since the acceptance of a project from the deciders occurs by unanimous approval we can assert that, in this case, **envy-freeness** is surely satisfied since, in presence of any envy, the envious deciders would not join the approval.
Similarly to what is done in [3] and [4] we state the definition from the point of view of the single decider. With this we mean that[11]:

[10] In both [3] and [4] equitability is defined only for the case of two players so we need either to abandon this property as non usable in our context or to adapt it to our context where we usually have more than two deciders. In this technical report, as we show in the closing part of the present section, we decided to adopt the second approach.

[11] We underline how the value b_{k_i} is obtained by e_k as a proxy measure of the benefits that he derives from $x_{i_s}^k$ as filtered through his individual values system whereas c_{k_i} is similarly obtained from the costs. On the other hand b_i and c_i represent similar quantities that e_k attributes to the deciders as a whole.

- if b_{k_i} is the benefit that e_k thinks to get from p_i to which he assign a global benefit equal to b_i (to be shared among all the members of E),

- if c_{k_i} is the cost that e_k thinks to get from p_i to which he assign a global cost equal to c_i (to be shared among all the members of E)

then an allocation of benefits and costs satisfies **envy-freeness** for the decider e_k if and only if the following inequalities hold for every $j \in [1, e]$:

$$b_{k_i} \geq b_{j_i} \tag{7}$$

$$c_{k_i} \leq c_{j_i} \tag{8}$$

In relation (7) b_{j_i} is the benefit that, according to e_k, is obtained by e_j from p_i whereas, in relation (8), c_{j_i} is the corresponding cost.
We note how:

- if relations (7) and (8) are satisfied for all the deciders $e_k \in E$ then the allocation is envy-free and the associated project gains the unanimous approval;

- if a project gains unanimous approval this means the absence of any envious decider and so that relations (7) and (8) are satisfied for all the deciders $e_k \in E$.

In this way we justify the assertion that we have made before. After envy-freeness we define **proportionality** in our context. We state that an allocation of benefits and costs satisfies **proportionality** for the decider e_k if and only if the following inequalities hold:

$$\frac{b_{k_i}}{b_i} \geq \frac{1}{e} \tag{9}$$

$$\frac{c_{k_i}}{c_i} \leq \frac{1}{e} \tag{10}$$

or, equivalently:

$$b_{k_i} \geq \frac{b_i}{e} \tag{11}$$

$$c_{k_i} \leq \frac{c_i}{e} \tag{12}$$

with $e = |E|$.
At this point, according to [3] and [4], we have to verify the relations that exist between envy-freeness and proportionality so to keep the fact that the former implies the second but the converse is true only for the case of two deciders.
In order to verify the direct implication we can start from relations (7) and (8) and sum over the set $[1, e]$ so to get[12]:

$$eb_{k_i} \geq \sum_{j=1}^{e} b_{j_i} = b_i \tag{13}$$

[12]We note how the values b_{j_i} are attributed by e_k to the other deciders so that they are numerical values on the same cardinal scale and can summed up without any problem. The same holds also for the values c_{j_i}.

$$ec_{k_i} \leq \sum_{j=1}^{e} c_{j_i} = c_i \tag{14}$$

from where we derive easily both relations (11) and (12) and, hence, both relations (9) and (10). In this way, having assumed that the devised solution verifies envy-freeness from its having gained unanimous approval, we have that it also verifies proportionality.

We note how the converse, in general, is not true. From relation (11), for instance, we cannot be sure to derive relation (7) for every $j \in [1, e]$ and the same holds also for relations (12) and (8).

We can easily justify this fact for the former pair of relations with the following argument that works similarly for the latter pair. If we have:

$$b_{k_i} \geq \frac{b_i}{e} \tag{15}$$

with $b_i = \sum_{j=1}^{e} b_{j_i}$ we can have values $b_{j_i} > b_{k_i}$ so violating the definition of envy-freeness since, in those cases, e_k would envy e_j.

If however $e = 2$ we have, from the definition of proportionality:

$$b_{k_1} \geq \frac{b_i}{2} \tag{16}$$

$$c_{k_1} \leq \frac{c_i}{2} \tag{17}$$

so that e_1 cannot envy the other decider e_2 (and vice versa e_2 cannot envy e_1 if the allocation is proportional also for the second decider). In this way, as it is obtained also in [3] and [4], we have seen how proportionality implies envy-freeness for the case of two deciders.

At this point we have verified that, from two plausible definitions of envy-freeness and proportionality, if the unanimous approval requirement is satisfied we have envy-freeness and, as a consequence, also proportionality.

The last two properties we are interested in are **Pareto efficiency** and, possibly, **equitability**.

For what concerns **Pareto efficiency** we can verify it as an ex-post property either on the vector x_{i_s} or on the values (such as b_{k_i}, c_{k_i}, b_i and c_i) that we have used for the definition of both envy-freeness and proportionality.

