Università di Pisa

Dipartimento di Informatica

Technical Report: TR-01-15

What is duality gap in vector optimization ?

Giancarlo Bigi   Massimo Pappalardo

July 18, 2001

ADDRESS: Corso Italia 40, 56125 Pisa, Italy.   TEL: +39 050 2212700   FAX: +39 050 2212726

What is duality gap in vector optimization ?

Giancarlo Bigi[a,1], Massimo Pappalardo[b]

[a] *Dipartimento di Informatica, Università di Pisa*
Corso Italia, 40, 56125 Pisa, Italia.

[b] *Dipartimento di Matematica Applicata, Università di Pisa*
Via Bonanno Pisano, 25/b, 56126 Pisa, Italia.

Abstract.

Since a vector program has not just an optimal value but a set of optimal ones, the analysis of duality gaps requires at least the comparison between two sets of vector optimal values. Relying only on a weak duality property, the situations that can occur are analysed in detail and some concepts of duality gap are proposed.

Keywords: vector optimization, duality gaps, Lagrangian duality.

1 Introduction

Duality is one of the most important topics in optimization both from a theoretical and algorithmic point of view. In scalar optimization, one generally looks for a dual problem in a way such that the difference between the optimal values is small and possibly zero. This difference is usually called duality gap. However, such a definition cannot be applied to vector optimization, since a vector program has not just an optimal value but a set of optimal ones. A first attempt to analyse the vector case appeared in [1] but it was based on scalarization techniques and the considered duality gaps were between scalar problems. Later on, Tanino and Sawaragi [4] developed a Lagrangian duality theory for vector optimization problems: they established weak duality in the general case and a strong duality theorem under

[1] Corresponding author. Tel: +39 050 2212789; fax: +39 050 2212726.
E-mail address: giancarlo.bigi@di.unipi.it.

cone–convexity and regularity assumptions. Since then, a large number of papers dealing with duality for vector optimization have been published. However, to the best of our knowledge, studies about the duality gap have not been developed. The first difficulty to overcome is the definition of duality gap itself: the aim of this note is to address possible answers to this question. Relying on a weak duality property, the relationships that may hold between the sets of optimal values of the given problem and its dual are analysed in detail; therefore, some concepts of duality gap are proposed and some examples within the Lagrangian duality framework are presented.

2 Duality Gaps

In this section the main situations that can occur in a general vector optimization framework are addressed and some duality gap measures are introduced in order to distinguish them.

Let E be a real normed linear space and $C \subseteq E$ a pointed convex cone; such a cone induces a partial ordering $\geq_C$ on E in the following way: for any pair $y_1, y_2 \in E$ the inequality $y_1 \geq_C y_2$ means $y_1 - y_2 \in C$; moreover, the strict inequality $y_1 >_C y_2$ means $y_1 - y_2 \in C \setminus \{0\}$. Using this ordering, the following concept of minimal (maximal) element of a given set is generally considered in vector optimization.

Definition 2.1. Let $A \subseteq E$ be nonempty; a point $\bar{x} \in A$ is a minimal (maximal) element of A if there is no $x \in A$ such that $\bar{x} >_C x$ ($x >_C \bar{x}$).

We will denote by $\min_C A$ ($\max_C A$) the set of all minimal (maximal) elements of A.

In many concrete duality schemes the dual problem is actually a set–valued program, i.e. vector optimization is considered with respect to a set–valued objective. Thus, we suppose also the primal problem to have a set–valued objective. Hence, given a nonempty set X and a set–valued mapping $f : X \rightrightarrows E$, we consider the following vector optimization problem

$$(P) \quad \min_C f(x) \quad \text{subject to} \quad x \in X.$$

Definition 2.2. $\bar{x} \in X$ is a minimum point of (P) if $f(\bar{x}) \cap \min_C f(X) \neq \emptyset$.

Moreover, the minimal elements of $f(X)$ will be referred to as optimal values of (P) and analogous definitions can be introduced for maximization problems.

