Università di Pisa
Dipartimento di Informatica

Technical Report: TR-01-16

Some New Results for k-Dense Trees: Graph Theory

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4 September 2001

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Abstract

Let k be a positive integer. A *k-cycle* is a connected graph in which each vertex has degree greater than k. A *k-dense forest* is a graph for which no subgraph is a k-cycle; if a k-dense forest is connected, then it is *k-dense tree*. A *k-leaf* is a vertex of a k-dense forest with degree less than or equal to k. Any k-dense forest has at least one k-leaf. If a k-leaf is removed, the resulting graph is still a k-dense forest. This fact is on the basis of another characterization of k-dense forests which make use of the concept of *k-elimination*, a particular ordering of removal for the vertices of a k-dense forest. In this paper, we study some new properties of the *complete* k-dense trees, a subclass of the one of k-dense trees. Such properties reveal some interesting relations between the class of complete k-dense trees and the widely studied class of k-trees.

Keywords

Graph Theory, k-cycle, k-dense forest, k-dense tree, complete k-dense tree, k-elimination, k-tree, triangulated graphs, triangulated constituent, connectivity, k-connected, k-edge-connected, k-linked, core, partial order.

1 Introduction

In this work we introduce "some new results for k-dense trees", as the title claim, pertinent to the pure Graph Theory. Before beginning the formal presentation of the results, we give a small summary about this quite young class of graphs. The concepts relative to Graph Theory that we are going to use in this context will be formally defined in §2.

Let k be a positive integer.

- A *cycle* is a connected graph in which each vertex has the degree equal to 2.
- A *k-cycle* is a connected graph in which each vertex has the degree greater or equal to $k+1$

Therefore, a cycle is also a 1-cycle. Obviously the converse is not true; consider, for instance, a graph formed by two cycles having one vertex v in common: this graph is a 1-cycle, but it is not a cycle, since the degree of vertex v is greater than 2.

- An undirected graph is a *tree* if it is connected and if none of its subgraph is a cycle.
- An undirected graph is a *k-dense tree* if it is connected and if none of its subgraph is a k-cycle.

Therefore, the concept of k-dense tree is a generalization of the classic concept of tree; this generalization is based on a natural extension (k-cycle) of the concept of cycle.

Although k-cycles and k-dense trees are theoretical objects of recent introduction (in 1999, see [LP99]), there were some insights, strictly correlated with such classes of graphs, that began to appear since 1968. In that year there were two works that were drawing near the concept of k-cycle for the first time: [SW68] by Szekeres and Wilf and [Mat68] by Matula. In both works the problem of graphs' colouring was considered and a generic upper bound of chromatic number was proved.

In 1982 Freuder, in [Fre82], based on the ideas contained in [SW68] and [Mat68], extended them to the more general case of constraints satisfaction problems. The Freuder's work is not related to Graph Theory, the author studied the sufficient conditions for which a generic ordering of the variables of a problem allow to find a solution through a *backtrack-free* search. The results of that Freuder's work are constructed on the concept of *width* of a *constraint graph* and on the concept of "linkage" of a subgraph of a "constraint graph". These two concept are equivalent to those of k-dense tree and k-cycle: if we extend the Freuder's terminology, we can say that a k-dense tree is a graph with width equal to k and that a k-cycle is a graph with linkage equal to $k+1$.

As we already know, the k-dense trees appeared, in the form that we saw at the beginning of this section, in 1999 in [LP99].We list below the contributions of that work:

- definitions of k-cycle, k-dense tree, k-elimination, and complete k-dense tree
- some properties of 2-dense trees,
- algorithm of construction (if it exists) of a spanning complete 2-dense tree for a generic graph,
- approximate algorithm of construction of a spanning 2-dense tree with the maximum number of edge for a generic graph,

The second work about k-dense trees is [LMP00]. In that paper a distributed data structure called *Dense Search Tree* was introduced (DST for short). A DST is constructed from a graph consisting of k identical binary trees: for any two not adjacent vertices, an edge is added between them if and only if the two vertices are contained in two different trees and their positions in this trees are identical. The result of this construction is a k-dense tree. Moreover, a variation of this scheme was contempled; this variation, called *reinforced dst*, has a greater number of edges.

We were pleased to contribute with some of the results of this work (or better, of the Master's thesis on which the present work is based) towards another paper about k-dense trees ([PFLP01]). Here, some new theoretic results for the k-dense trees and the k-trees are introduced, such as the Theorem of the Core, (see Theorem 7.1) together with two forms of decomposition for the k-dense trees, both obtainable in linear time.

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This paper is organized as follows. We begin (§2) with a short introduction to the Graph Theory in which we give a set of fundamental notations and we formally define the basic concepts used in the remainder of this work. In §3 we make a quick presentation of part of the basic theory concerning the class of k-dense trees; the whole theory is contained in [Fra01]. In §4 we present some results about the connectivity of the complete k-dense trees: we show that any complete k-dense tree is k-connected, k-edge-connected and $\lfloor \frac{k+1}{2} \rfloor$-linked. In §5 we introduce two properties of those subgraphs of a complete k-dense tree that are complete k-dense trees in their turn. In §6 we define the concept of *triangulated constituent* of a complete k-dense tree, which is a fundamental element of the remainder of this work, and we prove some properties of maximal triangulated constituents of a complete k-dense tree: the three most important of the above properties give some strong constraints on the set of vertices shared by any two maximal triangulated constituents of a complete k-dense tree. In §7 we are able to prove that any complete k-dense tree T own a particular maximal triangulated constituent (the *core* of T) such that each k-clique C of T is included in it if and only if C is a seed of T. In §8 we are able to prove (Theorem of the order) that any two adjacent vertices of a complete k-dense tree, not both included in the core, are in the same relative order in any k-elimination for the complete k-dense tree. This property does not hold for the k-trees for which however we present a similar but more weak property. In §9 we prove that any non-trivial maximal triangulated constituent of a complete k-dense tree own a k-clique with a particular property. After acquiring the above results, in §10 we can face up the problem of the search for the maximal triangulated constituents of a complete k-dense tree. We introduce an algorithm that, given a complete k-dense tree T and a k-elimination for this, finds all the maximal triangulated constituents of T in $O\left(k2^k\left|T\right|\right)$ time, that is in linear time on the order of the graph. We moreover introduce an algorithm which finds the core of a complete k-dense tree T in $O\left(k^2\left|T\right|\right)$ time. The remainder of this work (§11) is entirely devoted to the study of a particular partial order, defined on the vertices of a generic complete k-dense tree by the set of k-elimination for such graph. At the end of the section, we will have a complete description of the partial order which specify (together with the Theorem of the core and the Theorem of the order) the constraints that any k-elimination for a complete k-dense tree must satisfy.

2 Basic Definitions and Notations

Let I be a set, by $[I]^r$ we denote the set of all subsets of I with cardinality equal to r.

An *undirected graph* G is a pair (V, A) of sets satisfying $A \subseteq [V]^2$.

A *directed graph* (or *oriented*) G is a pair (V, A) of disjoint sets with two functions:

$$t : A \to V \quad \text{and} \quad h : A \to V$$

which assign to each $a \in A$ a *tail* $t\left(a\right)$ and a *head* $h\left(a\right)$.

In both cases the elements of V are the *vertices* (or *nodes*) of G, whereas the element of A are the *edges* (or *arcs*).

Given a graph G, the set of its vertices will be denoted by $V\left(G\right)$, whereas by $E\left(G\right)$ we will denote the set of edges of G.

The number of the vertices of a graph is its *order*. The order of a graph G will be denoted by $|G|$ and, similarly, we will indicate the number of edges of G with $\|G\|$.

Let G be a graph and let a be an edge of G, the definition of *ends* of a vary slightly in the two kinds of graphs that we have just introduced. Let u and v be two vertices ($u \neq v$): in case G is undirected, u and v are the *ends* of a if $a = \{u, v\}$; whereas, if G is directed, u and v are the *ends* of a if $\{t\left(a\right), h\left(a\right)\} = \{u, v\}$.

An edge of a generic graph is *incident* in a vertex if the vertex is one of its ends. Extending the notation given to denote the set of the edges of a graph G, we will denote the set of the edges incident in a vertex u, by $E\left(u\right)$.

Given a graph G, an edge a with x and y as ends will be denoted by xy, quite apart from the kind of the graph and only in order to highlight its ends. If G is an undirected graph, we could use the notation $\{x, y\}$. Finally, in case G is a directed graph, the edge a could be denoted by (x, y) if $t\left(a\right) = x$ and $h\left(a\right) = y$. An edge a of a directed graph will be represented graphically by an arrow directed to the vertex $h\left(a\right)$.

Two vertices of a generic graph are *adjacent* if such vertices are the ends of an edge, that is, by using the notation just introduced, if $E\left(u\right) \cap E\left(v\right) \neq \emptyset$. Two edges are *adjacent* if they share an end.

Let $G := (V, A)$ and $G' := (V', A')$ be two undirected graphs; G and G' are *isomorphic*, or more synthetically $G \simeq G'$, if there is a biunique function $\varphi : V \to V'$ such that:

$$xy \in A \iff \varphi\left(x\right)\varphi\left(y\right) \in A'$$

for every $x, y \in V$. Such a function is an *isomorphism* between G and G'.

An undirected graph G is *complete* if for any two vertices x, y of G we have that $xy \in E\left(G\right)$. A complete graph with r vertices is usually denoted by K^r. It can be easily derived from the definitions that a complete graph K^r has $\frac{r(r-1)}{2}$ edges.

A set of vertices or edges of a graph is *independent* (o *stable*) if every pair of elements of the set is not constituted by adjacent objects.

Let $G := (V, A)$ and $G' := (V', A')$ be undirected graphs. If $V' \subseteq V$ and $A' \subseteq A$ then G' is a *subgraph* of G (and G is a *supergraph* di G').

Let $G := (V, A)$ and $G' := (V', A')$ be directed graphs. If $V' \subseteq V$, $A' \subseteq A$ and every edge in A' has its ends in V' then G' is a is a *subgraph* of G (and G is a *supergraph* di G'); obviously, the functions associated with G' ($t_{G'}$ and $h_{G'}$) are the ones of G restricted to A'.

The relation "G' is a subgraph of G" will be denoted by $G' \subseteq_{\mathcal{G}} G$ or it will be indicated by saying that "G' is contained by G". If $G' \subseteq_{\mathcal{G}} G$ and $G' \neq G$ then G' is a *proper* or *non-trivial* subgraph of G (the same holds for G as supergraph of G') and in this case we will use the notation $G' \subset_{\mathcal{G}} G$. A graph $G'' := (V'', A'')$ such that $V'' \subseteq V$ and

$$A'' := \{xy \in A \mid x, y \in V''\}$$

is an *induced subgraph* of G and, in particular, is *the* subgraph of G induced by V''. In order to denote the subgraph of G by a set of vertices M we will write G_M. A graph $G''' := (V, A''')$ such that $A''' \subseteq A$ is a *partial subgraph* (or *spanning subgraph*) of G. Given a graph G, a set of r vertices in $V(G)$ which induces on G a complete subgraph is an *r-clique*.

We will say that a graph G is *minimal* (*maximal*) with some property $\mathbb{P}$ if no subgraph (supergraph) of G exists for which $\mathbb{P}$ holds, by refering to the relation $\subseteq_{\mathcal{G}}$ implicitly. In a similar way, we will refer to the set inclusion relation by writing about minimal or maximal sets of vertices or edges.

Let G be a graph, and let u be a vertex of G. The set of the vertices of G adjacent to u will be denoted by $adj_G(u)$ (or, more briefly, by $adj(u)$, if G is clear by the context). If G is directed, the notation can be extended in $adj_G^+(u)$ ($adj_G^-(u)$) in order to denote the vertices which are adjacent with u through an edge which has u as tail (head).

The *degree*, $deg_G(u)$ (or $deg(u)$), of a vertex u is $|E(u)|$, that is the number of the edges of G which have u as end. If G is an undirected graph or a simple directed graph (that is a directed graph without loop and parallel edges) then the condition $deg_G(u) = |adj_G(u)|$ is obviously valid. If G is directed, we extend the notation in $deg_G^+(u)$ and $deg_G^-(u)$ whose meanings are obvious. If a vertex has the degree equal to zero, it is *isolated*.

The number $\delta(G) := min\ \{deg_G(v) \mid v \in V(G)\}$ is the *minimum degree* of G, whereas the number $\Delta(G) := max\ \{deg_G(v) \mid v \in V(G)\}$ is the *maximum degree* of G.

A *path* is a non-empty graph $P := (V, A)$ of the form:

$$\begin{aligned} V &= \{x_0, x_1, x_2, \ldots, x_r\} \\ A &= \{x_0x_1, x_1x_2, \ldots, x_{r-1}x_r\} \end{aligned}$$

where the vertices x_i are all distinct. The vertices x_0 and x_r are *linked* (or *connected*) by P; the vertices $x_1, \ldots, x_{r-1}$ are the *inner vertices* of P. We will write about the *length* of a path P by refering to $\|P\|$, that is the number of the edges of P. We will often identify a path with one of the two natural sequences of its vertices, for instance $P := x_0x_1x_2 \ldots x_r$.

In order to reference subpaths of a path, we will write (for a path P and for $0 \leq i \leq j \leq r$):

$$\begin{gathered} Px_j := x_0 \ldots x_j \\ x_iP := x_i \ldots x_r \\ x_iPx_j := x_i \ldots x_j \end{gathered}$$

and

$$\begin{gathered} P\ \mathring{x}_j := x_0 \ldots x_{j-1} \\ \mathring{x}_i\ P := x_{i+1} \ldots x_r \\ \mathring{x}_i\ P\ \mathring{x}_j := x_{i+1} \ldots x_{j-1} \end{gathered}$$

To denote the concatenation of paths, we will write, say, $PxQyR$ instead of $Px \cup xQy \cup yR$. Given a graph G, a path $P := x_0 \ldots x_r$ in G and a vertex f such that $f \in V(G)$, $f \notin P$ and there is $fx_0 \in E(G)$ ($fx_r \in E(G)$), we will denote the path:

$$P' := fx_0 \ldots x_r \quad (P' := x_0 \ldots x_r f)$$

by fP (Pf).

Two or more paths are *independent* if none of them contains an inner vertex of another. Given a graph H, a path $P := x_0 \ldots x_r$ which shares with H only x_0 and x_r, is an *H-path*.

