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Abstract. Exploiting different tangent cones, many derivatives for set–valued functions have been introduced and considered to study optimality. The main goal of the paper is to address a general concept of K–epiderivative and to employ it to develop a quite general scheme for necesary optimality conditions in set–valued problems.

Keywords. Tangent cones, K–derivatives and K–epiderivatives, set–valued optimization, optimality conditions.

Mathematics subject classification. 49K27, 90C29, 90C46.

1 Introduction

In the last years set–valued optimization problems have been considered by many researchers. Besides intrinsic interest, this type of programs arise quite naturally in the context of duality for vector optimization (see, for instance, [18, 23]). Moreover, when the data of a single–valued optimization problem are not exactly known, it is reasonable to replace the values of the involved functions with sets representing their fuzzy outcomes.

In order to study set–valued problems, some notion of derivative for set–valued functions is required. The first concept of derivative had been introduced by Aubin [1], relying on the Bouligand contingent cone: given two real normed linear spaces E_1 and E_2, the contingent derivative of a set–valued function $H : E_1 \rightrightarrows E_2$ at the point (x, y) is the map $D_T^g H(x, y)$ whose graph equals the Bouligand contingent cone T of the graph of H, i.e.

$$\operatorname{graph} D_T^g H(x, y) = T(\operatorname{graph} H, (x, y)) \tag{1}$$

where

$$\operatorname{graph} H := \{(x, y) \in E_1 \times E_2 \ : \ y \in H(x)\}.$$

Even if it was originally employed within the context of differential inclusions, since then many applications also to the study of optimality conditions for vector and set–valued optimization problems have been provided (see for instance [8, 19, 26]). Recently, Jahn and Rauh [16] introduced the contingent epiderivative of a set–valued function, extending the

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concept of "upper contingent derivative" of real–valued ones [1]. The main difference between the definitions of contingent derivative and epiderivative is that the graph is replaced by the epigraph and the epiderivative is single–valued. When E_2 is partially ordered by a pointed convex cone C_{E_2}, a single–valued map $DH(x,y) : E_1 \longrightarrow E_2$ such that

$$\operatorname{epi} DH(x,y) = T(\operatorname{epi} H, (x,y)) \tag{2}$$

is called contingent epiderivative of H, where

$$\operatorname{epi} H := \{(x,y) \in E_1 \times E_2 \ : \ y \in H(x) + C_{E_2}\}.$$

Though single–valuedness seems useful to develop calculus rules [15], we believe that replacing the graph with the epigraph is even more important: approximating just the graph with the contingent cone may not preserve enough information about the function. In fact, as pointed out in [16], necessary and sufficient optimality conditions based on the contingent derivative do not coincide under convexity assumptions. Moreover, sometimes the domain of the contingent derivative is reduced to only one point (see Example 2.2).

We stress that both these derivatives rely on the well–known concept of Bouligand contingent cone to a set. Actually, several kinds of derivatives have been developed exploiting different types of concrete tangent cones [2, 8, 24, 26]. Moreover, relying on standard properties, general definitions of tangent cone have been proposed and employed to define generalized derivatives of real–valued functions [10, 13, 29].

Following these ideas, we propose a definition of generalized epiderivative for set–valued functions and we employ it to achieve a general scheme for necessary optimality conditions of set–valued optimization problems. Finally, we show how already known conditions can be recovered within this scheme quite easily.

2 Tangent cones and K–epiderivatives

We start this section recalling some of the main concrete tangent cones, which are useful in optimization theory. All these cones fall within the following general definition, which has been often considered (see, for instance, [30]).

Definition 2.1. Let E be a real normed linear space. A set–valued mapping $K : 2^E \times E \rightrightarrows E$ is a *tangent cone* on E if for all $A \subseteq E$ and $x \in E$ the set $K(A,x)$ is a cone such that $0^+A \subseteq 0^+K(A,x)$ where

$$0^+A := \{d \in E \ : \ x + td \in A, \ \forall x \in A, \forall t \geq 0\}$$

is the recession cone of A. A tangent cone K is called *local* if for any neighbourhood $I(x)$ of x the following equality holds:

$$K(A \cap I(x), x) = K(A,x).$$

Actually, other concepts of tangent cones can be considered: an axiomatic definition given through six reasonable properties has been proposed in [10] and the mappings satisfying them have been referred to as local cone approximations; another way to introduce general tangent cones is based on the use of quantificational expressions [13, 29].

Definition 2.2. We say that K is

- convex (open, closed) if $K(A,x)$ is convex (open, closed) for all $A \subseteq E$ and $x \in E$;
- convex preserving if $K(A,x)$ is convex whenever A is convex;
- isotone if $A \subseteq B$ implies $K(A,x) \subseteq K(B,x)$.

The disjunction of two sets is a recurrent situation in optimization theory (see Lemma 3.1); in order to analyze it, the knowledge of approximations preserving disjunction can be very helpful.

Definition 2.3. Let K_1, K_2 be tangent cones on E. The pair (K_1, K_2) is *admissible* if for all $A, B \subseteq E$ and for all $x \in E$ the following implication holds:

$$A \cap B = \emptyset \implies K_1(A,x) \cap K_2(B,x) = \emptyset.$$

A standard way to obtain a new tangent cone from a known one is the following [28].