Though many different duality schemes have been developed, they all satisfy the following weak duality property (see also the picture below): no value of the dual of (P) is greater (in the sense of $>_C$) than any value of (P), i.e.

$$d(Y) \cap (f(X) + C \setminus \{0\}) = \emptyset \tag{1}$$

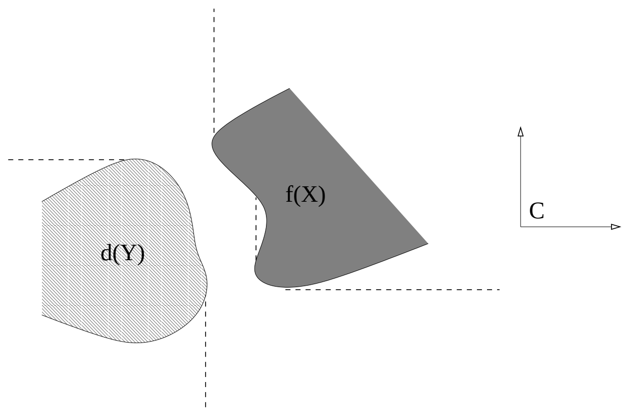
or equivalently

$$f(X) \cap (d(Y) - C \setminus \{0\}) = \emptyset \tag{2}$$

where $d : Y \rightrightarrows E$ is the dual mapping and Y is the set of dual variables. Thus, to achieve full generality, in the following we consider (P) together with the vector maximization problem

$$(D) \quad \max_C d(y) \quad \text{subject to} \quad y \in Y \tag{3}$$

and we just suppose that (1) holds.



Proposition 2.1. *Let $\bar{x} \in X$ and $\bar{y} \in Y$. If $f(\bar{x}) \cap d(\bar{y}) \neq \emptyset$, then $\bar{x}$ is a minimum point of (P) and $\bar{y}$ is a maximum point of (D) and all the elements of $f(\bar{x}) \cap d(\bar{y})$ are optimal values both for (P) and (D).*

Proof. Consider any $\bar{z} \in f(\bar{x}) \cap d(\bar{y})$. Ab absurdo, suppose that it is not an optimal value for (P); thus, there exists $y \in f(X)$ such that $\bar{z} \in y + C \setminus \{0\}$. Hence, (1) implies that $\bar{z} \notin d(Y)$ in contradiction with the assumption. Arguing in the same way, we can prove that $\bar{z}$ is an optimal value also for (D). □

Therefore, if the images have nonempty intersection, i.e.

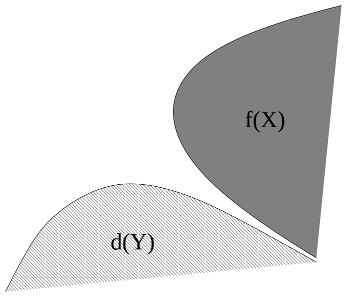
$$d(Y) \cap f(X) \neq \emptyset, \tag{4}$$

the two problems possess at least a common optimal value. To analyse whether situation (4) occurs or not, we can measure the distance between these two sets; hence, let us consider the quantity

$$\Delta_1 := \inf \{ \ ||v - u|| : \ v \in d(Y), \ u \in f(X) \ \},$$

where $|| \cdot ||$ denotes the norm E is endowed with.

Obviously, $\Delta_1 = 0$ if (4) holds. Unfortunately, the vice versa does not hold, unless suitable compactness assumptions are made. Thus, Δ_1 be may be zero even if no common value exists. However, by definition $\Delta_1 = 0$ means that the two objective functions achieve values whose difference has arbitrary small norm: in scalar optimization (i.e. $E = \mathbb{R}$) this means exactly that the optimal values coincide but in the vector framework this is not the case, as the following picture suggests.