A graph G is *connected* if for any $\{u, v\} \in [V(G)]^2$ there is a path $P \subseteq_{\mathcal{G}} G$ which link u and v. The *connected components* of a graph are its maximal connected subgraphs.

Let $P := (V, A) = x_0 x_1 \ldots x_r$ be a path with $r \geq 2$, the graph:

$$C := (V, A \cup \{x_r x_0\})$$

is a *cycle* and we will often denote a cycle with a cyclic sequence of its vertices, for instance $C := x_0 x_1 x_2 \ldots x_r x_0$.

There is another characterization of cycles which has relevance to the concept of k-cycle. A graph C is a *cycle* if and only if the following conditions hold:

- C is connected,
- $deg_C(v) = 2$, for every $v \in V(C)$.

The *length* of a cycle C is the number of its edges (or its vertices, since $\|C\| = |C|$). A graph G is *acyclic* if there is not any cycle C such that $C \subseteq_{\mathcal{G}} G$. An *oriented cycle* is an oriented graph C for which the following conditions hold:

- C is a cycle,
- there are not any two edges $a, a' \in E(C)$ such that $h_C(a) = h_C(a')$.

A *forest* is an acyclic graph. A connected forest is a *tree*

Given a graph G, a set of vertices $S \subset V(G)$ is a *vertex separator* for two non-adjacent vertices $u, v \in V(G) - S$, if u and v are contained in two different connected components of the subgraph $G_{V(G)-S}$; S is also called (u, v)-*separator*. If no proper subset of S is an (u, v)-separator then S is a *minimal* (u, v)-*separator*. A set $S \subset V(G)$ is a *minimal vertex separator* of G if there are two non-adjacent vertices of G for which S is a minimal vertex separator.

From now on, if we write about "graphs", we will refer to "undirected graphs" implicitly; we will specify a particular kind only when we write about directed graphs.

3 Dense Forests & Dense Trees

Definition 3.1
*A connected graph $C := (V_C, A_C)$ is a k-*cycle*, with k integer $1 \leq k < |V_C| - 1$, if, for each $v \in V_C$, the following condition holds:*

$$deg_C(v) > k.$$

Observation 3.2
*For all integer $k \geq 2$, a k-*cycle* is also a r-*cycle* with $1 \leq r < k$.*

Definition 3.3
*A k-*dense forest* F, with $k \geq 1$ integer, is a graph without k-cycles.*

Definition 3.4
*A k-*dense tree* T, with $k \geq 1$ integer, is a connected graph without k-cycles.*

Moreover, any connected graph of order $n \leq k$ satisfies the conditions of the definition of k-dense tree, in fact it is obvious that a graph with these characteristics cannot contain a k-cycle. Dense trees of this kind are *trivial*.

We can derive the following, simple observation.

Observation 3.5
The connected components of a k-dense forest are k-dense trees.

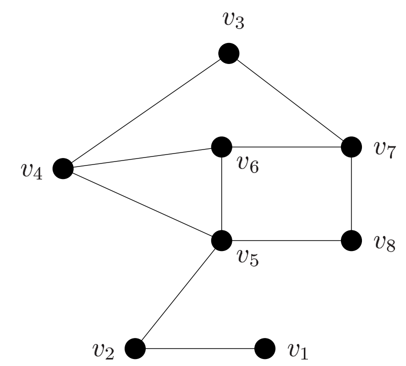

Figure 3.1: A 2-dense tree.

Consequently we will call *trivial* those k-dense forests which have only trivial k-dense trees as connected components.

By Observation 3.2 we have:

Observation 3.6
A k-dense forest *is also a* r-dense forest*, with* $r > k$*.*

And similarly:

Observation 3.7
A k-dense tree *is also a* r-dense tree*, with* $r > k$*.*

Proposition 3.8
A k-dense tree T *must have a vertex* f *such that* $deg_T(f) \leq k$*.*

PROOF. By contradiction. Suppose that all the vertices $v \in V(T)$ have $deg_T(v) > k$. Then T, which is connected, is a k-cycle, against the hypothesis. □

Definition 3.9
A k-leaf *of a* k*-dense forest* F *is a vertex* $f \in V(F)$ *such that* $deg_F(f) \leq k$*. If* $deg_F(f) = k$*, the* k*-leaf is* complete*. The set of the* k*-leaves of* F *is denoted by* $\mathcal{L}_F$*.*

Lemma 3.10
Let $F := (V, A)$ *be a* k*-dense forest. Any subgraph of* F *is a* k*-dense forest.*

PROOF. Immediate: if a subgraph of F contained a k-cycle, then F would also contain it, against the hypotheses. □

The following result is only a particular case of the previous one. We introduce it so as to mantain the uniformity with the subsequent theory.

Lemma 3.11
Let $F := (V, A)$ *be a* k*-dense forest and let* f *be a* k*-leaf of* F*. The subgraph* $F_{V-\{f\}}$ *of* F *induced by* $V - \{f\}$ *still is a* k*-dense forest.*

PROOF. It follows directly from Lemma 3.10. □

In general, a property analogous to the above one, does not hold for k-dense trees. For example consider the 2-dense tree of Figure 3.1: deleting the 2-leaf v_2 the resulting graph is not connected (hence is only a k-dense forest). We will see that a property analogous to the one of Lemma 3.11 holds for an interesting subclass of k-dense trees.

Definition 3.12
Let $G := (V, A)$ *be a graph of order* n*. A* k-elimination $\sigma := [v_1, \ldots, v_n]$*, with* $k \geq 1$ *integer, is an ordering of the vertices of* G *such that, for all integer* i *with* $1 \leq i \leq n$*,*

$$deg_{G_{V_i}}(v_i) \leq k$$

where G_{V_i} *is the subgraph of* G *induced by* $V_i := \{v_j \mid i \leq j\}$*.*

Notation 3.13
Let $G := (V, A)$ be a graph with n vertices and let $\sigma := [v_1, v_2, \ldots, v_n]$ be a k-elimination for G, we introduce the following notations:

- *by $\mathbf{E}_G$ we denote the set of k-eliminations for G;*
- *by $|\sigma|$ we denote the number of vertices contained in σ;*
- *by $\sigma(i)$ we denote the vertex of G in the i-th position in σ (from v_1);*
- *given a set of vertices $U \subseteq V$, by σ_U we denote the sub-sequence of σ induced by the vertices in U;*
- *given a vertex $v \in V$, by $\langle\sigma\rangle^{-1}(v)$ we denote the position (the index) of v in σ;*
- *given an integer i, with $1 \leq i \leq n$, by G_i we denote the subgraph of G induced by $\{v_j \mid i \leq j \leq n\}$;*
- *given a vertex $v \in V$, we set $\Lambda^\sigma(v) := \Lambda^\sigma_G(v) := \left\{u \in adj_G(v) \,\middle|\, \langle\sigma\rangle^{-1}(u) > \langle\sigma\rangle^{-1}(v)\right\}$.*

In order to denote some particular k-eliminations, we use some "list-like" notations. Given $\sigma := [v_1, \ldots, v_n]$, $\sigma' := [u_1, \ldots, u_m]$ and a vertex x (or, more generally, t vertices $x_1, \ldots, x_t$) not contained in σ, by:

$$[x \mid \sigma] \qquad ([x_1, \ldots, x_t \mid \sigma])$$

we denote:

$$[x, v_1, \ldots, v_n] \qquad ([x_1, \ldots, x_t, v_1, \ldots, v_n])$$

whereas by $app(\sigma, \sigma')$ we denote:

$$[v_1, \ldots, v_n, u_1, \ldots, u_m].$$

Finally, we denote $[v_2, \ldots, v_n]$ by $tail(\sigma)$.

Definition 3.14
*A k-*dense forest *F, with $k \geq 1$ integer, is a graph for which a k-elimination exists.*

Theorem 3.15 ([Fra01]**)**
A graph is a k-dense forest according to Definition 3.3 if and only if it is a k-dense forest according to Definition 3.14.

Definition 3.16
Let F be a non-trivial k-dense forest and let σ be a k-elimination for F, we call seed *of σ the set $\Gamma(\sigma) := \{\sigma(n-k+1), \ldots, \sigma(n)\}$ (that is the set of the last k vertices in σ). If F is trivial, we call* seed *of all σ the set of the vertices of F. We say that a set S of k vertices of F is a* seed *of F if there is a k-elimination σ for F such that $\Gamma(\sigma) = S$.*

Proposition 3.17 ([Fra01]**)**
Let T be a k-dense tree of order n, the following condition holds:

$$n - 1 \leq \|T\| \leq M \quad \textit{where} \quad M := \begin{cases} nk - \frac{k(k+1)}{2} & n \geq k+1 \\ \frac{n(n-1)}{2} & n < k+1 \end{cases}$$

and, both for the lower and upper bound, there are cases in which the equality holds. Moreover, the upper bound holds for k-dense forests also.

Definition 3.18
A complete k-dense tree *is a k-dense tree with the maximum number of edges (see Proposition 3.17).*

Directly from Definition 3.18, we have:

Observation 3.19
The set of trivial complete k-dense trees is the set of complete graphs of order r with $1 \leq r \leq k$.

Observation 3.20
A complete k-dense tree of order $k+1$ is a complete graph, has $k+1$ k-leaves and each of these has maximum degree (is a complete k-leaf).

Lemma 3.21
Each k-leaf of a non-trivial complete k-dense tree has degree k.

Proof. Let T be a non-trivial complete k-dense tree of order n. Suppose, by contrary, that there are some k-leaves of T of degree $r < k$ and let f be the one of minimum degree. From Lemma 3.11, the graph $T_{V(T)-\{f\}}$ still is a k-dense forest. But such a graph has $nk - \frac{k(k+1)}{2} - r$ edges, against that we established in Proposition 3.17 for a k-dense tree (and for a k-dense forest) of order $n-1$. □

Observation 3.22
For each non-trivial complete k-dense tree T, $\delta(T) = k$.

A property, analogous to the one of Lemma 3.11, holds for the class of complete k-dense trees. We have:

Lemma 3.23
Let $T := (V, A)$ be a complete k-dense tree and let f be a k-leaf of T. The subgraph $T_{V-\{f\}}$ of T induced by $V - \{f\}$ still is a complete k-dense tree.

Proof. If $|T| \leq k+1$, by Observations 3.19 and 3.20 we know that T is a complete graph, hence the thesis obviously holds. If $|T| > k+1$, by Lemma 3.21 we know that $deg_T(f) = k$ and hence $\|T'\| = |T'|\,k - \frac{k(k+1)}{2}$. Since T' has no k-cycles, it is enough to prove that it is connected. Let $C_1 = (V_1, E_1), \ldots, C_m = (V_m, E_m)$ be the $m \geq 1$ connected components of $T_{V-\{f\}}$. For all $1 \leq i \leq m$, C_i is a k-dense tree, since it is a subgraph of a k-dense tree and it is connected by hypothesis. Moreover C_i is a complete k-dense tree. In fact, if this was not true, there would exist two non-adjacent vertices v', v'' of C_i such that $C'_i := (V_i, E_i \cup \{v'v''\})$ would still be a k-dense tree. Since T is a complete k-dense tree, the graph $T'' = (V, E \cup \{v'v''\})$ would contain a k-cycle in which the vertex f would be necessarily involved. But f is a k-leaf and hence $deg_T(f) = k$, against Definition 3.1. Therefore, for all $1 \leq i \leq m$, C_i is a complete k-dense tree and, moreover, $|C_i| \geq k+1$, since otherwise T would have at least a k-leaf of degree $< k$, against Lemma 3.21. Therefore, by Proposition 3.17, we have:

$$
\begin{aligned}
\|T'\| = (|T|-1)k - \frac{k(k+1)}{2} &= |C_1|\,k - \frac{k(k+1)}{2} + \cdots + |C_m|\,k - \frac{k(k+1)}{2} \\
&= (|C_1| + \cdots + |C_m|)k - m\frac{k(k+1)}{2}
\end{aligned}
$$

Since $|T| - 1 = |C_1| + \cdots + |C_m|$, we have that $\frac{k(k+1)}{2} = m\frac{k(k+1)}{2}$ and therefore $m = 1$. □

Definition 3.24
A complete k-dense tree *T, with $k \geq 1$ integer, is a graph for which there is a k-elimination σ such that, given $n := |T|$:*

(i) $\forall i \,.\, 1 \leq i \leq n-k \Rightarrow deg_{T_i}(\sigma(i)) = k$,

(ii) $\forall i \,.\, max\,\{n-k+1, 1\} \leq i \leq n-1 \Rightarrow deg_{T_i}(\sigma(i)) = n-i$.

Condition (ii) can be replaced with: $T_{max\,\{n-k+1,1\}} \simeq K^{min\,\{n,k\}}$.

Theorem 3.25 ([Fra01])
A graph is a complete k-dense tree according to Definition 3.18 if and only if it is a complete k-dense tree according to Definition 3.24.

Proposition 3.26 ([Fra01])
Any k-elimination for a complete k-dense tree has the form stated Definition 3.24.

There is a third characterization of the class of complete k-dense trees.

Definition 3.27
A graph $G := (V, A)$ is maximally devoid of k-cycles *if it has no k-cycles and for any two non-adjacent vertices $x, y \in V$, the graph $G' := (V, A \cup \{xy\})$ has some k-cycles.*

Theorem 3.28 ([Fra01])
A graph is a complete k-dense tree if and only if is maximally devoid of k-cycles.

Observation 3.29
Let T be a non-trivial complete k-dense tree. For each k-elimination σ for T, the seed of σ is a k-clique.

Observation 3.30
Let T be a complete k-dense tree of order $n \geq k+1$. For each k-elimination for T $\sigma := \left[v_1, \ldots, v_{n-k-1} \mid \sigma^0\right]$ satisfying the conditions of Definition 3.24, any ordering of the vertices of T like $[v_1, \ldots, v_{n-k-1} \mid \sigma']$ still is a k-elimination for T satisfying the conditions of Definition 3.24; that is, any permutation of the vertices of T in the last $k+1$ positions of σ, give rise to a k-elimination for T, which satisfies the conditions of Definition 3.24.

Proposition 3.31 ([Fra01])
Any two k-leaves of a complete k-dense tree of order $\geq k+2$ are not adjacent.

We now introduce a new definition of k-trees derived from the one of complete k-dense trees.