Definition 2.4. Given a tangent cone K on E, let K_c denote the mapping defined by

$$K_c(A,x) := (K(A^c,x))^c, \quad \forall A \subset E,\ x \in E.$$

Remark 2.1. It is easy to check that K_c is actually a tangent cone. Obviously, if K is closed (open), then K_c is open (closed). Moreover, if K is isotone, also K_c is isotone.

Lemma 2.1. *[28] If K is isotone, then the pair (K, K_c) is admissible.*

The family of tangent cones is very wide; we recall the concrete ones that we will use in the following sections. The Bouligand tangent cone is

$$T(A,x) := \{w \in E \ : \ \exists \{t_n\} \downarrow 0,\ \exists \{w_n\} \to w \ \text{ s.t. } \ x + t_n w_n \in A\},$$

and it is well–known [22] that T is closed and isotone. The weak feasible tangent cone is

$$WF(A,x) := \{w \in E \ : \ \exists \{t_n\} \downarrow 0 \ \text{ s.t. } \ x + t_n w \in A\},$$

and it may be neither closed nor open; however, it is always isotone (see [29]). Using Definition 2.4, we obtain the following tangent cones. The Dubovitskij–Miljutin tangent cone is

$$DM(A,x) := T_c(A,x) = \{w \in E \ : \ \forall \{t_n\} \downarrow 0,\ \forall \{w_n\} \to w,\ x + t_n w_n \in A\},$$

and obviously it is open and isotone (see Remark 2.1). The feasible tangent cone is

$$F(A,x) := WF_c(A,x) = \{w \in E \ : \ \forall \{t_n\} \downarrow 0,\ x + t_n w \in A\},$$

and obviously it is isotone. Another widely used tangent cone is the Clarke one [7]:

$$C(A,x) := \{w \in E \ : \ \forall \{t_n\} \downarrow 0, \ \forall \{x_n\} \to_A x, \ \exists \{w_n\} \to w \ \text{ s.t. } \ x_n + t_n w_n \in A\},$$

where $x_n \to_A x$ means $x_n \to x$ and $x_n \in A$. Its main interesting feature is that it is always convex and closed, but unfortunately it is not isotone; however, $C \subseteq T$. Much less popular is the closed radial tangent cone (see, for instance, [11]):

$$R(A,x) := \{w \in E \ : \ \exists \{t_n\} \subseteq (0,+\infty), \ \exists \{w_n\} \to w \ \text{ s.t. } \ x + t_n w_n \in A\} = \operatorname{cl} \operatorname{cone}(A - x).$$

This cone is isotone but obviously it can not be local; moreover, $T \subseteq R$ and they coincide for star–shaped sets.

Since the approach developed in this paper is mainly based on tangent cones of epigraphs (see Definition 2.5), it is interesting to consider also tangent cones on product spaces. Relying on the structure of product spaces, some particular ones can be introduced. However, they seem not to be so popular; to the best of our knowledge, only the following one has been considered in a few papers [24, 25]. Let $A \subseteq E_1 \times E_2$ be a subset of a product space; the Shi tangent cone is

$$\begin{aligned} T_{\mathrm{Shi}}(A,(x,y)) \ := \ & \{(u,v) \in E_1 \times E_2 \ : \ \exists \{t_n\} \subseteq (0,+\infty), \ \exists \{u_n\} \to u, \ \exists \{v_n\} \to v \ \text{ s.t.} \\ & \qquad t_n u_n \to 0, \ (x + t_n u_n, y + t_n v_n) \in A\}. \end{aligned}$$

It is easy to check that S is closed and isotone but not local.

Slightly modifying classical ones, fairly new tangent cones on product spaces can be introduced. For instance, the following ones will be used in the sequel:

- a modified Bouligand tangent cone

$$\begin{aligned} T_m(A,(x,y)) \ := \ & \{(u,v) \in E_1 \times E_2 \ : \ \exists \{t_n\} \downarrow 0, \ \exists \{u_n\} \to u \ \text{ s.t. } \forall \{v_n\} \to v \\ & \qquad (x + t_n u_n, y + t_n v_n) \in A\}; \end{aligned}$$

- a modified Dubovitskij–Miljutin tangent cone

$$\begin{aligned} DM_m(A,(x,y)) := (T_m)_c(A,(x,y)) \ = \ & \{(u,v) \in E_1 \times E_2 \ : \ \forall \{t_n\} \downarrow 0, \ \forall \{u_n\} \to u, \\ & \ \exists \{v_n\} \to v \ \text{ s.t. } (x + t_n u_n, y + t_n v_n) \in A\}. \end{aligned}$$

As the original ones they are isotone.

Remark 2.2. The pairs (F, WF), (T, DM) and (T_m, DM_m) are admissible.

Directional derivatives of real–valued functions have been studied also from a geometrical point of view using conical approximations of the epigraph. The first results in this direction involved the Bouligand and the Clarke tangent cones (see, for instance, [7, 22]) and later on similar ones were developed with other cones [21, 27]. In more recent papers [10, 13, 29] this way to obtain directional derivatives has been developed into an axiomatic approach and optimality conditions have been considered in this framework.