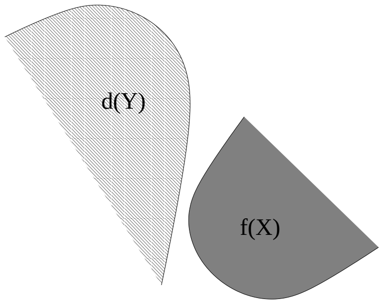
It worth noting that (4) is actually equivalent to the similar condition

$$\max_C \ d(Y) \cap \min_C \ f(X) \neq \emptyset. \tag{5}$$

However, they are some disadvantages to consider the distance between these two sets, i.e.

$$\Delta_1' := \inf \{ \ ||v - u|| : \ v \in \max_C \ d(Y), \ u \in \min_C \ f(X) \ \},$$

instead of Δ_1. In fact, Δ_1' requires the set of optimal values to be nonempty, Δ_1' can be zero even when (5) does not hold, Δ_1' is greater than Δ_1 and above all Δ_1' may be arbitrary large even if Δ_1 is small, as the following picture suggests.



We stress that in scalar optimization we always have $\Delta_1 = \Delta_1'$ since optimal values are unique. In the vector case, there is not uniqueness and therefore the situation is more complex: Δ_1 is zero whenever a common (optimal) value exists although there may be other values which are optimal for the given problem but not for its dual or vice versa. In fact,

$$\max_C d(Y) = \min_C f(X) \tag{6}$$

is a much stronger duality requirement than (4), although they coincide in the scalar case. Thus, in order to study whether all optimal values are common or not, the following quantity can be considered:

$$\Delta_2 := \max\{ \Delta_2^P, \ \Delta_2^D \},$$

where

$$\Delta_2^P := \sup \{ \inf \{ ||v-u|| : \ v \in d(Y) \} \ : \ u \in \min_C f(X) \}$$

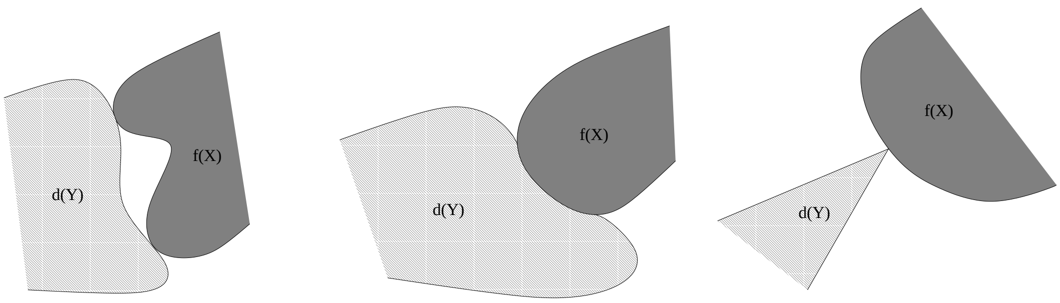
and

$$\Delta_2^D := \sup \{ \inf \{ ||v-u|| : \ u \in f(X) \} \ : \ v \in \max_C d(Y) \}.$$

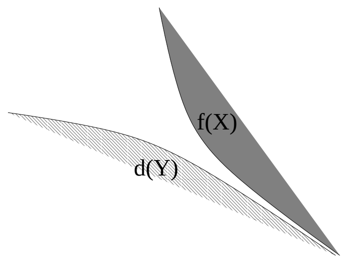
We stress that these definitions require that optimal values of the considered problem exist. If they do, obviously we have

$$\Delta_1 \leq \Delta_2^P, \quad \Delta_1 \leq \Delta_2^D.$$

Moreover, as for Δ_1 and (4), we have that $\Delta_2 = 0$ if all optimal values are common to both problems (i.e. (6) holds) but not vice versa. However, by definition $\Delta_2 = 0$ implies that an optimal value, if it is not a common one, is nevertheless the limit of a sequence of values of the other problem. Thus, when the images are closed sets, $\Delta_2 = 0$ and (6) are actually equivalent. Obviously, the quantities Δ_2^P and Δ_2^D provide the same kind of information of Δ_2 but only with respect to the optimal values of the corresponding problem. Situation where $\Delta_2 > 0$ can hence be achieved in three different ways, as shown by the following pictures.



Moreover, it may also happen that $\Delta_2 = +\infty$ as the following picture suggests.