In the former or **objective case** we proceed as follows. If p_i gets unanimous approval we have the associated vector $x_{i_s} = (x_{i_s j})_{j \in [1,h]}$ (of $h \leq c$ shareable benefits and costs) that is shared among the deciders of the set E. In this case every $e_k \in E$ gets his own vector $x_{i_s}^k$ so that:

$$x_{i_s}^k = (x_{i_s j}^k)_{j \in [1,h]} \tag{18}$$

and:

$$\sum_{k=1}^{e} x_{i_s j}^k = x_{i_s j} \tag{19}$$

At this point we may note that if $x_{i_s j}^k$ corresponds to a benefit e_k is better off if such a value is increased whereas if it is a cost e_k is better off if such a value is decreased. The presence of the constraints represented by relation (19) implies that to any of such variations for e_k there must correspond the opposite variations for at least another decider e_h that, consequently, would be worse off. From such considerations we derive how the resulting allocation is Pareto efficient in this objective sense.
In the latter or **subjective case** we have that every decider e_k assigns to the project p_i that gets unanimous approval the values b_{k_i}, b_i, c_{k_i}, c_i with the following constraints:

$$b_i = \sum_{j=1}^{e} b_{j_i} \tag{20}$$

$$c_i = \sum_{j=1}^{e} c_{j_i} \tag{21}$$

so that if e_k claims either higher benefits or lower costs for himself (so to be better off) there is for sure at least another decider who suffers the opposite variations and so is worse off. From these considerations we derive how the allocation associated to p_i satisfies Pareto efficiency also in this subjective sense.
Now we have to say something about **equitability**. In [3] and [4] for two players A and B an allocation is equitable if A thinks that the portion he got is worth the same, in his valuation, as the portion that B got, in B's valuation.
The key points of equitability are therefore the following[13]:

- the property can be verified only as an ex-post condition;
- it is based on publicly revealed values expressed on a common scale and not on subjective valuations.

In order to follow the same guidelines we modified the definition of equality so that it measures, for each decider, the total balance between the wins and the losses that he gets in pairwise comparisons with the other deciders on every single shareable feature. In this way we say that an allocation is equitable if no decider gets more losses than wins.
Before showing how this seemingly vague definition can be applied to our context we recall that:

- to the selected project p_i there corresponds a global vector x_{i_s} and, for each decider, an individual vector $x_{i_s}^k$;
- the global vector contains the scalar values $x_{i_s j}$ whereas the individual vectors contain the scalar values $x_{i_s j}^k$ so that relation (19) holds.

[13]We follow the indications of [3] and [4]. Especially of [4] where a method called *AW* or **Adjusted Winner** is defined and fully discussed.

At this point we consider a decider e_k and perform a comparison between $x^k_{i_s}$ and $x^l_{i_s}$ for every other decider e_l with $l \neq k$. Such comparison is performed by confronting $x^k_{i_s j}$ with $x^l_{i_s j}$ for every $j \in [1, h]$ with $h = |x_{i_s}|$.
We have that:

- j may correspond to a benefit so that if $x^k_{i_s j} \geq x^l_{i_s j}$ this counts as 1 otherwise it counts as -1;

- j may correspond to a cost so that if $x^k_{i_s j} \leq x^l_{i_s j}$ this counts as 1 otherwise it counts as -1.

Now for a given j we sum the 1s (that correspond to wins) and the -1s (that correspond to losses) so to get an integer in the range $[-(e-1), (e-1)]$ that is assigned to the j-th element of the vector v_k (of h elements) of the valuations that we associate to every decider e_k. As the last step we sum all the components of every vector v_k so to get, for each of them, a single value $s_k \in [-h(e-i), h(e-1)]$ as a cumulative value, one for each decider, in the spirit of AW (see footnote 13).
The values s_k, as a last step, can be used to evaluate the equitability of the allocations associated to the selected project p_i as follows:

- if for every e_k we have $s_k = 0$ (and so we have a perfect balance of wins and losses) we can state that the allocation satisfies equitability;

- if for every e_k we have $s_k \geq 0$ and there are some deciders $e_j \in \hat{E} \subseteq E$ for which we have $s_j > 0$ we can state that the allocation still satisfies equitability;

- if we have at least one decider e_k such that $s_k < 0$ we can state that the allocation fails equitability.

Summing up we have that if a project $p_i \in P$ gets **unanimous approval** then we are sure that the associated allocation x_{i_s} of shareable benefits and costs among the deciders satisfies **envy-freeness** (and therefore **proportionality**) and **Pareto efficiency** without any guarantee that it satisfies also **equitability**. On the ground of [3] and [4] the last fact prevents us from declaring that the allocation is surely fair although we think that the lack of a guarantee of equitability is a minor problem, at least within our context.
The key point, indeed, is the fact that there is no guarantee that at least one $p_i \in P$ gets unanimous approval from the deciders. In the literature that we have cited so far (but see also [24, 12, 23, 5]) the conditions under which a division is effectively carried out are given for granted and satisfied[14]. In our case the situation is quite different since the effective execution of a sharing (and so of a division) requires the satisfaction of the necessary step of the unanimous

[14]For instance in the case of the cake cutting algorithms there is no discussion about the conditions that make the division a necessary step but similar considerations hold in most of the cases, if not in all. With this we mean that the division is a given and has no prerequisite to be satisfied but, for instance, the presence of the cake.

approval.
The only guarantee that we have that this step is satisfied is that:

- the presence of a common problem gives strong incentives to the deciders in order to have them reach the necessary unanimous approval on at least one of the projects $p_i \in P$;
- the wider is the set of the deciders that are affected by a problem in relation to the whole set of the deciders the higher is the probability that the deciders reach the necessary unanimous approval;
- the same holds the more serious and impacting the common problem is.