Inspired by the axiomatic approach for the real–valued case [10], we propose the following definition.

Definition 2.5. The K–epiderivative of H at $(x, y) \in \operatorname{graph} H$ is the set–valued map $D_K H(x, y) : E_1 \rightrightarrows E_2$ such that

$$\operatorname{graph} D_K H(x, y) = K(\operatorname{epi} H, (x, y)),$$

that is

$$v \in D_K H(x, y)(u) \iff (u, v) \in K(\operatorname{epi} H, (x, y)).$$

It worth noting that this K–epiderivative does not collapse into the one of [10] when H is a real single–valued map; in fact, the former is always set–valued while the latter involves the infimum of the "vertical" lines of $K(\operatorname{epi} H, (x, y))$ to achieve single–valuedness, i.e.

$$D_K H(x, y)(u) := \inf \{\beta \in \mathbb{R} \ : \ (u, \beta) \in K(\operatorname{epi} H, (x, y))\}.$$

This definition is suitable only for real–valued maps. Since it is equivalent to require

$$K(\operatorname{epi} H, (x, y)) \subseteq \operatorname{epi} D_K H(x, y) \subseteq \operatorname{cl} K(\operatorname{epi} H, (x, y)), \tag{3}$$

it could be reasonable to consider just the above chain of inclusions as a definition of the K–epiderivative also in the set–valued case; such a definition would not guarantee uniqueness. On the contrary, Definition 2.5 allows to achieve uniqueness, satisfying (3); in fact, we have

$$\begin{aligned} \operatorname{epi} D_K H(x, y) &= \operatorname{graph} D_K H(x, y) + (\{0\} \times C_{E_2}) \\ &= K(\operatorname{epi} H, (x, y)) + (\{0\} \times C_{E_2}) \\ &= K(\operatorname{epi} H, (x, y)) \end{aligned}$$

where the last equality is due to the inclusions $\{0\} \times C_{E_2} \subseteq 0^+ \operatorname{epi} H \subseteq 0^+ K(\operatorname{epi} H, (x, y))$. Hence, the reason why it does not collapse in the scalar case is the lack of single–valuedness. To recover it, the contingent epiderivative of a set–valued map has been introduced in [16] but unfortunately it may not exist (see, for instance, the example in [4]); in fact, to achieve single–valuedness, some particular assumptions are needed [6, 15]. Obviously, the contingent epiderivative can not coincide with our T–epiderivative; however, their epigraphs always do.

Relying on (1), also the concept of contingent derivative can be generalized in the same way.

Definition 2.6. The K–derivative of H at $(x, y) \in \operatorname{graph} H$ is the set–valued map $D_K^g H(x, y) : E_1 \rightrightarrows E_2$ such that

$$\operatorname{graph} D_K^g H(x, y) = K(\operatorname{graph} H, (x, y)),$$

that is

$$v \in D_K^g H(x, y)(u) \iff (u, v) \in K(\operatorname{graph} H, (x, y)).$$

It is clear that K–epiderivatives and K–derivatives are strictly connected. In fact, we have

$$D_K H(x, y) = D_K^g (H + C_{E_2})(x, y),$$

where $(H + C_{E_2})(x) := H(x) + C_{E_2}$. However, some differences do exist. For instance, necessary optimality conditions based on K–derivatives do not turn out to be sufficient in the convex case while the ones based on K–epiderivatives do (see [8] for the cases $K = T, C$). Moreover, it may happen that K–derivatives do not exist or have very small domains, as shown by the following examples.

Example 2.1. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be the function

$$H(x) = [x^2, 2x^2].$$

Considering the point $(0, 0)$, we have

$$DM(\operatorname{graph} H, (0,0)) = \emptyset \quad \text{and} \quad DM(\operatorname{epi} H, (0,0)) = \mathbb{R} \times \operatorname{int} \mathbb{R}_+ .$$

Example 2.2. Let $H : \mathbb{R} \rightrightarrows \mathbb{R}$ be the function

$$H(x) = [-2\sqrt{|x|}, -\sqrt{|x|}].$$

Considering the point $(0, 0)$, we have

$$T(\operatorname{graph} H, (0,0)) = \{0\} \times \mathbb{R}_- \quad \text{and} \quad T(\operatorname{epi} H, (0,0)) = \mathbb{R}^2$$

and hence

$$D^g_T H(0,0)(u) = \begin{cases} \mathbb{R}_- & \text{if } u = 0 \\ \emptyset & \text{if } u \neq 0 \end{cases} \quad \text{and} \quad D_T H(0,0)(u) = \mathbb{R}.$$

3 A general approach to optimality conditions

Starting from the seminal results presented in [8], in recent years many authors studied optimality conditions for set–valued problems, using different concepts of derivative. Though the most used one has been the contingent derivative (see [8, 19, 25, 26]), also other ones have been employed [8, 25, 26]. More recently concepts of epiderivatives [4, 6, 11, 12, 16] have been introduced and applied in the same framework.