3 Examples in Lagrangian Duality

The aim of this section is to make clearer the ideas discussed in the previous one through some examples within the classical Lagrangian duality scheme (see [3, 4, 5]). We recall that in this scheme the primal problem (P) has a single–valued objective $f : S \to E$ and the constraint is described as follows

$$X = \{ \ x \in S \ : \ -g(x) \in Q \ \},$$

where $g : S \to F$ with F a real normed linear space, $Q \subseteq F$ a convex cone and S a set. Relying on the Lagrangian function

$$L(x, \Lambda) := f(x) + \Lambda g(x),$$

where Λ is a continuous linear mapping from F to E carrying Q into C, i.e.

$$\Lambda \in \mathcal{L}(F, E) \quad \text{and} \quad \Lambda(Q) \subseteq C, \tag{7}$$

a dual problem (D) is built as (3) with Y being the set of all mappings Λ satisfying (7) and with the dual set–valued map given by

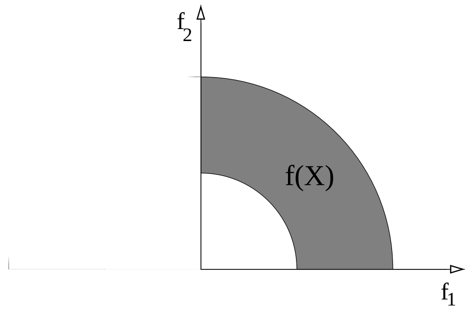
$$d(\Lambda) := \min_C \ \{ \ L(x, \Lambda) \ : \ x \in S \ \}.$$

It is well–known [3, 4, 5] that such a pair $(P) - (D)$ satisfies (1) and that, under cone-convexity and some regularity assumptions, all optimal values of the primal (P) are common with the dual (D), i.e. $\Delta_2^P = 0$.

In all the following examples we will consider a bicriteria objective (i.e. $E = \mathbb{R}^2$) of two real variables (i.e. $S \subseteq \mathbb{R}^2$) subject to just one inequality constraint (i.e. $F = \mathbb{R}$ and $Q = \mathbb{R}_+$); therefore, the space of dual variables will be $Y = \mathbb{R}^2_+$. We will use both the so called Pareto (i.e. $C = \mathbb{R}^2_+$) and weak Pareto (i.e. $C = \operatorname{int} \mathbb{R}^2_+ \cup \{0\}$) orderings.

Example 3.1. Let

$$f(x_1, x_2) = (x_2 \cos x_1, x_2 \sin x_1), \quad g(x_1, x_2) = 1 - x_2, \quad S = [0, \pi/2] \times [0, 2].$$



Considering $C = \operatorname{int} \mathbb{R}^2_+ \cup \{0\}$, the set of optimal values of (P) is

$$(\{0\} \times [1,2]) \cup ([1,2] \times \{0\}) \cup \{ \ (\cos\theta, \sin\theta) \ : \ \theta \in [0, \pi/2] \ \}.$$

Obviously, we have

$$d(0,0) = ([0,2] \times \{0\}) \cup (\{0\} \times [0,2]).$$

Thus, (4) holds and $\Delta_1 = 0$. On the contrary, we can prove that

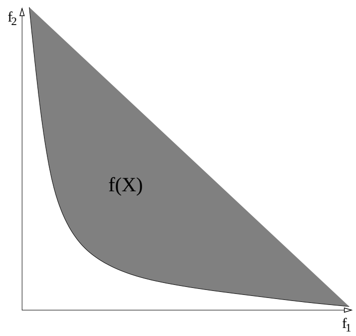
$$d(Y) \cap \mathbb{R}^2_+ = d(0,0);$$

hence, $\Delta_2^P > 0$ and $\Delta_2^D > 0$.

Example 3.2. It can be shown that considering $C = \mathbb{R}^2_+$ we get also $\Delta_1 > 0$.