Definition 3.32
A graph T is a k-tree, *with $k \geq 1$ integer, if the following conditions hold:*

- *T is a complete k-dense tree,*
- *if $|T| \geq k+1$, each minimal vertex separator of T is a k-clique.*

Observation 3.33
For each $k \geq 2$, there exist complete k-dense trees of any order which are not k-trees.

Lemma 3.34 ([Fra01])
Let $T := (V, A)$ be a k-tree and let f be a k-leaf of T. The subgraph $T_{V-\{f\}}$ of T induced by $V - \{f\}$ still is a k-tree.

Definition 3.35
A k-tree *T, with $k \geq 1$ integer, is a graph for which there is a k-elimination σ such that, given $n := |T|$:*

(i) $\forall i \,.\, 1 \leq i \leq n-k \Rightarrow deg_{T_i}(\sigma(i)) = k,$

(ii) $\forall i \,.\, max\,\{n-k+1, 1\} \leq i \leq n-1 \Rightarrow deg_{T_i}(\sigma(i)) = n-i,$

(iii) $\forall i \,.\, 1 \leq i \leq n-k \Rightarrow adj_{T_i}(\sigma(i))$ *is a clique.*

The second condition can be replaced with: $T_{max\{n-k+1,1\}} \simeq K^{min\{n,k\}}$.

Theorem 3.36 ([Fra01])
A graph is a k-tree according to Definition 3.32 if and only if it is a k-tree according to Definition 3.35.

A graph G of order n is *triangulated* if none of its induced subgraphs is a cycle of length $n > 3$. An ordering $\epsilon := [v_1, \ldots, v_n]$ of the vertices of G is a *perfect elimination* if for each v_i the set $\{v_j \in adj_G(v_i) \mid j > i\}$ is a clique. So, by Definition 3.35, the k-trees are triangulated.

Proposition 3.37 ([Fra01])
Any k-elimination for a k-tree has the form stated in Definition 3.35.

Theorem 3.38 ([Fra01])
Let T be a non-trivial k-tree, a set of k vertices S is a seed of T if and only if it is a k-clique of T.

Proposition 3.39 ([Fra01])
Every k-tree has at least two k-leaves.

4 Connectivity of Complete Dense Trees

A graph G is *r-connected*, with $r \in \mathbf{N}$, if the following conditions hold:

(i) $|G| > r$,

(ii) for each $X \subseteq V(G)$ such that $|X| < r$, the graph $G_{V(G)-X}$ is connected.

The greatest integer r for which G is r-connected is the *connectivity* of G, which we denote by $\kappa(G)$.

Similarly, a graph $G := (V, A)$ is *l-edge-connected*, with $l \in \mathbf{N}$, if the following conditions hold:

(i) $|G| > 1$,

(ii) for each $Y \subseteq E(G)$ such that $|Y| < l$, the partial subgraph of G $(V(G), A - Y)$ is connected.

The greatest integer l for which G is l-edge-connected is the *edge-connectivity* of G, which we denote by $\lambda(G)$.

Observation 4.1
For each graph G non-trivial ($|G| > 1$) we have $\kappa(G) \leq \lambda(G) \leq \delta(G)$.

Proposition 4.2
For each non-trivial complete k-dense tree $T := (V, A)$, we have $\kappa(T) = k$.

Proof. The inequality $\kappa(T) \leq k$ follows directly from Observation 3.22 and Observation 4.1.

We will prove that $\kappa(T) \geq k$ by induction on $|T|$. If $|T| = k + 1$ then, from Observation 3.20, $T \simeq K^{k+1}$ and so the thesis is trivial.

If $|T| = n > k + 1$, let us suppose that for each complete k-dense tree T' of order $n - 1$, $\kappa(T') \geq k$ holds.

Let $X \subseteq V(T)$ such that $|X| < k$ and let f be a k-leaf of T. From Lemma 3.21, f has degree k in T. The subgraph $T'' := T_{V-\{f\}}$ is a non-trivial complete k-dense tree of order $n - 1$ and hence, by the inductive hypotesis, $\kappa(T'') \geq k$ holds that is, for each $Y \subseteq V(T)$ such that $|Y| < k$, the subgraph of T'':

$$T''_{V(T'')-Y} = T_{V-(\{f\}\cup Y)}$$

is connected. We have two cases.

- $f \notin X$. From what we know about T'', $T''_{V(T'')-X} = T_{V-(\{f\}\cup X)}$ is certainly connected. Since $|X| < k$, at least one of the vertices in $adj_T(f)$ belongs to $T''_{V(T'')-X} = T_{V-(\{f\}\cup X)}$ and therefore also T_{V-X} is connected.
- $f \in X$. From what we know about T'' and since $|X - \{f\}| < k - 1$, the graph:

  $$T''_{V(T'')-(X-\{f\})} = T_{(V-\{f\})-(X-\{f\})} = T_{V-X}$$

  is connected.

Proving that $\kappa(T) = k$ now amounts to prove that T is not $(k + 1)$-connected, which is trivial because each k-leaf of T is adjacent to k vertices of T. □

Proposition 4.3
For each non-trivial complete k-dense tree $T := (V, A)$, we have $\lambda(T) = k$.

Proof. We have that $k = \kappa(G) \leq \lambda(G) \leq \delta(G) = k$, directly from Observation 3.22, Observation 4.1 and Proposition 4.2. □

Given $r \in \mathbf{N}$, a graph G with $|G| \geq 2r$ is *r-linked* if, for every $2r$ distinct vertices $s_1, \ldots, s_r, z_1, \ldots, z_r$, there exist r disjoint paths $P_1, \ldots, P_r$ in G such that $P_i = s_i \ldots z_i$, for all $1 \leq i \leq r$. Obviously, a r-linked graph is r-connected also. However, the converse is not true.

Theorem 4.4
Every complete k-dense tree T, with $|T| \geq 2\left\lfloor \frac{k+1}{2} \right\rfloor$, is $\left\lfloor \frac{k+1}{2} \right\rfloor$-linked and $\left\lfloor \frac{k+1}{2} \right\rfloor$ is the greatest integer for which this is possible.

Proof. Let T be a complete k-dense tree with $|T| \geq 2\left\lfloor \frac{k+1}{2} \right\rfloor$. Given $r := \left\lfloor \frac{k+1}{2} \right\rfloor$, we will prove by induction on $|T|$ that T is r-linked.

Let us suppose that $|T| = n = 2r$. We have two cases:

- k is even, hence $r = \frac{k}{2}$ and $n = k$,
- k is odd, hence $r = \frac{k+1}{2}$ and $n = k+1$.

In both cases we have that $T \simeq K^n$ and hence, whatever pairing of the n vertices of T we choose, for each pair there is a path of length 1 linking the two vertices and, obviously, all these paths are disjoint.

If $|T| = n > 2r$, let us suppose that each complete k-dense tree of order $n-1$ is r-linked. Let f be a k-leaf of T. By Lemma 3.21, we know that $deg_T(f) = k$. Let $s_1, \ldots, s_r, z_1, \ldots, z_r$ be $2r$ distinct vertices, let us set $S := \{s_i \mid 1 \leq i \leq r\}$ and $Z := \{z_i \mid 1 \leq i \leq r\}$.
We have two cases.

(i) $f \notin S$ and $f \notin Z$.

Hence, by the inductive hypothesis, there are r disjoint paths $P_1, \ldots, P_r$ in $T_{V(T)-\{f\}}$ such that, for all i, $P_i = s_i \ldots z_i$. Obviously, such paths are contained in T.

(ii) $f \in S$ or $f \in Z$.

Let us suppose that $f = s_i \in S$ (the case $f \in Z$ is analogous). We have two cases.

(a) If $z_i \in adj_T(f)$, let us consider the path P_i between s_i and z_i with the only edge $\{s_i, z_i\}$. We will use the inductive hypothesis for the other pairs of vertices:

$$\{(s_j, z_j) \mid 1 \leq j \leq r \wedge j \neq i\}.$$

Let us choose a vertex $h \in V(T)-(S \cup Z)$. By inductive hypothesis, we know that in $T_{V(T)-\{f\}}$ there are r disjoint paths $P_1, \ldots, P_i', \ldots, P_r$ such that $P_i' = h \ldots z_i$ and $P_j = s_j \ldots z_j$, for all $j \neq i$ ($1 \leq j \leq r$). Obviously, the $r-1$ paths P_j with $j \neq i$ are disjoint in T as well. Moreover, since they are disjoint from the path P_i', none of them contains the vertex z_i and then they are disjoint from P_i. Therefore the r paths $P_1, \ldots, P_r$ prove that T is r-linked.

(b) If $z_i \notin adj_T(f)$, since $r = \left\lfloor \frac{k+1}{2} \right\rfloor$, there is a vertex u such that $u \in adj_T(f) - (S \cup Z)$. By the inductive hypothesis, we know that in $T_{V(T)-\{f\}}$ there are r disjoint paths $P_1, \ldots, P_i'', \ldots, P_r$ such that $P_i'' = u \ldots z_i$ and $P_j = s_j \ldots z_j$, for all j with $j \neq i$ and $1 \leq j \leq r$. Obviously, such paths are disjoint in T as well. Since f and u are adjacent, the path $P_i := fP_i''$ link f and z_i in T and is disjoint from the path P_j, for all j with $j \neq i$ and $1 \leq j \leq r$. Therefore, in this case also, the r paths $P_1, \ldots, P_r$ prove that T is r-linked.

Finally, we have to prove that the greatest integer p for which T is p-linked is $\left\lfloor \frac{k+1}{2} \right\rfloor$. We begin proving that the following condition

$$p < \frac{k}{2} + 1$$

is necessary in order that T can be p-linked. Let f be a k-leaf of T, let $adj_T(f) = \{d_1, \ldots, d_k\}$ (we know that $deg_T(f) = k$ from Lemma 3.21) and let $s_1, \ldots, s_r, z_1, \ldots, z_r$ be $2r$ distinct vertices of T, with $S := \{s_i \mid 1 \leq i \leq r\}$ e $Z := \{z_i \mid 1 \leq i \leq r\}$. Let us suppose that $f = s_i \in S$ (the case $f \in Z$ is analogous). So that T can be p-linked, it is necessary that a situation for which:

$$d_j \in S - \{s_i\} \quad \text{or} \quad d_j \in Z - \{z_i\},$$

for all j with $1 \leq j \leq k$, doesn't happen. Otherwise each path between f and z_i should include one of the vertices in the set:

$$(S - \{s_i\}) \cup (Z - \{z_i\})$$

and so there could not be p disjoint paths which link the pairs of vertices. Therefore it is necessary that:

$$|(S - \{s_i\}) \cup (Z - \{z_i\})| = 2p - 2 < k$$

that is:

$$p < \frac{k}{2} + 1.$$

What is the greatest integer p for which such inequalities holds? If k is odd, we have that $\frac{k}{2} + 1 = \frac{k-1}{2} + \frac{1}{2} + 1$ and so $p = \frac{k+1}{2}$. If k is even, $\frac{k}{2} = \left\lfloor \frac{k+1}{2} \right\rfloor$ holds and so $p = \frac{k}{2}$ in this case.

If k is even, $\frac{k}{2} = \left\lfloor \frac{k+1}{2} \right\rfloor$ holds and therefore we can join together both cases and we can conclude that the greatest integer for which the condition holds is $p = \left\lfloor \frac{k+1}{2} \right\rfloor$. □

5 Complete Sub-dense trees of a Complete Dense Tree

Proposition 5.1
If T and T' are complete k-dense trees such that $T' \subseteq_{\mathcal{G}} T$, then T' is the subgraph of T induced by $V(T')$.

Proof. We will prove the thesis by induction on $|T|$.

If $|T| \leq k + 1$, then, by Observation 3.19 and Observation 3.20, both T and T' are complete graphs and hence the thesis is trivial.

If $|T| = n > k + 1$, let us suppose that the thesis holds for each complete k-dense tree of order $n - 1$. Let f be a k-leaf of T. The graph $T'' := T_{V(T)-\{f\}}$ is a complete k-dense tree of order $n - 1$. If T' is trivial then it is a complete graph and hence the thesis is obviously true. If T' is non-trivial, we have two cases.

- $f \notin V(T')$. Hence T' is a subgraph of T'' and, by inductive hypothesis, it is the subgraph of T'' induced by $V(T')$. Therefore, since T'' is the subgraph of T induced by $V(T) - \{f\}$, T' is the subgraph of T induced by $V(T')$.

- $f \in V(T')$. Since T' is a non-trivial complete k-dense tree, by Observation 3.22 we know that $\delta(T') = k$ and then any edge $a := fv \in E(T)$ is contained in $E(T')$. Therefore f is a k-leaf of T' and $T''' := T'_{V(T')-\{f\}}$ is a complete k-dense tree and is also a subgraph of T''. Then, by the inductive hypothesis, T''' is the subgraph of T'' induced by $V(T''') = V(T') - \{f\}$. Therefore, since $T'' = T_{V(T)-\{f\}}$ and each edge on f is also contained in $E(T')$, we can conclude that T' is the subgraph of T induced by $V(T')$. □

By Proposition 5.1 we have that a complete k-dense tree which is subgraph of another complete k-dense tree is certainly an induced subgraph. However we will continue to use the adjective "induced" in the remainder of this work, for clearness purposes. The result of Proposition 5.1 is a strengthening of the theory which we will describe in the remainder of this work.

Proposition 5.2
Let $T := (V, A)$ and $T' := (V', A')$ be complete k-dense trees and let σ be a k-elimination for T. If $V' \subseteq V$ and T' is the subgraph of T, induced by V', then the ordering $\sigma_{V'}$ (see Notation 3.13) of the vertices in V' is a k-elimination for T'.

Proof. If T' is trivial then it is a complete graph (Observation 3.19), hence every ordering of its vertices is a k-elimination. For the same reasons, also if T is trivial, the thesis is obviously true. Therefore, let us suppose that $|T| \geq k + 1$ and $|T'| \geq k + 1$. We will prove the thesis by induction on $|T'|$.

If $|T'| = k + 1$ then, by Observation 3.20, $T' \simeq K^{k+1}$ and hence the thesis is true.