In this section we aim to present a general scheme to obtain necessary optimality conditions based on K–epiderivatives. Such an approach recalls the one presented in [5] for the real single–valued case.

Let X, Y, Z be real normed linear spaces, $C_Y \subseteq Y$, $C_Z \subseteq Z$ be pointed convex cones with nonempty interior and $F : X \rightrightarrows Y$, $G : X \rightrightarrows Z$ be set–valued functions. We will consider the following constrained set–valued optimization program:

$$\begin{cases} \min_{\operatorname{int} C_Y} F(x) \\ \text{subject to} \\ G(x) \cap -C_Z \neq \emptyset \\ x \in S. \end{cases} \tag{4}$$

For simplicity let $\hat{S} := \{x \in S \ : \ G(x) \cap -C_Z \neq \emptyset\}$ denote the feasible region of problem (4), which is supposed to be nonempty.

Definition 3.1. Let $A \subseteq Y$ be nonempty. A point $a \in A$ is called a *minimal element of* A if

$$(\{a\} - C_Y) \cap A = \{a\};$$

it is called *weak minimal element of* A if

$$(\{a\} - \operatorname{int} C_Y) \cap A = \emptyset.$$

The set of all (weak) minimal elements of A is denoted by $\min_{C_Y} A$ ($\min_{\operatorname{int} C_Y} A$).

Definition 3.2. The pair $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ is a weak minimizer of (4) if $\bar{x} \in \hat{S}$ and $\bar{y}$ is a weak minimal element of $F(\hat{S})$, i.e. there is no $x \in \hat{S}$ such that

$$F(x) \cap (\bar{y} - \operatorname{int} C_Y) \neq \emptyset.$$

Optimality can be characterized as the disjunction of two sets, one of which involving the graph or the epigraph of the cartesian product of the objective and constraining functions.

Lemma 3.1. *Let* $(F, G)(x) := F(x) \times G(x)$ *and* $(\bar{x}, \bar{y}) \in \operatorname{graph} F$*. The following assertions are equivalent:*

(i) $(\bar{x}, \bar{y})$ *is a weak minimizer of (4),*

(ii) $\operatorname{graph}(F, G) \cap [S \times (\bar{y} - \operatorname{int} C_Y) \times -C_Z] = \emptyset$,

(iii) $\operatorname{epi}(F, G) \cap [S \times (\bar{y} - \operatorname{int} C_Y) \times -C_Z] = \emptyset$.

Proof. The equivalence between (i) and (ii) is straightforward and obviously (iii) implies (ii). We just prove that (i) implies (iii). Ab absurdo, suppose that

$$\operatorname{epi}(F, G) \cap [S \times (\bar{y} - \operatorname{int} C_Y) \times -C_Z] \neq \emptyset.$$

Thus, there exist $x \in S,\ y \in F(x) + C_Y,\ z \in G(x) + C_Z$ such that

$$y \in \bar{y} - \operatorname{int} C_Z, \qquad z \in -C_Z.$$

Let $\hat{y} \in F(x)$ and $\hat{z} \in G(x)$ be such that $y \in \hat{y} + C_Y,\ z \in \hat{z} + C_Z$. Then,

$$\begin{array}{rcl} \hat{y} & \in & y - C_Y \subseteq \bar{y} - \operatorname{int} C_Y - C_Y = \bar{y} - \operatorname{int} C_Y, \\ \hat{z} & \in & z - C_Z \subseteq -(C_Z + C_Z) = -C_Z, \end{array}$$

contradicting optimality. □

The above lemma is the key to obtain the following optimality condition based on general K–epiderivatives.

Theorem 3.1. *Let* (K_1, K_2) *be an admissible pair. Suppose there exists* K_3 *tangent cone on* X *such that the inclusion*

$$K_3(S, x) \times -\operatorname{int} C_Y \times -\operatorname{int} \operatorname{cone}(C_Z + z) \subseteq K_2(S \times (y - \operatorname{int} C_Y) \times -C_Z, (x, y, z)) \tag{5}$$

holds for all $x \in S,\ y \in Y,\ z \in -C_Z$. *If* $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ *is a weak minimizer of (4), then for any given* $\bar{z} \in G(\bar{x}) \cap -C_Z$ *the following condition*

$$D_{K_1}(F,G)(\bar{x},(\bar{y},\bar{z}))(u) \cap -[\operatorname{int} C_Y \times \operatorname{int} \operatorname{cone}(C_Z+\bar{z})] = \emptyset \tag{6}$$

holds for all $u \in K_3(S,\bar{x})$.