Example 3.3. Let

$$f(x_1, x_2) = (x_1, x_2), \quad g(x_1, x_2) = 1 - x_1 x_2, \quad S = \mathbb{R}^2_+.$$



Considering $C = \mathbb{R}^2_+$, the set of optimal values of (P) is

$$\{ \ (f_1, f_2) \in \mathbb{R}^2_+ \ : \ f_1 f_2 = 1 \ \}.$$

It is easy to evaluate that

$$\begin{array}{ll} d(\lambda_1,\lambda_2) = (\lambda_1,\lambda_2) & \text{if } \lambda_1\lambda_2 = 0 \\ d(\lambda_1,\lambda_2) \cap \mathbb{R}^2_+ = \emptyset & \text{if } \lambda_1\lambda_2 \neq 0 \end{array}$$

Thus,

$$\mathbb{R}^2_+ \setminus \operatorname{int} \mathbb{R}^2_+ \subseteq d(Y) \subseteq \mathbb{R}^2 \setminus \operatorname{int} \mathbb{R}^2_+$$

and hence $\Delta_1 = 0$ although (4) does not hold; moreover, it is easy to check that $\Delta_2^P = 1$ and $\Delta_2^D = -\infty$ (adopting the usual notation $\sup \emptyset = -\infty$).

Example 3.4. Consider the previous example with $C = \operatorname{int} \mathbb{R}^2_+ \cup \{0\}$; the set of optimal values of (P) does not change, but we have

$$d(0,\lambda_2) = (\{0\} \times [\lambda_2,+\infty)) \cup ([0,1/\lambda_2) \times \{\lambda_2\})$$

$$d(\lambda_1,0) = ([\lambda_2,+\infty) \times \{0\}) \cup (\{\lambda_1\} \times [0,1/\lambda_2))$$

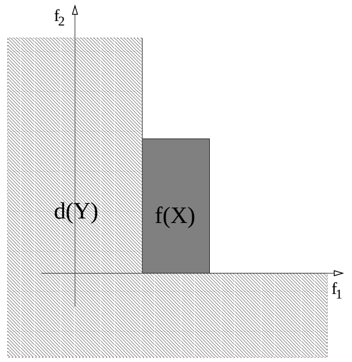
for any $\lambda_1,\lambda_2 \in \mathbb{R}_+$; once again for any $(\lambda_1,\lambda_2) \in \operatorname{int} \mathbb{R}^2_+$ we have

$$d(\lambda_1,\lambda_2) \cap \mathbb{R}^2_+ = \emptyset.$$

Thus, $\Delta_1 = \Delta_2 = 0$ although both (4) and (6) do not hold.

Example 3.5. Let

$$f(x_1,x_2) = (x_1,x_2), \quad g(x_1,x_2) = 1 - x_1, \quad S = [0,2] \times [0,2].$$



Considering $C = \operatorname{int} \mathbb{R}^2_+ \cup \{0\}$, the set of optimal values of (P) is

$$([1,2] \times \{0\}) \cup (\{1\} \times [0,2]).$$

It is easy to check that $\Delta_1 = \Delta_2^P = 0$ (the functions are linear) but $\Delta_2^D = +\infty$.

Example 3.6. Considering $C = \mathbb{R}^2_+$, we get also $\Delta_2^D = 0$.

References

[1] Di Guglielmo, F. (1977), "Nonconvex Duality in Multiobjective Optimization", *Mathematics of Operations Research*, Vol. 2, pp. 285-291.

[2] Luc, D.T. and Jahn, J. (1991), "Axiomatic Approach to Duality in Optimization", *Numerical Functional Analysis and Optimization*, Vol. 13, pp. 305-326.

[3] Sawaragi, Y., Nakayama, H. and Tanino, T. (1985), *Theory of Multiobjective Optimization*, Academic, Orlando.

[4] Tanino, T. and Sawaragi, Y. (1979), "Duality Theory in Multiobjective Programming", *Journal of Optimization Theory and Applications*, Vol. 27, pp. 509-529.

[5] Wang, S.Y. and Li, Z.F. (1992), "Scalarization and Lagrange Duality in Multiobjective Optimization", *Optimization*, Vol. 26, pp. 315-324.