If $|T'| = n > k+1$, let us suppose that the thesis holds for each complete k-dense tree with $n-1$ vertices which is an induced subgraph of T. Let $f \in V'$ be the vertex such that:

$$\langle\sigma\rangle^{-1}(f) = min\left\{\langle\sigma\rangle^{-1}(v) \mid v \in V'\right\}.$$

Since σ is a k-elimination for T, since $|T'| = n > k+1$ and for the definition of f, we have that $|\Lambda^{\sigma}(f)| = k$. For the choice of f, we know that $adj_{T'}(f) \subseteq \Lambda^{\sigma}(f)$. Moreover, since $\delta(T') = k$ (T' is a complete k-dense tree), we know that $deg_{T'}(f) \geq k$. Joining together the above relations, we have:

$$\Lambda^{\sigma}(f) = adj_{T'}(f)$$

and then f is a k-leaf of T'. By Lemma 3.23 we know that the graph $T'_{V'-\{f\}} = T_{V'-\{f\}}$ is a complete k-dense tree with $n-1$ vertices. Therefore, by the inductive hypothesis, the sub-sequence $\sigma_{V'-\{f\}}$ is a k-elimination for $T_{V'-\{f\}}$. Now, we can conclude that:

$$\sigma_{V'} = \left[f \;\middle|\; \sigma_{V'-\{f\}}\right]$$

is a k-elimination for T'. □

As a consequence, from now on, for any two complete k-dense trees T and T', such that $T' \subseteq_{\mathcal{G}} T$, and for any k-elimination σ for T, we will apply to $\sigma_{V(T')}$ all the notations that we have already introduced (and we will introduce) for the k-eliminations.

6 Maximal Triangulated Constituents of a Complete Dense Tree

Definition 6.1
Let $T := (V, A)$ be a complete k-dense tree and let $R := (V_R, A_R)$ be a k-tree. If $R \subseteq_{\mathcal{G}} T$ then R is a triangulated constituent of T. *If there is not any triangulated constituent of T R' such that $R \subset_{\mathcal{G}} R'$ then R is a* maximal triangulated constituent of T.

By Proposition 5.1, we know that Definition 6.1 is equivalent to the following.

Definition 6.2
Let $T := (V, A)$ be a complete k-dense tree and let $R := (V_R, A_R)$ be a k-tree. If $V_R \subseteq V$ and $R = T_{V_R}$ then R is a triangulated constituent of T. *If there is not any triangulated constituent of T R' such that $R \subset_{\mathcal{G}} R'$ then R is a* maximal triangulated constituent of T.

Since the two above definitions are equivalent, from now on we will make use of the second one exclusively to avoid to repeatedly reference Proposition 5.1.

Lemma 6.3
Let T be a complete k-dense tree. If there exists a maximal triangulated constituent R of T containing all the k-leaves of T, then $T = R$.

Proof. We will prove the thesis by induction on $|T|$.

If $|T| \leq k+1$ then T is a complete graph and hence it is also a k-tree.

If $|T| = n > k+1$, let us suppose that the thesis holds for each complete k-dense tree of order $n-1$ having a maximal triangulated constituent containing all its k-leaves. Let f be a k-leaf of T; by Lemma 3.23 we know that the graph $T' := T_{V(T)-\{f\}}$ is a complete k-dense tree with $n-1$ vertices. Since $f \in V(R)$, R is a maximal triangulated constituent of T, and the following holds:

$$\mathcal{L}_{T'} \subseteq adj_T(f) \cup (\mathcal{L}_T - \{f\})$$

then the graph $R' := R_{V(R)-\{f\}}$ is a maximal triangulated constituent of T' containing all its k-leaves. Then, by inductive hypothesis, $T' = R'$ and hence $T = R$. □

Any two maximal triangulated constituents of a complete k-dense tree can share $k-1$ vertices at most, as we will prove in the following result.

Theorem 6.4
Let $T := (V, A)$ be a complete k-dense tree and let $R' := (V', A')$ and $R'' := (V'', A'')$ be maximal triangulated constituents of T. The following condition holds:

$$|V' \cap V''| \leq k - 1$$

PROOF. If one of the two triangulated constituents of T is trivial then the thesis is obviously true. So, let us suppose that both R' and R'' are non-trivial. Let $n' := |R'|$ and $n'' := |R''|$. Moreover, let $I := V' \cap V''$ and let us suppose that $|I| = s \geq k$. Finally, let σ be a k-elimination for T. By Proposition 5.2 we know that $\sigma' := \sigma_{V'}$ is a k-elimination for R' and $\sigma'' := \sigma_{V''}$ is a k-elimination for R''. Therefore the two k-eliminations share the vertices of I which certainly mantain in σ' and in σ'' the same relative order that they have in σ. So, for each vertex $v \in I$, the following fundamental condition holds:

$$|\Lambda^{\sigma}(v)| \geq \left|\Lambda^{\sigma'}(v) \cup \Lambda^{\sigma''}(v)\right|. \tag{1}$$

Let us set

$$I_k := \left\{v \in I \;\middle|\; s - k + 1 \leq \langle\sigma_I\rangle^{-1}(v) \leq s\right\},$$

so we denote by I_k the set of the last k vertices of I in the order given by σ. Let $t \in I_k$ be the vertex such that

$$\langle\sigma'\rangle^{-1}(t) = min\left\{\langle\sigma'\rangle^{-1}(x) \mid x \in I_k\right\}.$$

We have the following cases.

- $t \notin \Gamma(\sigma')$ and $t \notin \Gamma(\sigma'')$, that is:

  $$\begin{aligned} \langle\sigma'\rangle^{-1}(t) &< n' - k + 1 \\ \langle\sigma''\rangle^{-1}(t) &< n'' - k + 1. \end{aligned}$$

  In this case we have

  $$\left|\Lambda^{\sigma'}(t)\right| = \left|\Lambda^{\sigma''}(t)\right| = k$$

  then there exist at least two vertices $u', u'' \notin I_k$ such that:

  $$u' \in \Lambda^{\sigma'}(t) \quad \text{e} \quad u'' \in \Lambda^{\sigma''}(t).$$

  For the definition of the set I_k, It is clear that the two vertices u' and u'' cannot belong to I. Hence, we have:

  $$|\Lambda^{\sigma}(t)| = k < k + 1 \leq \left|\Lambda^{\sigma'}(t) \cup \Lambda^{\sigma''}(t)\right|$$

  against the condition 1.

- $t \in \Gamma(\sigma')$ and $t \in \Gamma(\sigma'')$. Therefore, for the choice of t, we have:

  $$\begin{aligned} \sigma' &= \left[x_1, \ldots, x_{n'-k} \mid \sigma_{I_k}\right] \\ \sigma'' &= \left[y_1, \ldots, y_{n''-k} \mid \sigma_{I_k}\right]. \end{aligned}$$

  Then, the ordering:

  $$\sigma''' := [y_1, \ldots, y_{n''-k} \mid \sigma']$$

  of the vertices in $V' \cup V''$ is a k-elimination, with the properties required by Definition 3.35 for the subgraph of T induced by $V' \cup V''$. Therefore, this subgraph is a k-tree. Such a k-tree contains both R' and R'', against the hypotheses about the maximality of these triangulated constituents of T.

- $t \notin \Gamma(\sigma')$ and $t \in \Gamma(\sigma'')$. Because of the choice of t, we have:

$$\sigma'' = \left[y_1, \ldots, y_{n''-k} \;\middle|\; \sigma_{I_k}\right].$$

  So, the following holds:

$$\left|\Lambda^{\sigma'}(t)\right| = k$$

  and hence there is at least a vertex u' such that:

$$u' \notin I_k \quad \text{e} \quad u' \in \Lambda^{\sigma'}(t).$$

  By the definition of the set I_k, it is clear that such vertices cannot belong to I. If such vertices are more than one, the following inequality holds

$$|\Lambda^{\sigma}(t)| = k < k+1 \leq \left|\Lambda^{\sigma'}(t) \cup \Lambda^{\sigma''}(t)\right|$$

  then condition 1 is violated once again.

  Conversely, if the vertex u' is the unique with such characteristics then the ordering:

$$\sigma''' := [y_1, \ldots, y_{n''-k} \mid \sigma']$$

  of the vertices in $V' \cup V''$ is a k-elimination, with the properties required by Definition 3.35, for the subgraph of T induced by $V' \cup V''$. Then, this subgraph is a k-tree. Such a k-tree contains both R' and R'', once again in contradiction with the hypotheses about the maximality of these triangulated constituents of T.

- $t \in \Gamma(\sigma')$ and $t \notin \Gamma(\sigma'')$. This case is symmetric to the previous one. □

Proposition 6.5 ([Fra01])
For each $k \geq 2$ and for each integer s such that $0 \leq s \leq k-1$, there is a complete k-dense tree T having two non-trivial maximal triangulated constituents $R' := (V', A')$, $R'' := (V'', A'')$ such that $|V' \cap V''| = s$.

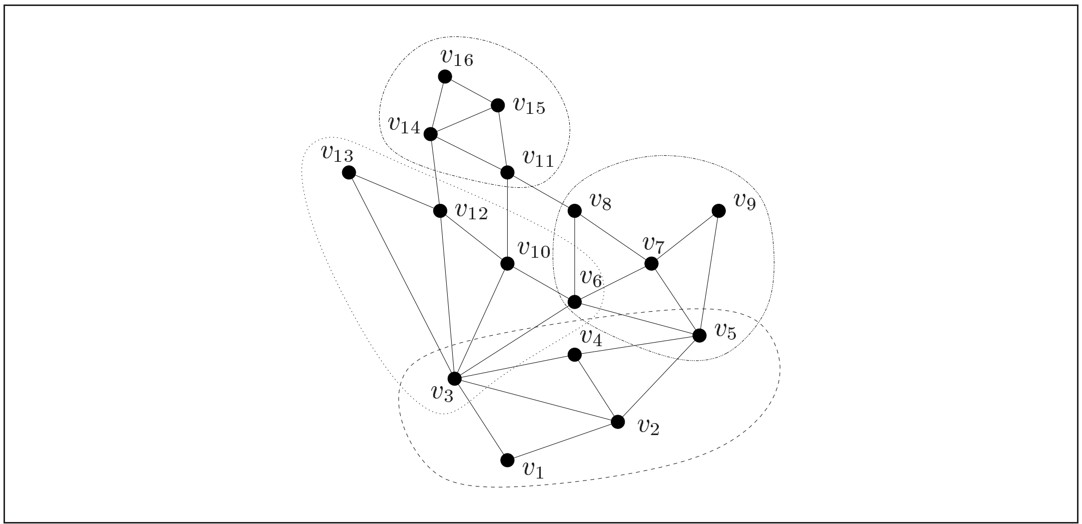

Figure 6.1: A complete 2-dense tree with four non-trivial maximal triangulated constituents induced by $\{v_1, v_2, v_3, v_4, v_5\}$, $\{v_5, v_6, v_7, v_8, v_9\}$, $\{v_3, v_6, v_{10}, v_{12}, v_{13}\}$, $\{v_{11}, v_{15}, v_{14}, v_{16}\}$.

By Theorem 6.4, we know that any two maximal triangulated constituents of a complete k-dense tree can share a set of vertices with $k-1$ members at most; this is not the unique constraint that this set must respect.

Proposition 6.6
Let $T := (V, A)$ be a complete k-dense tree and let $R' := (V', A')$ and $R'' := (V'', A'')$ be maximal triangulated constituents of T. The set of vertices: $I := V' \cap V''$ is a clique.

PROOF. If one of the two triangulated constituents is trivial, it is a complete graph and the thesis is obviously true. Let us suppose that both R' are R'' are not trivial. Let $s = |I|$ and let σ be a k-elimination for T. By Proposition 5.2 we know that $\sigma' := \sigma_{V'}$ is a k-elimination for R' and $\sigma'' := \sigma_{V''}$ is a k-elimination for R''. Since σ is a k-elimination for a complete k-dense tree, for each vertex $v \in I$ the following condition holds:

$$\left|\Lambda^{\sigma}(v)\right| \geq \left|\Lambda^{\sigma'}(v) \cup \Lambda^{\sigma''}(v)\right|. \tag{2}$$

Let us suppose that there are two vertices $x, y \in I$ which are not adjacent. Moreover, let us suppose that $\langle\sigma\rangle^{-1}(x) < \langle\sigma\rangle^{-1}(y)$ (the other case is symmetric to this one). Since σ' and σ'' are sub-sequences of σ, obviously $\langle\sigma'\rangle^{-1}(x) < \langle\sigma'\rangle^{-1}(y)$ and $\langle\sigma''\rangle^{-1}(x) < \langle\sigma''\rangle^{-1}(y)$ holds. We have two cases.

- $x \in \Gamma(\sigma')$ or $x \in \Gamma(\sigma'')$. In this case, since both $\Gamma(\sigma')$ and $\Gamma(\sigma'')$ are k-cliques (by Observation 3.29), x and y are certainly adjacent.
- $x \notin \Gamma(\sigma')$ and $x \notin \Gamma(\sigma'')$. Therefore, by Proposition 3.37, we have:

  $$\left|\Lambda^{\sigma'}(x)\right| = \left|\Lambda^{\sigma''}(x)\right| = k.$$

  Let $I_x := \{v \in I \mid \sigma(x) < \sigma(v)\}$ and let $s' := |I_x|$, because of the hypothesis on x and y, we have:

  $$\left|\Lambda^{\sigma'}(x) \cap \Lambda^{\sigma''}(x)\right| \leq s' - 1$$

  and, for the hypotheses of the case, the following holds:

  $$\left|\Lambda^{\sigma'}(x) \cup \Lambda^{\sigma''}(x)\right| \geq 2k - s' + 1$$

  however, by Theorem 6.4, we know that $s' < s < k$ and, since $\left|\Lambda^{\sigma}(x)\right| \leq k$, we have:

  $$\left|\Lambda^{\sigma'}(x) \cup \Lambda^{\sigma''}(x)\right| \geq 2k - s' + 1 > 2k - k + 1 = k + 1 > \left|\Lambda^{\sigma}(x)\right|$$

  against the condition 2. □

The set of the vertices shared by any two maximal triangulated constituents, must respect a constraint about the order of the vertex in any k-elimination, in the same way it has to respect the constraints on cardinality and connectivity.

Proposition 6.7
Let $T := (V, A)$ be a complete k-dense tree and let $R' := (V', A')$ and $R'' := (V'', A'')$ be maximal triangulated constituents of T. Let $I := V' \cap V''$.