Proof. The definition of admissible pair implies that given any $\bar{z} \in G(\bar{x}) \cap -C_Z$, we have

$$K_1(\operatorname{epi}(F,G),(\bar{x},\bar{y},\bar{z})) \cap K_2(S \times (\bar{y} - \operatorname{int} C_Y) \times -C_Z,(\bar{x},\bar{y},\bar{z})) = \emptyset.$$

Thus, assumption (5) implies

$$K_1(\operatorname{epi}(F,G),(\bar{x},\bar{y},\bar{z})) \cap (K_3(S,\bar{x}) \times -\operatorname{int} C_Y \times -\operatorname{int}\operatorname{cone}(C_Z+\bar{z})) = \emptyset. \tag{7}$$

Therefore, given any $u \in K_3(S,\bar{x})$ and $v \in D_{K_1}(F,G)(\bar{x},\bar{y})(u)$, we have

$$(u,v) \in K_1(\operatorname{epi}(F,G),(\bar{x},\bar{y},\bar{z}))$$

and thus (7) implies that

$$v \notin -[\operatorname{int} C_Y \times -\operatorname{int}\operatorname{cone}(C_Z+\bar{z})].$$

Hence, (6) follows. □

Actually, assumption (5) is not so restrictive as it may seem. On the contrary, it is quite natural, requiring (roughly speaking) that the tangent cone of a "cylinder" with base S contains the "cylinder" whose base is a suitable tangent cone of S; in fact, this assumption is satisfied by a wide range of tangent cones (see Section 4 for some important cases).

Moreover, when $S = X$, a class of tangent cones such that (5) always holds is given by the local cone approximations. In fact, taking into account that $K_3(X,x) = X$, we have

$$\begin{aligned}
X \times -\operatorname{int} C_Y \times -\operatorname{int}\operatorname{cone}(C_Z+z) &\subseteq \operatorname{int} 0^+(X \times -\operatorname{int} C_Y \times -\operatorname{cone}(C_Z+z)) \\
&\subseteq \operatorname{int} 0^+K_2(X \times -\operatorname{int} C_Y \times -\operatorname{cone}(C_Z+z),(x,0,0)) \\
&\subseteq K_2(X \times -\operatorname{int} C_Y \times -\operatorname{cone}(C_Z+z) \cap I(0),(x,0,0)) \\
&= K_2(X \times -\operatorname{int} C_Y \times -(C_Z+z) \cap I(0),(x,0,0)) \\
&= K_2(X \times (y - \operatorname{int} C_Y) \times -C_Z,(x,y,z))
\end{aligned}$$

where the neighbourbood $I(0)$ is small enough to have $(C_Z+z)\cap I(0) = \operatorname{cone}(C_Z+z)\cap I(0)$. The other crucial steps of the above chain of inclusions are the third and the last one: the former holds since K_2 is local and $\operatorname{int} 0^+C \subseteq C$ whenever C is a cone; while the latter follows from the traslation property of local cone approximations (see [10]).

Remark 3.1. A set–valued optimization problem with no explicit constraint can be viewed as a particular case of problem (4) with $G \equiv \{0\}$; in fact, in such a case, the constraint $G(x) \cap -C_Z \neq \emptyset$ is satified by any $x \in X$. Thus

$$D_{K_1}F(\bar{x},\bar{y})(u) \cap -\operatorname{int} C_Y = \emptyset, \tag{8}$$

is a necessary optimality condition for all $u \in K_3(S,\bar{x})$ whenever (K_1,K_2) is an admissible pair such that there exists K_3 tangent cone on X satisfying

$$K_3(S,x) \times -\operatorname{int} C_Y \subseteq K_2(S \times (y - \operatorname{int} C_Y), (x,y)), \qquad \forall x \in S,\ y \in Y.$$

Remark 3.2. An unconstrained set–valued optimization problem can be viewed as a particular case of problem (4) with $G \equiv \{0\}$ and $S = X$; thus, at a weak minimizer of (4), condition (8) holds for all $u \in X$ whenever K_2 is a local cone approximation such that the pair (K_1,K_2) is admissible.

The optimality condition of Theorem 3.1 is expressed as the impossibility of a suitable system and it recalls the well–known Abadie Linearization Lemma of nonlinear programming; thus, no multipliers are involved. Minumum principle type optimality conditions for problem (4) can be obtained from Theorem 3.1 by applying standard separation arguments; some results in this direction have already been presented in [8] using the C–derivative and in [12] using the contingent epiderivative. The following minimum principle theorem is based on general K–epiderivatives.

To achieve a set of multipliers, we need to recall that the positive dual cone of a given cone $C \subseteq E$ is

$$C^* := \{x^* \in E^* \ :\ \langle x^*, x\rangle \geq 0,\ \forall x \in C\},$$

where $\langle \cdot,\cdot \rangle$ denotes the duality paring between E and its topological dual E^*.

Theorem 3.2. *Let K_1, K_2, K_3 satisfy the assumptions of the previous theorem. Suppose that S and $\operatorname{epi}(F,G)$ are convex, K_1 and K_3 are convex preserving. If $(\bar{x},\bar{y})$ is a weak minimizer of (4), then for all $\bar{z} \in G(\bar{x}) \cap -C_Z$ there exist $t \in C_Y^*,\ s \in C_Z^*$ not both zero such that $\langle s, \bar{z}\rangle = 0$ and*

$$\langle t, v\rangle + \langle s, w\rangle \geq 0$$

for all $(v,w) \in D_{K_1}(F,G)(\bar{x},(\bar{y},\bar{z}))(u)$ and $u \in K_3(S,\bar{x})$.