11 The use of different weights for the criteria

In section 3 we have assumed that the criteria of the set C have the same weight or importance. In this way the criteria give the same contribution to the evaluation of the projects and so to the definition of the set P_e.
There may be cases, however, where either the normative context or the political will of the deciders impose, possibly through a negotiated agreement among the deciders ([18]), a differentiation of the weights to be associated to the criteria.
In this case to each criterion c_j we associate a weight w_j such that:

(1) $w_j > 0$ for each $j \in [1, c]$

(2) $\sum_{j=1}^{c} w_j = 1$

The key point is represented, however, by the the presence of distinct values for the weights instead of their effective values. Such distinct values allow us to partition the set C in a certain number of disjoint subsets C_j of criteria with the same value of the associated weight. If we suppose to have $k \leq c$ distinct weights $\tilde{w}_j$ we may assume that they are ordered as:

$$\tilde{w}_1 > \tilde{w}_2 > \cdots > \tilde{w}_k \tag{22}$$

where each value $\tilde{w}_j$ identifies a subset $C_j \subseteq C$.
In this case the procedure that we have presented in section 4 must be modified in order to account for the presence of disjoint subsets C_j of criteria with decreasing importances $\tilde{w}_j$ (with $j \in [1, k]$) where the criteria of the set C_1 have the highest importance and those of the set C_k have the lowest importance.
For this purpose we note that:

- the first two steps are modified since the deciders firstly associate to every project $p_i \in P$ a vector $x_i^1 \in \mathbb{R}^{|C_1|}$ by using the criteria of the set C_1;
- by using such a vector the deciders filter the set P so to obtain the set P_{e_1};

- then the deciders use the criteria of the set C_2 and associate to every project $p_i \in P_{e_1}$ a vector $x_i^2 \in \mathbb{R}^{|C_2|}$ in order to filter the set P_{e_1} and obtain the new set P_{e_2};
- this procedure goes on until the deciders
    - either determine, at some step i, a set P_{e_i} such that $|P_{e_i}| = 1$
    - or use all the sets C_j with $j \in [1, k]$ and determine the final set P_{e_k}.
- in both cases the final set of Pareto equivalent projects represents the set P_e that is used from the step labeled as (3) on (see section 4).

In addition to such modifications we note that if the procedure of section 4 jumps back from step (11) to step (1) the deciders have the possibility to modify also the weights of the criteria of the set C in order to produce a different set P_e on which to repeat iteratively the steps from (3) to (8).
As a last point we must consider if the recovery step must be modified or not. Since the recovery step define a subset of P_e as containing the projects that, according to the deciders, are worth being reexamined it should be evident how the procedure that we have seen in section 9 can remain unchanged.
As a last step we can show how the use of different weights for the criteria works in the case of our toy example as compared to the use of equal weights.
We recall how in our toy example we assumed to have:

$C = \{c_1, c_2\}$

$P = \{p_1, p_2, p_3, p_4\}$

We may have that:

c_1 represents the implementation costs of a project and has a weight $w_1 = 2/3$,

c_2 represents the functioning costs of a project and has a weight $w_2 = 1/3$.

In this scenario the deciders are more focused on the short term whereas, in the opposite case of $w_1 = 1/3$ and $w_2 = 2/3$, they would be more focused on the long term.
If, as the second step of the procedure that we saw in section 4, we filter the projects of P with c_1 we are interested in the values x_{j1} with $j \in [1, 4]$. In this way we can get, for instance, $P_{e_1} = \{p_1, p_3\}$ since we have $x_{11} = x_{31} < x_{21} \leq x_{41}$. By so doing we are going to not consider any more the projects p_2 and p_4. If we filter the projects of P_{e_1} with c_2 we are interested in the values x_{j2} so that, if we have $x_{32} < x_{12}$ we end up with $P_{e_2} = \{p_3\}$ so that p_3 is the only Pareto equivalent project in this case.
If, on the other hand, we assume equal weights for the criteria of C we have that both p_2 and p_4 are to be included in P_e if we have $x_{22} < x_{42} < x_{32}$ whereas p_1 would be strongly dominated by p_3 and so it would be discarded also in this case.

12 Conclusions and future plans

In this technical report we have presented a procedure for the ranking and selection of a project from a set P of projects through the use of the criteria of the set C that may have either equal or different weights and through the application of concepts of Pareto dominance and Pareto equivalence or indifference. The use of such concepts allows the iterative definition of the set P_e of the Pareto equivalent projects from the set P of the projects that the deciders have to analyze and rank.

The procedure has been presented, examined in details and its properties have been stated, justified and verified.

Future plans include the definition and implementation of a dynamic setting (see section 1) as well as a more strict formalization of the alleged properties of the proposed procedure and its application to more complex and more realistic examples.

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