For each k-elimination σ for T, given $\sigma' := \sigma_{V'}$ and $\sigma'' := \sigma_{V''}$, if the following conditions hold:

(i) R' is not trivial

(ii) there is a vertex d such that $d \in I$ and $d \notin \Gamma(\sigma')$

then we have:

$$I = \left\{v \in V'' \,\middle|\, |V''| - |I| + 1 \leq \langle\sigma''\rangle^{-1}(v) \leq |V''|\right\}$$

then, the vertices of I are the last $|I|$ vertices of σ''.

PROOF. Let $t \in I$ be the vertex such that:

$$\langle\sigma'\rangle^{-1}(t) = min\left\{\langle\sigma'\rangle^{-1}(x) \mid x \in I\right\}.$$

Since the vertex $d \in I$ does not belong to $\Gamma(\sigma')$ and since:

$$\langle\sigma\rangle^{-1}(t) \leq \langle\sigma\rangle^{-1}(d)$$

t also does not belong to $\Gamma(\sigma')$ then:

$$\left|\Lambda^{\sigma'}(t)\right| = k.$$

By Proposition 6.6, we know that the set I is a clique and hence, given $J := I - \{t\}$, we have that $J \subseteq \Lambda^{\sigma''}(t)$ and that $J \subset \Lambda^{\sigma'}(t)$, since $\left|\Lambda^{\sigma'}(t)\right| = k$ and since $|I| \leq k-1$, by Theorem 6.4. Therefore, because of the hypotheses on the set I, we have:

$$\Lambda^{\sigma'}(t) \cap \Lambda^{\sigma''}(t) = J.$$

Since σ is a k-elimination for a complete k-dense tree, for each vertex $v \in I$ the following condition holds:

$$k \geq |\Lambda^{\sigma}(v)| \geq \left|\Lambda^{\sigma'}(v) \cup \Lambda^{\sigma''}(v)\right|. \tag{3}$$

Let $s := |I|$. From the above reasonings, t certainly belongs to $\Gamma(\sigma'')$; in fact, by contrary, the equality $\left|\Lambda^{\sigma''}(t)\right| = k$ would hold and then we would have:

$$\begin{aligned}
\left|\Lambda^{\sigma'}(t) \cup \Lambda^{\sigma''}(t)\right| &= \left|(\Lambda^{\sigma'}(t) - J) \cup J \cup (\Lambda^{\sigma''}(t) - J)\right| = \\
&= \left|\Lambda^{\sigma'}(t) - J\right| + |J| + \left|\Lambda^{\sigma''}(t) - J\right| = \\
&= k - s + 1 + s - 1 + k - s + 1 = 2k - s + 1 = \\
&\geq 2k - k + 2 = k + 2
\end{aligned}$$

against the condition 3.

Let us suppose that there is a vertex $m \in V'' - I$ such that:

$$|V''| - |I| + 1 \leq \langle\sigma''\rangle^{-1}(m) \leq |V''|.$$

Since $t \in \Gamma(\sigma'')$ and since the last k vertices of a k-elimination for a complete k-dense tree form a k-clique, by the hypotheses on t we have:

$$(J \cup \{m\}) \subseteq \Lambda^{\sigma''}(t).$$

And then:

$$\begin{aligned}
\left|\Lambda^{\sigma'}(t) \cup \Lambda^{\sigma''}(t)\right| &= \left|(\Lambda^{\sigma'}(t) - J) \cup J \cup (\Lambda^{\sigma''}(t) - J)\right| = \\
&= \left|\Lambda^{\sigma'}(t) - J\right| + |J| + \left|\Lambda^{\sigma''}(t) - J\right| = \\
&\geq k - s + 1 + s - 1 + 1 = k + 1
\end{aligned}$$

once again in contradiction with the condition 3. □

Lemma 6.8

Let $T := (V, A)$ be a complete k-dense tree and let σ be k-elimination for T. Let R be the maximal triangulated constituent of T such that $\Gamma(\sigma) \subseteq V(R)$. The ordering $app\left(\sigma_{V-V(R)}, \sigma_{V(R)}\right)$ of the vertices of V is a k-elimination for T.

Proof. We prove the thesis by induction on $|T|$.

If $|T| \leq k+1$, the thesis is trivial, since T is k-tree.

If $|T| = n > k+1$, let us suppose that the thesis holds for each complete k-dense tree of order $n-1$. Let $f := \sigma(1)$. By the definition of k-elimination, we know that such a vertex is a k-leaf of T. Therefore the graph $T' := T_{V-\{f\}}$ is a complete k-dense tree of order $n-1$ and $\sigma' := \sigma_{V-\{f\}}$ is a k-elimination for T'.

Let R' be the maximal triangulated constituent of T' such that $\Gamma(\sigma') \subseteq V(R')$. By the inductive hypothesis we know that

$$\sigma'' := app\left(\sigma'_{V(T')-V(R')}, \sigma'_{V(R')}\right) = app\left(\sigma_{V(T')-V(R')}, \sigma_{V(R')}\right)$$

is a k-elimination for T'. We have two cases.

- $f \in V(R)$. By Proposition 5.2, we know that the sub-sequence $\sigma_{V(R)}$ is a k-elimination for R. Then, since σ'' is a k-elimination for T' (by the inductive hypothesis), the following ordering of the vertices of T:

$$\sigma''' := app\left(\sigma_{V(T')-V(R')}, \left[f \,\middle|\, \sigma_{V(R')}\right]\right)$$

is a k-elimination for T. However, since $\sigma_{V(R)} = \left[f \,\middle|\, \sigma_{V(R')}\right]$ and $V(R') = V(R) - \{f\}$, we have:

$$\begin{aligned} \sigma''' &= app\left(\sigma_{V(T')-V(R')}, \left[f \,\middle|\, \sigma_{V(R')}\right]\right) = \\ &= app\left(\sigma_{(V-\{f\})-(V(R)-\{f\})}, \left[f \,\middle|\, \sigma_{V(R')}\right]\right) = \\ &= app\left(\sigma_{V-V(R)}, \sigma_{V(R)}\right) \end{aligned}$$

and the thesis is proved.

- $f \notin V(R)$. Then, the ordering $\sigma''' := [f \mid \sigma'']$ of the vertices of T is a k-elimination for T. However, since $f \in V - V(R)$, the equality $V(R) = V(R')$ certainly holds then:

$$\begin{aligned} \sigma''' &= \left[f \,\middle|\, app\left(\sigma_{V(T')-V(R')}, \sigma_{V(R')}\right)\right] = \\ &= app\left(\left[f \,\middle|\, \sigma_{V(T')-V(R')}\right], \sigma_{V(R')}\right) = \\ &= app\left(\sigma_{V-V(R)}, \sigma_{V(R)}\right) \end{aligned}$$

and the thesis is proved in this case also. □

As we will see later (§10 (i)), the maximal triangulated constituents of any complete k-dense tree can be found in linear time in the order of the graph once a k-elimination is given.

7 The Theorem of the Core

The Theorem which we are going to introduce is one of the fundamental theoretic results about the complete k-dense trees. It proves that, for each complete k-dense tree T, there exists a particular maximal triangulated constituent

– which contains all the seeds of T

– and such that, for any of its k-clique, there exists a k-elimination for T with such a k-clique as seed.

Therefore, the Theorem 7.1 furnishes a complete characterization of the set of the seeds of a complete k-dense tree.

Theorem 7.1
Let T be a complete k-dense tree. There is a maximal triangulated constituent R of T such that:

(i) for each k-elimination σ for T, $\Gamma(\sigma) \subseteq V(R)$,

(ii) for each k-clique $C \subseteq V(R)$ there is a k-elimination σ' for T such that $\Gamma(\sigma') = C$.

PROOF. If T is trivial, the thesis is obviously true. If T is not trivial, we prove by induction on $|T|$, the existence of a maximal triangulated constituent R which satisfies the property (i).

If $|T| = k + 1$ then T is a k-tree as well and then, given $R := T$, the property (i) is obviously true.

If $|T| = n > k + 1$, let us suppose that for each complete k-dense tree of order $n - 1$ there is a maximal triangulated constituent which satisfies the property (i).

Let f be a k-leaf of T. By Lemma 3.23, we know that $T' := T_{V(T)-\{f\}}$ is a complete k-dense tree of order $n - 1$. Therefore, by the inductive hypothesis, there is a maximal triangulated constituent R' of T' which satisfies the property (i) (for T'). The k-tree R' is a triangulated constituent of T as well and then we have to distinguish between two cases.

- R' is maximal in T. Therefore, by Definition 6.2, the subgraph of T induced by $V(R') \cup \{f\}$ is not a k-tree. Since R' is a k-tree, the set $adj_T(f)$ cannot be a k-clique of R', otherwise the subgraph of T induced by $V(R') \cup \{f\}$ would be a k-tree and a proper supergraph of R', against the hypothesis of maximality in T for that last graph.

  Let σ be a k-elimination for T. We can certainly affirm that $f \notin \Gamma(\sigma)$. If this was not true, the set $adj_T(f)$ would be a k-clique of T, since the last $k+1$ vertices of σ are a $(k+1)$-clique of T and $|adj_T(f)| = k$ (f is a k-leaf of T). Moreover, still supposing that $f \in \Gamma(\sigma)$, there would be a $(k-1)$-clique $C' \subset adj_T(f)$ such that:

  $$C' \cup \{f\} = \Gamma(\sigma)$$

  and then, for the vertex u such that $adj_T(f) = C' \cup \{u\}$, we would have:

  $$\langle\sigma\rangle^{-1}(u) = n - k.$$

  Since the set $adj_T(f)$ is not a k-clique of R', there would be at least one vertex v such that:

  $$v \in adj_T(f) \quad \text{e} \quad v \notin V(R').$$

  Finally, for the above reasonings and by Proposition 5.2, the sub-sequence $\sigma' := \sigma_{V(T)-\{f\}}$ would be a k-elimination for T' with $v \in \Gamma(\sigma')$ although $v \notin V(R')$, against the inductive hypothesis.

  Therefore we can conclude that $f \notin \Gamma(\sigma)$. By Proposition 5.2, we know that the sub-sequence $\sigma' := \sigma_{V(T)-\{f\}}$ is a k-elimination for T' and then, by inductive hypothesis, we have:

  $$\Gamma(\sigma') \subseteq V(R')$$

  and, since $f \notin \Gamma(\sigma)$, we have:

  $$\Gamma(\sigma) \subseteq V(R').$$

  Therefore the maximal triangulated constituent $R := R'$ of T satisfies the property (i).

- R' is not maximal in T. Therefore, by Definition 6.2, the graph $R := T_{V(R')\cup\{f\}}$ is a maximal triangulated constituent of T. Let σ be a k-elimination for T. By Proposition 5.2, we know that the sub-sequence $\sigma' := \sigma_{V(T)-\{f\}}$ is a k-elimination for T' and then, by the inductive hypothesis, we have:

  $$\Gamma(\sigma') \subseteq V(R').$$

  Therefore, it is clear that, whatever the position of f in σ may be, the following condition holds:

  $$\Gamma(\sigma) \subseteq V(R)$$

  then the maximal triangulated constituent R of T satisfies the property (i).

We have proved the existence of a maximal triangulated constituent R of T which satisfies the property (i). Now, we have to prove the property (ii). Let $C \subseteq V(R)$ be a k-clique of R. By Theorem 3.38, we know that C is a seed of R, so we know that there is a k-elimination σ^R for R such that $\Gamma\left(\sigma^R\right) = C$. Let σ be a k-elimination for T. For what we have proved so far, we know that $\Gamma(\sigma) \subseteq V(R)$. So, by Lemma 6.8, the ordering of the vertices of T:

$$app\left(\sigma_{V(T)-V(R)}, \sigma_{V(R)}\right)$$

is a k-elimination. Therefore, the ordering

$$\sigma'' := app\left(\sigma_{V(T)-V(R)}, \sigma^R\right)$$

is a k-elimination for T. Since $\Gamma(\sigma'') = C$, the property (ii) also is proved. □

Definition 7.2
Let T be a complete k-dense tree. The maximal triangulated constituent of T, refered in Theorem 7.1, is the core *of T, denoted it by $\mathcal{C}(T)$.*

We will see later (§10 (ii)) that the core of any complete k-dense tree can be found in linear time in the order of the graph, if a k-elimination is given.

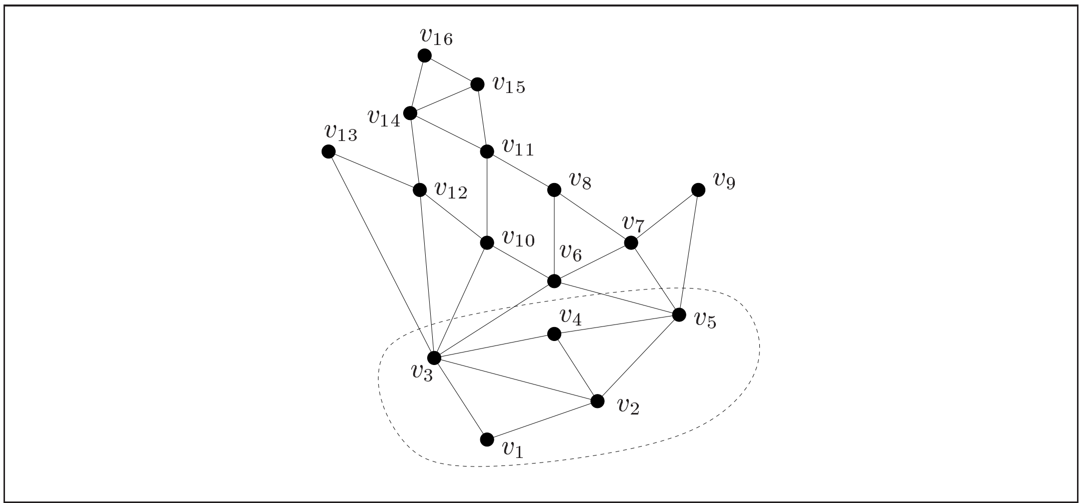

Figure 7.1: A complete 2-dense tree and its core (dotted region)

Lemma 7.3 ([Fra01])
If T is a complete k-dense tree with $|T| \geq k+2$ then $|\mathcal{C}(T)| \geq k+2$.

8 The Theorem of the Order

One of the most meaningful differences between complete k-dense trees and k-trees concerns the relative order of any two adjacent vertices in a k-elimination. In fact, we will prove that, for any two adjacent vertices of a complete k-dense tree, if at least one of them does not belong to the core, then their relative order is the same in any k-elimination.