Proof. Given any $\bar{z} \in G(\bar{x}) \cap -C_Z$, by the previous theorem we have

$$D_{K_1}(F,G)(\bar{x},\bar{y})(K_3(S,\bar{x})) \cap -[\operatorname{int} C_Y \times (\operatorname{int} \operatorname{cone}(C_Z + \bar{z})] = \emptyset \tag{9}$$

where

$$D_{K_1}(F,G)(\bar{x},\bar{y})(K_3(S,\bar{x})) := \bigcup_{u \in K_3(S,\bar{x})} D_{K_1}(F,G)(\bar{x},\bar{y})(u) \tag{10}$$

To prove that the above set is a convex cone, consider any $\lambda_i \geq 0$, $v_i \in D_{K_1}(F,G)(\bar{x},\bar{y})(u_i)$ and $u_i \in K_3(S,\bar{x})$, $i = 1,...,n$. Since

$$(u_i, v_i) \in K_1(\operatorname{epi}(F,G), (\bar{x},(\bar{y},\bar{z}))) \qquad i = 1,...,n$$

and the assumptions imply that $K_1(\operatorname{epi}(F,G),(\bar{x},(\bar{y},\bar{z})))$ is a convex cone, we have

$$\sum_{i=1}^{n} \lambda_i (u_i, v_i) \in K_1(\operatorname{epi}(F,G), (\bar{x},(\bar{y},\bar{z}))).$$

Moreover, also $K_3(S,\bar{x})$ is a convex cone; therefore, we have

$$\sum_{i=1}^{n} \lambda_i v_i \in D_{K_1}(F,G)(\bar{x},\bar{y})(K_3(S,\bar{x})).$$

Hence, (10) is a convex cone and a well–known Separation Theorem (see [14]) implies that there exist $t \in Y^*$ and $s \in Z^*$ not both zero such that

$$\langle t, c_Y \rangle + \langle s, c_Z + \bar{z} \rangle \geq 0, \qquad \forall\, c_Y \in C_Y,\ \forall\, c_Z \in C_Z, \tag{11}$$

$$\langle t, v \rangle + \langle s, w \rangle \geq 0, \qquad \forall\, (v,w) \in D_{K_1}(F,G)(\bar{x},\bar{y},\bar{z})(u),\ \forall\, u \in K_3(S,\bar{x}). \tag{12}$$

It is clear that (11) implies $t \in C_Y^*$, $s \in C_Z^*$ and $\langle s, \bar{z} \rangle \geq 0$. Since $\bar{z} \in -C_Z$, we have also the complementarity slackness condition $\langle s, \bar{z} \rangle = 0$. □

Since separation theorems require convexity, what is really needed in the above theorem is that the sets in (9) are convex; hence, the assumptions on (F,G) and S can be obviously replaced by the convexity of the tangent cones K_1, K_3.

Remark 3.3. The proof of Theorem 3.1 (and consequently also Theorem 3.2) involves characterization (iii) given in Lemma 3.1, which relies on epigraphs; considering instead characterization (ii), we can achieve the analogous results for K–derivatives. However, approximating the epigraph with a tangent cone generally gives more information: in the following example necessary optimality conditions based on the T–epiderivative allow to drop a pair, which is not a weak minimizer, while the ones based on the T–derivative do not. Let $F : \mathbb{R} \rightrightarrows \mathbb{R}$ be the function

$$F(x) = \begin{cases} [2,3] & \text{if } x \in \mathbb{Q} \\ [0,1] & \text{if } x \notin \mathbb{Q}. \end{cases}$$

Obviously, the pair $(0,2)$ is not a weak minimizer of F over $\mathbb{R}$ with respect to the ordering cone $\mathbb{R}_+$. It is easy to check that

$$D_T^g F(0,2)(u) = \mathbb{R}_+ \qquad \text{and} \qquad D_T F(0,2)(u) = \mathbb{R}.$$

Thus, (8) is not satisfied by any $u \in \mathbb{R}$ while the following necessary condition holds

$$D_T^g F(0,2)(u) \cap -\operatorname{int} \mathbb{R}_+ = \emptyset, \qquad \forall u \in \mathbb{R}.$$

4 Some applications

The aim of this section is to show that many known optimality conditions can be obtained applying Theorem 3.1 with some particular choices of the tangent cones.

Since the most popular derivatives are based on Bouligand tangent cone [4, 6, 8, 12, 16], we start considering the choice $K_1 = T$. Thus, exploiting the suggestion in Lemma 2.1, we can take $K_2 = DM$. The following result gives the optimal choice for K_3.

Lemma 4.1. *The equality*

$$DM(A,x) \times -\mathrm{int}\, C_Y \times -\mathrm{int\,cone}\,(C_Z + z) = DM(A \times (y - \mathrm{int}\, C_Y) \times -C_Z, (x,y,z))$$

holds for all $A \subseteq X, x \in X, y \in Y, z \in -C_Z$.

Proof. Since DM is product preserving, i.e.

$$DM(A \times (y - \mathrm{int}\, C_Y) \times -C_Z, (x,y,z)) = DM(A,x) \times DM(-\mathrm{int}\, C_Y, 0) \times DM(-C_Z, z),$$

the thesis follows immediately from the equality

$$DM(C,x) = \mathrm{int\,cone}\,(C - x),$$

that holds for each convex set C and $x \in \mathrm{cl}\, C$. □

In [16] problem (4) with no explicit constraint is studied. Since the authors suppose that F is defined only on the set S, their problem can be equivalently expressed as an unconstrained one with objective function F_S defined as

$$F_S(x) := \begin{cases} F(x) & \text{if } x \in S \\ \emptyset & \text{if } x \notin S. \end{cases}$$

In their Theorem 7 the following necessary optimality condition has been proved

$$DF_S(\bar{x},\bar{y})(x - \bar{x}) \notin -\mathrm{int}\, C_Y, \qquad \forall\, x \in S, \tag{13}$$

where DF_S denotes the contingent epiderivative of F_S as defined in (2). This condition can also be deduced from Theorem 3.1 with our choice and Remark 3.2. In fact, we have

$$D_T F_S(\bar{x},\bar{y})(u) \cap -\mathrm{int}\, C_Y = \emptyset, \qquad \forall u \in X.$$

or equivalently

$$\mathrm{graph}\, D_T F_S(\bar{x},\bar{y}) \cap X \times -\mathrm{int}\, C_Y = \emptyset.$$

Since $\mathrm{graph}\, D_T F_S(\bar{x},\bar{y}) = \mathrm{epi}\, DF_S(\bar{x},\bar{y})$, then also

$$\mathrm{epi}\, DF_S(\bar{x},\bar{y}) \cap (S - \bar{x}) \times -\mathrm{int}\, C_Y = \emptyset$$

which implies (13).

It worth noting that condition (13) does not hold if $F : X \rightrightarrows Y$ and $S \neq X$ even in the real single–valued case, as the following example shows.

Example 4.1. Let $S = \mathbb{R}^2_+$ and $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined as follows

$$F(x) = \begin{cases} \|x\| & \text{if } x_2 \geq 0 \\ -\|x\| & \text{if } x_2 < 0. \end{cases}$$

The point $\bar{x} = (0,0)$ is a global minimum of F on S but, choosing $\bar{u} = (1,0)$, we have

$$DF(\bar{x},0)(\bar{u}) = \liminf_{(u,t)\to(\bar{u},0^+)} t^{-1} F(tu) = -\|\bar{u}\| = -1.$$

The same contingent epiderivative has been exploited in [12] to obtain a minimum principle optimality condition for problem (4). Since T and DM are convex preserving, this result can be deduced from Theorem 3.2 with the same reasoning used to obtain (13).

We notice that if (K_1, K_2) is an admissible pair then also (K_1', K_2) is admissible, whenever $K_1' \subseteq K_1$. Hence, the above choice can be considered with the Clarke tangent cone in place of the Bouligand one. In particular, the seminal results presented in [8] follow immediately, taking into account Remark 3.3.

Another concept of contingent epiderivative has been proposed in [6] and independently in [4].

Definition 4.1. The set–valued map $D_T^{\min} F(\bar{x}, \bar{y}) : X \rightrightarrows Y$ defined as

$$D_T^{\min} F(\bar{x}, \bar{y})(x) := \min\nolimits_{C_Y} \{ y \in Y \ : \ (x, y) \in T(\operatorname{epi} F, (\bar{x}, \bar{y})) \}$$

is called generalized contingent epiderivative of $F : X \rightrightarrows Y$.

Since $\operatorname{graph} D_T^{\min} F(\bar{x}, \bar{y}) \subseteq \operatorname{graph} D_T F(\bar{x}, \bar{y})$, Theorem 5 of [6] follows from Theorem 3.1 in a way analogous to Theorem 7 of [16].

Also necessary conditions for Benson proper minimum points can be recovered within our scheme; we recall that $(\bar{x}, \bar{y}) \in \operatorname{graph} F$ is called a Benson proper minimum point if

$$\operatorname{cl} \operatorname{cone} (F(S) + C_Y - \bar{y}) \cap -C_Y = \{0\}.$$

In [4] such a condition has been presented for problem (4) with no explicit constrains under the assumption that C_Y has a compact base. It is well–known (see [4]) that under this assumption any Benson proper minimum point is a weak one with respect to the ordering given by some convex cone $\hat{C}_Y$ such that $C_Y \setminus \{0\} \subseteq \operatorname{int} \hat{C}_Y$. Thus, by Theorem 3.1 we get that

$$D_T F(\bar{x}, \bar{y})(u) \cap -\operatorname{int} \hat{C}_Y = \emptyset, \qquad \forall u \in X,$$

and Theorem 1 of [4] follows.

Also other types of derivative have been employed in set–valued optimization; for instance, optimality conditions for problem (4) have been presented in [26], relying on the lower Dini derivative introduced in [20]:

$$\underline{D} F(\bar{x}, \bar{y})(\bar{u}) = \liminf_{(t,u) \to (0^+, \bar{u})} t^{-1} [F(\bar{x} + tu) - \bar{y}]$$

where $\liminf$ is intended in the sense of set–valued analysis [2], i.e.

$$\liminf_{n \to +\infty} A_n := \left\{ x \in X \ : \ \lim_{n \to +\infty} \inf_{y \in A_n} ||x - y|| = 0 \right\}.$$

It is easy to check that

$$\underline{D} F(\bar{x}, \bar{y})(u) + C_Y \subseteq D_{DM_m} F(\bar{x}, \bar{y})(u) = \underline{D}(F + C_Y)(\bar{x}, \bar{y})(u), \qquad \forall u \in X. \tag{14}$$

Thus, let us consider the choice $K_1 = DM_m$; exploiting Lemma 2.1, we can take $K_2 = T_m$. Since these cones are defined on product spaces, in the following we assume that the first space E_1 is always equal to X. The following result suggests a possible choice for K_3.