Theorem 8.1
Let T be a complete k-dense tree. Let s and t be vertices of T such that:

(i) $st \in E(T)$,

(ii) $s \notin V(\mathcal{C}(T))$ or $t \notin V(\mathcal{C}(T))$.

There is an ordering σ^0 of the vertices in $\{s,t\}$ such that, for each k-elimination σ for T:

$$\sigma_{\{s,t\}} = \sigma^0.$$

Less formally: the vertices s and t are in the same relative order in any k-elimination for T.

Proof. We can exclude that T is a k-tree, otherwise $\mathcal{C}(T) = T$ and the thesis is trivial.

Then, we can exclude the cases with $|T| \leq k+1$, in which T is obviously a k-tree. Moreover, we can exclude the case with $|T| = k+2$. In fact, also in this case T is a k-tree, since, by Lemma 7.3, we have that $|\mathcal{C}(T)| \geq k+2$.

Let us then suppose that $|T| \geq k+3$. We will prove the thesis by induction on $|T|$. If $|T| = k+3$ then T, which is not a k-tree, must have only one k-leaf and such a k-leaf cannot belong to the core of T. In fact, by Lemma 7.3, we know that $|\mathcal{C}(T)| \geq k+2$. Moreover, since T is not a k-tree, by Lemma 6.3, we know that there is a k-leaf $d \notin V(\mathcal{C}(T))$. Since $|T| = k+3$, d must be the unique k-leaf of T.

Since d is the unique vertex of T which does not belong to the core, we have to prove the thesis only for any pair of vertices formed by d and by a generic vertex $t \in adj_T(d)$.

Let us suppose that there is a k-elimination σ' such that $\langle\sigma'\rangle^{-1}(t) < \langle\sigma'\rangle^{-1}(d)$. Since d is a k-leaf of T (and so it is adjacent to k vertices only), we should have $d \in \Gamma(\sigma')$. For the definition of k-elimination, the vertex $\sigma'(1) \neq d$ should be a k-leaf of T. However, we know that d is the unique k-leaf of T and we have a contradiction.

If $|T| = n > k+3$, let us suppose that the thesis holds for each complete k-dense tree of order $n-1$. Since T is not a k-tree, by Lemma 6.3, there is a k-leaf f of T which does not belong to the core of T. By Lemma 3.23, we know that the subgraph $T' := T_{V(T)-\{f\}}$ is a complete k-dense tree with $n-1$ vertices. Let s and t be two adjacent vertices of T such that:

$$s \notin V(\mathcal{C}(T)) \quad \text{or} \quad t \notin V(\mathcal{C}(T)).$$

We have two cases.

- $f \notin \{s,t\}$. Let Π_T be the set of all the orderings of the vertices of T. Since $\mathcal{C}(T') \subseteq_{\mathcal{G}} \mathcal{C}(T)$ and since T' is a complete k-dense tree of order $n-1$, by the inductive hypothesis, we know that there is an ordering σ^0 of the vertices of $\{s,t\}$ such that, for each $\sigma' \in \mathbf{E}_{T'}$ (the set of k-eliminations for T') $\sigma'_{\{s,t\}} = \sigma^0$. Let $\Pi_T(\mathbf{E}_{T'})$ be the following set:

  $$\Pi_T(\mathbf{E}_{T'}) := \left\{\sigma \in \Pi_T \;\middle|\; \exists\sigma'.\, \sigma' \in \mathbf{E}_{T'} \;\wedge\; \sigma' = \sigma_{V(T')}\right\}.$$

  The set $\mathbf{E}_T$ of k-eliminations for T is certainly included in $\Pi_T(\mathbf{E}_{T'})$. Let σ be a k-elimination σ'' for T. From above, we have that $\sigma \in \Pi_T(\mathbf{E}_{T'})$, then there is a k-elimination σ'' for T' such that:

  $$\sigma'' = \sigma_{V(T')} = \sigma_{V(T)-\{f\}}.$$

  Therefore, since $\{s,t\} \subset (V(T) - \{f\})$, we have:

  $$\sigma_{\{s,t\}} = \sigma''_{\{s,t\}} = \sigma^0$$

  and the thesis is proved.

- $f \in \{s,t\}$. Let us suppose that $f = s$ (the other case is analogous) and so that $t \in adj_T(f)$. Let σ be a k-elimination for T, we will prove that:

  $$\sigma_{\{f,t\}} = [f,t]$$

  and so that:

  $$\langle\sigma\rangle^{-1}(f) < \langle\sigma\rangle^{-1}(t).$$

  Let us suppose that there is a k-elimination σ' for T such that $\langle\sigma'\rangle^{-1}(t) < \langle\sigma'\rangle^{-1}(f)$. In this hypothesis, f should belong to the seed of σ'. In fact, since $deg_T(f) = k$ and $\langle\sigma'\rangle^{-1}(t) < \langle\sigma'\rangle^{-1}(f)$, we have:

  $$\left|\Lambda^{\sigma'}(f)\right| \leq k-1$$

  and therefore, if f did not belong to $\Gamma(\sigma')$ then the k-elimination σ' for T would not satisfy the conditions of Definition 3.24, against Proposition 3.26 for which we know that each k-elimination of a complete k-dense tree must be of the kind required in Definition 3.24.

  Therefore $f \in \Gamma(\sigma')$ and consequently, since $f \notin V(\mathcal{C}(T))$, we should have that $\Gamma(\sigma') \nsubseteq V(\mathcal{C}(T))$ against the Theorem of the Core. $\square$

Corollary 8.2 ([Fra01]**)**
Let T be a complete k-dense tree and let f be a k-leaf of T not included in the core. For each k-elimination σ for T and for each $t \in adj_T(f)$, we have:

$$\langle\sigma\rangle^{-1}(f) < \langle\sigma\rangle^{-1}(t).$$

From Theorem 3.38 and Observation 3.30 it immediately follows that, given a generic k-tree R, for each vertex v, there is a k-elimination σ for R such that:

$$\sigma(|R|) = v.$$

Therefore, for a pair of adjacent vertices there not exists a unique relative order common to all the k-eliminations for R.

However, for any k-clique C of R, it is possible to prove that, for any pair of adjacent vertices, there exists a relative order, common to all the k-eliminations for R which have C as seed.

Theorem 8.3 ([Fra01]**)**
Let R be a non-trivial k-tree and let C be a k-clique of R. Let s and t be two vertices of T such that:

(i) $st \in E(R)$,

(ii) $s \notin C$ *or* $t \notin C$.

There is an ordering σ^0 of the vertices in $\{s,t\}$ such that, for each k-elimination σ for R with $\Gamma(\sigma) = C$:

$$\sigma_{\{s,t\}} = \sigma^0.$$

Less formally: the vertices s and t are in the same relative order in any k-elimination for R, which has the k-clique C as seed.

9 The Generator of a Maximal Triangulated Constituent

Let T be a complete k-dense tree. By Theorem 7.1, we know that, for each k-clique C of the core of T, there is a k-elimination σ for T such that:

$$\Gamma\left(\sigma_{V(\mathcal{C}(T))}\right) = C.$$

Different is the situation for the other maximal triangulated constituents. We have:

Theorem 9.1
Let T be a complete k-dense tree and let $R \neq \mathcal{C}(T)$ be a non-trivial maximal triangulated constituent of T. There is a k-clique C of R such that, for each k-elimination σ for T:

$$\Gamma\left(\sigma_{V(R)}\right) = C.$$

Proof. Let us suppose that there exists a k-clique $C' \neq C$ of R for which there exists a k-elimination σ' for T such that $\Gamma\left(\sigma'_{V(R)}\right) = C'$, and let σ be a k-elimination for T such that $\Gamma\left(\sigma_{V(R)}\right) = C$.

Since $C' \neq C$, there exists a vertex v of T such that:

$$v \in C' \quad \text{and} \quad v \notin C.$$

Since $v \notin C$ and $\Gamma\left(\sigma_{V(R)}\right) = C$, the vertex v can not stay in any of the last k positions of σ, then we have $|\Lambda^\sigma(v)| = k$ and hence there exists a vertex u such that:

$$u \in \Lambda^\sigma(v) \quad \text{and} \quad u \notin C'.$$

The vertex u certainly does not belong to the core of T. In fact, by Theorem 6.4, R and $\mathcal{C}(T)$ can share at most $k-1$ vertices and, since $\Lambda^\sigma(v) \subset V(R)$, there should be a vertex t such that:

$$t \in \Lambda^\sigma(v) \quad \text{and} \quad t \notin V(\mathcal{C}(T)).$$

Consequently, if v belonged to $V(\mathcal{C}(T))$, the ordering $\sigma_{V(\mathcal{C}(T))}$ of the vertices of $\mathcal{C}(T)$ would not be a k-elimination having the properties of Definition 3.24, against Proposition 5.2.

Finally, the vertices v and u satisfy the hypotheses of Theorem 8.1 and we have $\langle\sigma\rangle^{-1}(v) < \langle\sigma\rangle^{-1}(u)$. Since $u \notin C'$, we have $\langle\sigma'\rangle^{-1}(u) < \langle\sigma'\rangle^{-1}(v)$ in contradiction with Theorem 8.1. $\square$

Definition 9.2
Let T be a complete k-dense tree and let $R \neq \mathcal{C}(T)$ be a non-trivial maximal triangulated constituent of T. The k-clique of R with the property established in Theorem 9.1 is called the generator *of R and is denoted by $\Gamma_T(R)$.*

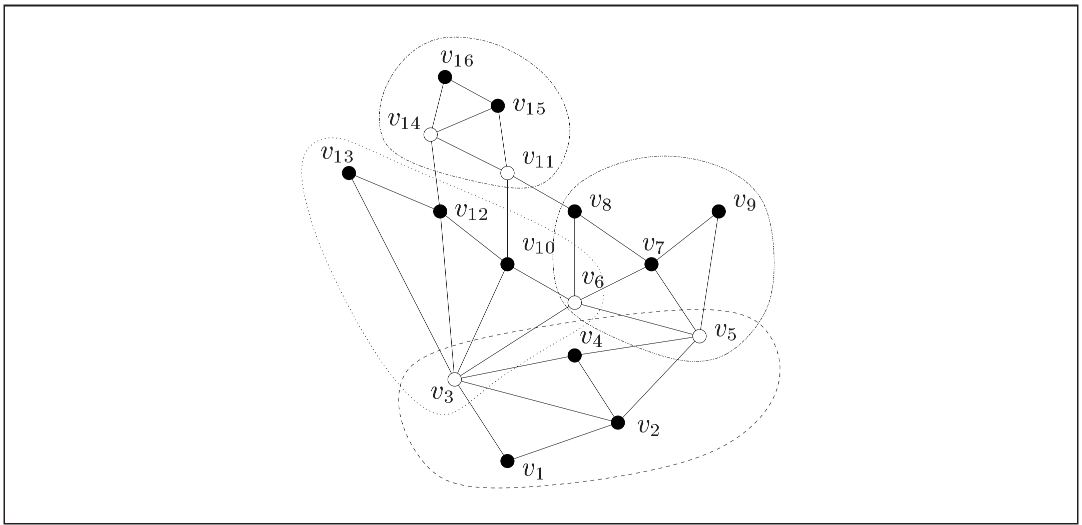

Figure 9.1: The generators of the non-trivial maximal triangulated constituents of a complete 2-dense tree.

Theorem 9.3 ([Fra01]**)**
Let T be a complete k-dense tree and let $R \neq \mathcal{C}(T)$ be a non-trivial maximal triangulated constituent of T.

$$|\Gamma_T(R) \cap V(\mathcal{C}(T))| \leq 1$$

$$\Longleftrightarrow$$

There exists an ordering σ^R of the vertices of $\Gamma_T(R)$ such that, for each k-elimination σ for T, $\sigma_{\Gamma_T(R)} = \sigma^R$.

10 Search for the Maximal Triangulated Constituents

10 (i) Algorithm to locate the maximal triangulated constituents of a complete k-dense tree

Theorem 10.1
Let T be a complete k-dense tree. For any k-elimination σ for T, the Algorithm 10.1, applied to T and σ, finds all the maximal triangulated constituents of T.

Algorithm 10.1 Location of the maximal triangulated constituents of a complete k-dense tree

Input: $\Rightarrow$ $T := (V, A)$, complete k-dense tree,

$\Rightarrow$ σ, k-elimination for T.

Output: $\Leftarrow$ D, array of the sets of vertices of the maximal triangulated constituents of T.

Data:

- N, array of sets of integer. $N[i]$ denote the set of indices in D of triangulated constituents to which the vertex $\sigma(i)$ belongs. Initially, for each i, $N[i] = \emptyset$.
- j, current number of found triangulated constituents.

```
j ← 1
n ← |T|
if n ≤ k + 1 then
  D[1] ← V(T)
  return D
end if
D[1] ← {σ(n − k − 1), ..., σ(n)}
for i = n − k − 1 to n do
  N[i] ← {1}
end for
for i = n − k − 2 downto 1 do
  v ← σ(i)
  ⟨Find the maximal cliques C_1, ..., C_r of the subgraph of T induced by Λ^σ(v)⟩
  if r = 1 (that is C_1 is a k-clique) then
    t ← Intersezione(C_1, σ, N)
    D[t] ← D[t] ∪ {v}
    N[i] ← {t}
  else
    for all C_l, 1 ≤ l ≤ r do
      t ← Intersection(C_l, σ, N)
      if |D[t]| = |C_l| then
        D[t] ← D[t] ∪ {v}
        N[i] ← {t}
      else
        j ← j + 1
        for all u ∈ C_l do
          N[⟨σ⟩^{-1}(u)] ← N[⟨σ⟩^{-1}(u)] ∪ {j}
        end for
        D[j] ← C_l ∪ {v}
        N[i] ← {j}
      end if
    end for
  end if
end for
return D
```

Function 10.2 Intersection

Input: $\Rightarrow C$, set of vertices,

$\Rightarrow \sigma$, ordering of vertices,

$\Rightarrow N$, array of sets of integer,

Output: $\Leftarrow$ any of the integer in I ($|I| > 0$ always holds).

$I \leftarrow \bigcap_{v \in C} N[\langle\sigma\rangle^{-1}(v)]$

return $i \in I$

PROOF. Let σ be a k-elimination for T. If $|T| \leq k+1$ then T is a complete graph and then it is a k-tree. Hence this case is treated correctly by the steps 3, 4 and 5.

If $|T| > k+1$, we will prove the thesis by induction on $|T|$.

If $|T| = k+2$ then T is a k-tree and then the unique maximal triangulated constituent is T itself. In this case, the step 7 of Algorithm 10.1 leads to a situation in which all the $k+1$ vertices of T are contained in $D[1]$. Since $|T| = k+2$, the body of the cycle of step 11 is not executed and therefore the execution ends and returns the triangulated constituent ($D[1]$) which contains all the vertices of T.

If $|T| = n > k+1$, let us suppose that the thesis holds for each complete k-dense tree of order $n-1$. Let us set $f := \sigma(1)$. By Definition 3.24, such a vertex is a k-leaf of T. By Lemma 3.23, the subgraph $T' := T_{V(T)-\{f\}}$ is a complete k-dense tree with $n-1$ vertices and hence, for each k-elimination for T', the Algorithm 10.1 finds all the maximal triangulated constituents of T'. The ordering $\sigma' := \sigma_{V(T)-\{f\}}$ is a k-elimination for T', so, if we apply the Algorithm 10.1 to T' and σ', we obtain an array D' whose elements are the sets of the vertices of all maximal triangulated constituents of T'.

The Algorithm 10.1, applied to T and σ, examines, by the cycle of step 11, the vertices of the graph in order inverted with respect to σ. Moreover, for each iteration of the cycle, all the performed operations are independent from the vertices subsequent the actual one. In fact, if we denote by i the index of a generic iteration of the cycle, the operations for the vertex $v := \sigma(i)$ involve the vertices in $\Lambda^{\sigma}(v)$ only.

Now, it is clear that the array D', obtained from the Algorithm 10.1 applied to T' and σ', is equal to the array D, obtained at the end of the iteration of index $i = 2$ of the cycle of step 11, in the execution of the Algorithm 10.1 with T and σ as input.

Therefore we have proved that, in the execution of the Algorithm 10.1 with T and σ as input, at the end of the iteration of index $i = 2$ of the cycle of step 11, all the maximal triangulated constituents of T' are in D. Then, we have to prove that, in the last iteration ($i = 1$) of the cycle, all the maximal triangulated constituents of T, containing the vertex $f := v := \sigma(1)$, are found. Let R be one of such maximal triangulated constituents. We have two cases.

- $R' := R_{V(R)-\{f\}}$ is a maximal triangulated constituent for T'. From the previous reasonings it derives that R' is in some position of the array D. We have two cases.
  - If $|R'| \geq k$, the set $\Lambda^{\sigma}(f) = adj_T(f)$ must be a k-clique of R': in fact, if this was not true, the ordering $\sigma_{V(R)}$ of the vertices of R would not be a k-elimination for R having the properties of Definition 3.35, against Proposition 3.37 and Proposition 5.2.

    Moreover, by Theorem 6.4, we know that any two maximal triangulated constituents of T' can share at most $k-1$ vertices and then $\Lambda^{\sigma}(f)$ is contained only in R'. Therefore, this case is treated correctly by the condition of the step 14 and by the steps 15, 16 and 17. In fact, the condition in 14 is true if $\Lambda^{\sigma}(f)$ is a k-clique and then, by the inductive hypothesis, it is clear that the index of R' in D is selected at step 15
  - If $|R'| \leq k-1$, since each trivial k-tree is a complete graph, $V(R')$ must be

a maximal clique of the subgraph of T induced by $\Lambda^{\sigma}(f) = adj_T(f)$, that is $V(R') = C_s$ for some clique C_s found at step 13.

Therefore such cases are treated in steps 22 and 23, reached if the condition of the step 21 is true, that is if C_s is a maximal clique of T' as well, that is if R' is a maximal triangulated constituent of T', since the graphs induced by the sets in D are maximal triangulated constituents of T'.

- $R' := R_{V(R)-\{f\}}$ is not a maximal triangulated constituent of T'. Let R'' be the maximal triangulated constituent of T' such that $R' \subset_{\mathcal{G}} R''$. We then have that $|R'| \leq k-1$. In fact, if this was not true, we would have that $adj_T(f) \subseteq V(R')$ and then the subgraph $T_{V(R'')\cup\{f\}}$ would be a triangulated constituent of T containing R, against the hypothesis of maximality. Therefore $|R'| \leq k-1$, then R' is a trivial k-tree. Moreover, since R is a trivial k-tree as well, we have that $V(R') \subset adj_T(f)$. By these arguments, $V(R')$ is one of the cliques, say C_m, found in step 13. Such cases are treated correctly in the steps $25, \ldots, 30$ reached if the condition of step 21 is false, that is if $T_{C_m} = R'$ is not maximal in T'. □

Theorem 10.2 ([Fra01])
Let T be a complete k-dense tree. The Algorithm 10.1 can be implemented to run in $O\left(k2^k n\right)$ time, where $n := |T|$.

10 (ii) Algorithm to locate the core

Algorithm 10.3 Location of the vertices of the core of a complete k-dense tree

Input: $\Rightarrow$ $T := (V, A)$, complete k-dense tree,

$\Rightarrow$ σ, k-elimination for T.

Output: $\Leftarrow$ C, set of the vertices of the core of T.

1: $j \leftarrow 1$
2: $n \leftarrow |T|$
3: **if** $n \leq k+1$ **then**
4: $\quad C \leftarrow V(T)$
5: $\quad$ return C
6: **end if**
7: $C \leftarrow \{\sigma(n-k-1), \ldots, \sigma(n)\}$
8: **for** $i = n-k-2$ downto 1 **do**
9: $\quad v \leftarrow \sigma(i)$
10: $\quad$ **if** $\langle \Lambda^{\sigma}(v)$ *is a k-clique of T_C*$\rangle$ **then**
11: $\quad\quad C \leftarrow C \cup \{v\}$
12: $\quad$ **end if**
13: **end for**
14: return C

Theorem 10.3
Let T be a complete k-dense tree. For each k-elimination σ for T, Algorithm 10.3, applied to T and σ, finds the vertices of the core of T.

Proof. Let σ be a k-elimination for T. If $|T| \leq k+1$ then T is a complete graph and then is a k-tree. Hence $\mathcal{C}(T) = T$: this case is treated correctly by steps 3, 4 and 5.

If $|T| > k+1$, the set of vertices C, obtained at the end of the cycle, induces a k-tree. In fact, if we denote by σ' the ordering of the vertices in $C - \{\sigma(n-k-1), \ldots, \sigma(n)\}$, established by the cycle of step 8, the following ordering of the vertices in C:

$$\sigma^C := app(\sigma', [\sigma(n-k-1), \ldots, \sigma(n)])$$

is a k-elimination for T_C, having the properties required by Definition 3.35.

T_C is also a maximal triangulated constituent of T. In fact, if there was a k-tree R which was subgraph of T such that $T_C \subset_{\mathcal{G}} R$, then, by Proposition 5.2, $\sigma_{V(R)}$ would be a k-elimination for R and there would be a vertex v such that:

$$v \in V(R) \quad \text{and} \quad v \notin C$$

and that $\Lambda^{\sigma}(v) \subseteq C$; moreover, by Proposition 5.2 and Definition 3.35, the set $\Lambda^{\sigma}(v)$ would be a k-clique and then Algorithm 10.3 would have to insert the vertex v in C, according the condition of step 10.

Finally, the graph T_C is a maximal triangulated constituent of T and we have that $\Gamma(\sigma) \subseteq C$. Therefore, from Theorem 7.1 follows that T_C is the core of T. □

Theorem 10.4 ([Fra01])
Let T be a complete k-dense tree. The Algorithm 10.3 has a time complexity of $O(k^2 n)$, where $n := |T|$.

11 The Partial Order $\preccurlyeq_T$

Definition 11.1
Let T be a complete k-dense tree. The relation $\preccurlyeq_T$ between any two vertices $x, y \in V(T)$ is defined in the following manner:

$$x \preccurlyeq_T y \equiv \forall \sigma . \sigma \in \mathbf{E}_T \Rightarrow \langle\sigma\rangle^{-1}(x) \geq \langle\sigma\rangle^{-1}(y)$$

Proposition 11.2 ([Fra01])
For each complete k-dense tree T, the relation $\preccurlyeq_T$ is a partial order.

Proposition 11.3 ([Fra01])
Let R be a k-tree. The partial order $\preccurlyeq_R$ yields a flat ordering of the vertices of R, that is, for any two distinct vertices $x, y \in V(R)$, $x \not\preccurlyeq_R y$.

For each complete k-dense tree T, we can provide a complete description of the partial order $\preccurlyeq_T$. This description make use of a directed graph which is obtained from a particular partial subgraph of T by orientating the edges and by the insertion of some additional oriented edges having the head in $V(\mathcal{C}(T))$ and the tail in $V(T) - V(\mathcal{C}(T))$. The following sub-sections are devoted to the development of the theoretic results which are fundamental in order to introduce such a description.

11 (i) Preliminary results for complete k-dense trees and k-trees

Proposition 11.4
Let R be a k-tree, let $V' \subseteq V(R)$ and let $R' \subseteq_{\mathcal{G}} R$ be a k-tree such that $V' \subseteq V(R')$. If all the k-leaves of R are contained in V' then $R = R'$.

Proof. By induction on $|R|$.

If $|R| \leq k+1$, R is a complete graph and each vertex is a k-leaf, and the thesis holds.

If $|R| = n > k+1$, assume that the thesis holds for each k-tree of order $n-1$. Let f be a k-leaf of R. By Lemma 3.34, we know that $R'' := R_{V(R)-\{f\}}$ is a k-tree of order $n-1$.

The k-tree R' is non-trivial. In fact, we know that R' contains all the k-leaves of R and, by Proposition 3.39, we know that R has at least two k-leaves. Let f_1 and f_2 be two k-leaves of R, by Proposition 3.31, we have that f_1 and f_2 are not adjacent; since any trivial k-tree is a complete graph and since R' contains two non-adjacent vertices, R' must be non-trivial.

Let L be the set of k-leaves of R'' which does not belong to R. For the definition of R'', we have that $L \subseteq adj_R(f)$. For the hypotheses on V', we know that $f \in V'$. Let us pose $V'' := (V' - \{f\}) \cup L$. The set V'' contains all the k-leaves of R'' then, by the inductive hypothesis, the following holds:

$$\text{if } R''' \subseteq_{\mathcal{G}} R'' \text{ is a } k\text{-tree such that } V'' \subseteq V(R''') \text{ then } R''' = R''.$$

Since R' is a non-trivial k-tree and $f \in V(R')$, it is clear that $adj_R(f) \subseteq V(R')$, since f is a k-leaf for R'.

Hence, it follows that $R'_{V(R')-\{f\}}$ is a subgraph of R'' which contains all the vertices in V'' and then it contains all the k-leaves of R''. Therefore, by the consequences of the inductive hypothesis, we can affirm that $R'_{V(R')-\{f\}} = R''$. Moreover, since f is a k-leaf of R', all the k edges incident on it are included in $E(R')$. We can then conclude that $R = R'$. □

Lemma 11.5
Let R ad R' be two k-trees such that $R' \subseteq_{\mathcal{G}} R$ and let σ' be a k-elimination for R'. There exists a k-elimination σ for R such that:

$$\sigma = app\left(\sigma_{V(R)-V(R')}, \sigma'\right).$$

PROOF. If $|R'| \leq k$, the thesis follows directly from Theorem 3.38.

If $|R'| \geq k+1$ (and $|R| \geq k+1$), we will prove the thesis by induction on $|R|$.

If $|R| = k+1$, both graphs are complete and the thesis is trivial.

If $|R| = n > k+1$, let us suppose that the thesis holds for each k-tree of order $n-1$. If $R = R'$, there is nothing to prove, obviously. Let us suppose that $R \neq R'$. In this case, by Proposition 11.4, there exists in R a k-leaf f which does not belong to R'. The graph $R'' := R_{V(R)-\{f\}}$ is a k-tree of order $n-1$ such that $R' \subseteq_{\mathcal{G}} R''$. Then, by inductive hypothesis, there is a k-elimination σ'' for R'' such that:

$$\sigma'' = app\left(\sigma''_{V(R'')-V(R')}, \sigma'\right) = app\left(\sigma''_{V(R)-(V(R')\cup\{f\})}, \sigma'\right).$$

Since f is a k-leaf, the ordering $\sigma := [f \mid \sigma'']$ is a k-elimination for R which satisfies the thesis. □

Proposition 11.6
Let T and T' be complete k-dense trees such that $T' \subseteq_{\mathcal{G}} T$. The graph $R := T'_{V(T')\cap V(\mathcal{C}(T))}$ is a k-tree.

PROOF. Let us set $S := V(T') \cap V(\mathcal{C}(T))$. Let σ be a k-elimination for T, from Lemma 6.8 it follows that:

$$\sigma' := app\left(\sigma_{V(T)-V(\mathcal{C}(T))}, \sigma_{V(\mathcal{C}(T))}\right)$$

is a k-elimination for T. By Proposition 5.2, we know that $\sigma'' := \sigma'_{V(T')}$ is a k-elimination for T'. Moreover, by the definition of σ', it is clear that:

$$\{\sigma''(|T'|-|S|), \ldots, \sigma''(|T'|)\} = S$$

and therefore, since σ'' is a k-elimination for T', the ordering $\sigma''' := \sigma''_S$ is a k-elimination for R, having the properties of Definition 3.24; hence, R is a complete k-dense tree.

If $|S| \leq k+1$, $R \simeq K^{k+1}$ and then R is a k-tree.

Conversely, if $|S| \geq k+2$, let us to suppose that R is not a k-tree. That is, since R is a complete k-dense tree, let us suppose that for some integer i, $1 \leq i \leq |S|-k$, the set of k vertices $\Lambda^{\sigma'''}(\sigma'''(i))$ is not a k-clique of R.

By Proposition 5.1, we know that F is the subgraph of T induced by S. Since the set $\Lambda^{\sigma'''}(\sigma'''(i))$ is not a k-clique of R, it cannot even be a k-clique of T.

The vertex $\sigma'''(i)$ belongs to R which, by definition, is a subgraph of $\mathcal{C}(T)$. Therefore, for the definition of σ' and since $1 \leq i \leq |S|-k$, the set $\Lambda^{\sigma'}(\sigma'''(i))$ must be a k-clique of T.