Lemma 4.2. *The inclusion*

$$T(A,x) \times -\mathrm{int}\, C_Y \times -\mathrm{int\,cone}\,(C_Z + z) \subseteq T_m(A \times (y - \mathrm{int}\, C_Y) \times -C_Z, (x,y,z))$$

holds for all $A \subseteq X, x \in X, y \in Y, z \in -C_Z$.

Proof. Let $(u,v,w) \in T(A,x) \times -\mathrm{int}\, C_Y \times -\mathrm{int\,cone}\,(C_Z + z)$; then, there exist two sequences $\{t_n\} \downarrow 0$ and $\{u_n\} \to u$ such that $x + t_n u_n \in A$. Let $\{v_n\}$ and $\{w_n\}$ be any two sequences, which converge respectively to v and w; hence, $y + t_n v_n \in y - \mathrm{int}\, C_Y$ and

$$z + t_n w_n \in z - \mathrm{int\,cone}\,(C_Z + z) \subseteq z - (C_Z + z) = -C_Z,$$

which proves the inclusion. □

Applying Theorem 3.1 to problem (4) with no explicit constraints (see Remark 3.1) and exploiting our choice, we get

$$D_{DM_m} F(\bar{x},\bar{y})(u) \cap -\mathrm{int}\, C_Y = \emptyset, \qquad \forall u \in T(S,\bar{x}).$$

Hence, (14) implies the following necessary optimality condition

$$\underline{D} F(\bar{x},\bar{y})(u) \cap -\mathrm{int}\, C_Y = \emptyset, \qquad \forall u \in T(S,\bar{x}).$$

presented in [26] as Proposition 3.1.

Moreover, in the same paper problem (4) has been considered with explicit constraints. Thus, Theorem 4.1 of [26] follows arguing as above and noting that

$$D_{DM_m}(F,G)(\bar{x},(\bar{y},\bar{z}))(u) = (D_{DM_m} F(\bar{x},\bar{y})(u), D_{DM_m} G(\bar{x},\bar{y})(u)).$$

Though introduced with a different aim [24], the Shi tangent cone has been used to obtain optimality conditions in [25] for unconstrained programs. Since it is isotone, we can choose $K_1 = T_{\mathrm{Shi}}$ and $K_2 = (T_{\mathrm{Shi}})_c$ to get

$$D_{T_{\mathrm{Shi}}}(F,G)(\bar{x},(\bar{y},\bar{z}))(u) \cap -[\mathrm{int}\, C_Y \times \mathrm{int\,cone}\,(C_Z + \bar{z})] = \emptyset, \qquad \forall u \in DM(S,\bar{x})$$

as a necessary optimality condition for problem (4). This follows from Theorem 3.1 with the choice $K_3 = DM$ suggested by the inclusion

$$DM(S,\bar{x}) \times -\mathrm{int}\, C_Y \times -\mathrm{int\,cone}\,(C_Z + \bar{z}) \subseteq (T_{\mathrm{Shi}})_c(S \times (\bar{y} - \mathrm{int}\, C_Y) \times -C_Z, (\bar{x},\bar{y},\bar{z})),$$

which can be proved just as Lemma 4.2.

Remark 4.1. For the sake of simplicity, we showed optimality conditions only for (global) weak minimizers; however, we stress that all the results presented in this paper still hold for local weak minimizers, just supposing that at least one of the involved tangent cones is local. In fact, the characterizations of Lemma 3.1 hold for local optima if disjunction is achieved considering the two sets and a suitable cylinder neighbourhood $I(\bar{x}) \times Y \times Z$. Moreover, it worth noting that all the concrete tangent cones, we considered, are local expect the Shi one: however, we stress that it satisfies the local property for the above type of cylinder neighbourhoods.

5 Conclusion

Relying on the concept of K–epiderivative of a set–valued map, we have proved necessary optimality conditions in a quite general form. In the last section we have shown how some concrete ones can be easily obtained with suitable choices. Actually, many others could be developed in an analogous way. For instance, the closed radial tangent cone R could be used following the ideas in [11] or the pair (F, WF) could be considered as already pointed out in [5] for the real single–valued case.

We stress that not all the known optimality conditions fall within this approach. In fact, it is based only on derivatives and tangent cones while some results rely on suitable subdifferentials; for instance, a quite different approach has been proposed in [9], considering the approximate subdifferential of a suitable distance function.

Moreover, some concepts of subdifferential for a set–valued function [3, 6, 17, 23] have also been considered; since they are naturally related to contingent derivatives and epiderivatives, they are surely connected with our approach and relationships could be the core of future investigations.

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