Since $1 \leq i \leq |S|-k$, it results:

$$\Lambda^{\sigma'''}(\sigma'''(i)) = \Lambda^{\sigma'}(\sigma'''(i)),$$

and therefore we have a contradiction. □

Proposition 11.7
Let T and T' be complete k-dense trees such that $T' \subseteq_{\mathcal{G}} T$ and $|V(T') \cap V(\mathcal{C}(T))| \geq k$. Let σ' be a k-elimination for T'. There is a k-elimination σ for T such that:

$$\sigma = app\left(\sigma_{V(T)-V(T')}, \sigma'\right).$$

PROOF. Let us set $S := V(T') \cap V(\mathcal{C}(T))$. Let σ^0 be a k-elimination for T, by Lemma 6.8, we know that:

$$\sigma'' := app\left(\sigma^0_{V(T)-V(\mathcal{C}(T))}, \sigma^0_{V(\mathcal{C}(T))}\right)$$

is a k-elimination for T.

If $|S| = k$ or $|S| = k+1$, we have that $V(T') = S$. In fact, by Proposition 5.1, T' is the subgraph of T induced by $V(T')$ and, by Proposition 5.2, the ordering $\sigma''' := \sigma''_{V(T')}$ is a k-elimination for T'.

Therefore, the vertices of S are in the last $|S|$ positions of σ'''. Let us assume that $V(T') \neq S$ and let us set $v := \sigma'''(|T'| - |S|)$. For such a vertex, the set $\Lambda^{\sigma'''}(v)$ is a k-clique contained in S, then we have $\Lambda^{\sigma'''}(v) = \Lambda^{\sigma''}(v)$. Then v is included in the core of T and also in S, against the assumptions.

Consequently, if $|S| = k$ or $|S| = k+1$, we have that $V(T') = S$ and the thesis follows directly from Lemma 11.5 and Lemma 6.8.

Now, we have to prove the thesis for $|S| \geq k+2$. By Proposition 11.6, we know that the graph $R := T'_S$ is a k-tree. Since $R \subseteq_{\mathcal{G}} \mathcal{C}(T)$, by Lemma 11.5 we know that there is a k-elimination $\sigma^{\mathcal{N}}$ for $\mathcal{C}(T)$ such that

$$\sigma^{\mathcal{N}} = app\left(\sigma^{\mathcal{N}}_{V(\mathcal{C}(T))-S}, \sigma'_S\right).$$

Since $|S| \geq k+2$, the ordering σ which is defined in the following way:

$$\sigma := app\left(\sigma''_{V(T)-(V(T')\cup V(\mathcal{C}(T)))}, app\left(\sigma^{\mathcal{N}}_{V(\mathcal{C}(T))-S}, \sigma'\right)\right)$$

is a k-elimination for T which satisfies the thesis. □

11 (ii) The graph $min_R^{k\text{-tree}}(V')$

Proposition 11.8
Let R be a k-tree and let V' be a subset of the vertices of R. There exists a unique minimal subgraph R' of R such that:

(i) R' is a k-tree,

(ii) $V' \subseteq V(R')$.

PROOF. We will prove the thesis by induction on $|R|$.

If $|R| \leq k+1$, both R and R' are complete graphs and so the thesis certainly holds, since R' is the complete subgraph of R induced by V'.

If $|R| = n > k+1$, let us suppose that the thesis holds for each k-tree with $n-1$ vertices. We have two cases.

- There is a k-leaf of R which does not belong to V'.

  Since R' is one of the minimal subgraphs respecting the conditions of the thesis, it is clear that $f \notin V(R')$. Otherwise, f would be a k-leaf of R' then, by Lemma 3.34, the graph $R'_{V(R')-\{f\}}$ would respect the conditions of the thesis, although it is a proper subgraph of R'.

  By Lemma 3.34, the graph $R'' := R_{V(R)-\{f\}}$ is a k-tree of order $n-1$. For the choice of f, we know that $V' \subseteq V(R'')$ and that $f \notin V(R')$. Then, from the inductive hypothesis it follows that R' is the unique minimal subgraph of R'' such that it is a k-tree and contains all the vertices of V'.

Since f is a k-leaf of R and since f does not belong to V', by the above reasonings for R', we can conclude that f cannot be contained in any minimal subgraph of R obeying the conditions of the thesis. Therefore, R' is the unique minimal subgraph R such that R is a k-tree and contains all the vertices of V'.

- All the k-leaves of R are contained in V'.

  By Proposition 11.4, we know that the unique subgraph of R such that it is a k-tree and contains all the vertices of V' is R itself. □

Notation 11.9
Let R be a k-tree and let V' be a subset of the vertices of R. We denote by $min_R^{k\text{-tree}}(V')$ the subgraph of Proposition 11.8, that is the minimal subgraph R' of R such that:

(i) R' is a k-tree,

(ii) $V' \subseteq V(R')$.

11 (iii) The sets $\mathcal{C}_T(v)$ and $\Omega_T(v)$

Definition 11.10
Let T be a complete k-dense tree, σ be a k-elimination for T and v be a vertex of T. The dependence on core *of v with respect to σ is the following subset of the vertices of $\mathcal{C}(T)$:*

$$\mathcal{C}_T^{\sigma}(v) := \begin{cases} \{v\} & v \in V(\mathcal{C}(T)) \\ V\left(min_{\mathcal{C}(T)}^{k\text{-tree}}\left(\bigcup_{u \in \Lambda^{\sigma}(v)} \mathcal{C}_T^{\sigma}(u)\right)\right) & v \notin V(\mathcal{C}(T)) \end{cases}$$

Proposition 11.11
Let T be a complete k-dense tree and let σ' and σ'' be k-eliminations for T. For each $v \in V(T)$, the following holds:

$$\mathcal{C}_T^{\sigma'}(v) = \mathcal{C}_T^{\sigma''}(v).$$

Proof. If $v \in V(\mathcal{C}(T))$, the thesis is certainly true. Conversely, if $v \notin V(\mathcal{C}(T))$ then, by Theorem 8.1, $S := \Lambda^{\sigma'}(v) = \Lambda^{\sigma''}(v)$. Let us assume inductively that, for each $u \in S$, $V_u := \mathcal{C}_T^{\sigma'}(u) = \mathcal{C}_T^{\sigma''}(u)$. By Definition 11.10 and by the inductive hypothesis, we have that:

$$\begin{aligned}
\mathcal{C}_T^{\sigma'}(v) &= \\
V\left(min_{\mathcal{C}(T)}^{k\text{-tree}}\left(\bigcup_{u \in \Lambda^{\sigma'}(v)} \mathcal{C}_T^{\sigma'}(u)\right)\right) &= V\left(min_{\mathcal{C}(T)}^{k\text{-tree}}\left(\bigcup_{u \in S} \mathcal{C}_T^{\sigma'}(u)\right)\right) = \\
V\left(min_{\mathcal{C}(T)}^{k\text{-tree}}\left(\bigcup_{u \in S} V_u\right)\right) &= \\
V\left(min_{\mathcal{C}(T)}^{k\text{-tree}}\left(\bigcup_{u \in S} \mathcal{C}_T^{\sigma''}(u)\right)\right) &= V\left(min_{\mathcal{C}(T)}^{k\text{-tree}}\left(\bigcup_{u \in \Lambda^{\sigma''}(v)} \mathcal{C}_T^{\sigma''}(u)\right)\right) \\
&= \mathcal{C}_T^{\sigma''}(v)
\end{aligned}$$

and so the thesis is proved. □

Therefore, by Proposition 11.11, we know that, for each vertex v of a complete k-dense tree T, the set of vertices $\mathcal{C}_T^{\sigma}(v)$ is independent of the particular k-elimination σ which is used to define it. From now on, this set will be denoted simply by $\mathcal{C}_T(v)$.

We have given definition of the set $\mathcal{C}_T(v)$ by making use of k-eliminations; in this way, it is immediately clear how this set can be calculated using the Algorithm 10.3 to obtain the core of T. As we will see, we will make similar choices for the definitions of the set $\Phi_T(v)$ and the graph $\langle T \rangle$.

Observation 11.12
Let T be a complete k-dense tree and let v be a vertex of T. For each $t \in \mathcal{C}_T(v)$, we have:

$$\mathcal{C}_T(t) \subseteq \mathcal{C}_T(v).$$

Lemma 11.13
Let T be a complete k-dense tree. For each $v \notin V(\mathcal{C}(T))$ we have:

$$|\mathcal{C}_T(v)| \geq k+2.$$

Proof. Let us assume the existence of some vertices which are not included in the core of T (otherwise the thesis holds obviously). From Lemma 7.3 it follows that $|T| \geq k+3$.

Let σ be a k-elimination for T. For each vertex v, the set $\mathcal{C}_T(v)$ is defined recursively by the sets $\mathcal{C}_T(u)$, for each $u \in \Lambda^\sigma(v)$. Therefore, we prove the thesis by induction on this sets.

Let c be the vertex such that:

$$\langle\sigma\rangle^{-1}(c) = max\left\{\langle\sigma\rangle^{-1}(t) \mid t \notin V(\mathcal{C}(T))\right\},$$

this vertex represents the induction's basis. By Theorem 7.1, we know that $\Gamma(\sigma) \subset V(\mathcal{C}(T))$ and then $|\Lambda^\sigma(c)| = k$. Moreover, by the choice of c, we know that $\Lambda^\sigma(c) \subset V(\mathcal{C}(T))$ and that $\Lambda^\sigma(c)$ is not a k-clique of $\mathcal{C}(T)$. Therefore, for any k-tree R' subgraph of $\mathcal{C}(T)$ and containing the vertices in $\Lambda^\sigma(c)$, it results: $|R'| \neq k$. Otherwise, R' would be a complete graph, against what we have established for $\Lambda^\sigma(c)$. Moreover, we have $|R'| \neq k+1$. Otherwise, in this case as well, R' would be a complete graph and then any subset of k vertices of $V(R')$ would be a k-clique, once again in contradiction with what we have established for $\Lambda^\sigma(c)$.

By Definition 11.10, we have:

$$\mathcal{C}_T(c) = V\left(min^{k\text{-tree}}_{\mathcal{C}(T)}\left(\bigcup_{u\in\Lambda^\sigma(c)} \mathcal{C}^\sigma_T(u)\right)\right) = V\left(min^{k\text{-tree}}_{\mathcal{C}(T)}\left(\Lambda^\sigma(c)\right)\right)$$

and hence, we can conclude that $|\mathcal{C}_T(c)| \geq k+2$.

Let $v \neq f$ be a vertex which does not belong to the core of T. Let us assume inductively that the thesis holds for each vertex $u \in \Lambda^\sigma(v)$ not included in the core. We have two cases.

- $\Lambda^\sigma(v) \subset V(\mathcal{C}(T))$. In this case, by the same arguments used for c, we have that $|\mathcal{C}_T(v)| \geq k+2$.
- $\Lambda^\sigma(v) \not\subset V(\mathcal{C}(T))$. Let $d \in \Lambda^\sigma(v)$ be a vertex not in the core. By the inductive hypothesis, we have:

  $$|\mathcal{C}_T(d)| \geq k+2,$$

  then:

  $$\left|\bigcup_{u\in\Lambda^\sigma(v)} \mathcal{C}^\sigma_T(u)\right| \geq k+2.$$

  Therefore, by Definition 11.10, we can conclude that $|\mathcal{C}_T(v)| \geq k+2$. □

Definition 11.14
Let T be a complete k-dense tree, let σ be a k-elimination for T and let v be a vertex of T. The indirect dependence on core *of v is the following subset of the vertices of $\mathcal{C}_T(v)$:*

$$\Omega_T(v) := \begin{cases} \emptyset & v \in V(\mathcal{C}(T)) \\ \mathcal{C}_T(v) - \bigcup_{u\in\Lambda^\sigma(v)} \mathcal{C}_T(u) & v \notin V(\mathcal{C}(T)) \end{cases}$$

By Theorem 8.1, we know that, for any complete k-dense tree T, the set $\Lambda^\sigma(v)$, with $v \notin V(\mathcal{C}(T))$, is independent from the particular k-elimination used to denote it, then we omit the reference to such a k-elimination from the notation of indirect dependence on core of v.

Lemma 11.15
Let T be a complete k-dense tree and let $v \notin V(\mathcal{C}(T))$ be a vertex of T. For each $u \in \Omega_T(v)$ we have:

$$deg_{T_{\mathcal{C}_T(v)}}(u) > k$$

and therefore, since $T_{\mathcal{C}_T(v)} \subseteq_{\mathcal{G}} \mathcal{C}(T)$, $deg_{\mathcal{C}(T)}(u) > k$.

Proof. Let σ be a k-elimination for T and let:

$$I := \bigcup_{u \in \Lambda^\sigma(v)} \mathcal{C}_T^\sigma(u).$$

By Definition 11.10 and Notation 11.9, $T_{\mathcal{C}_T(v)}$ is the minimal subgraph R' of $\mathcal{C}(T)$ such that R' is a k-tree and $I \subseteq V(R')$. Then, from Observation 3.22 it follows that $\delta\left(T_{\mathcal{C}_T(v)}\right) = k$. Since $\Omega_T(v) = \mathcal{C}_T(v) - I$, if there was a vertex $u \in \Omega_T(v)$ such that $deg_{T_{\mathcal{C}_T(v)}}(u) = k$, u would be a k-leaf of $T_{\mathcal{C}_T(v)}$ and the graph $R_u := T_{\mathcal{C}_T(v)-\{u\}}$ would be a k-tree such that $I \subseteq V(R_u)$ and $R_u \subset_{\mathcal{G}} T_{\mathcal{C}_T(v)}$ against the hypothesis on minimality of $T_{\mathcal{C}_T(v)}$. □

11 (iv) The cone of a vertex

Definition 11.16
Let T be a complete k-dense tree and let σ be a k-elimination for it. For each vertex v of T, we define recursively the following subset of the set of vertices $V(T) - V(\mathcal{C}(T))$:

$$\Phi_T^\sigma(v) := \begin{cases} \emptyset & v \in V(\mathcal{C}(T)) \\ (\Lambda^\sigma(v) - V(\mathcal{C}(T))) \cup \bigcup_{u \in \Lambda^\sigma(v)} \Phi_T^\sigma(u) & v \notin V(\mathcal{C}(T)) \end{cases}$$

Lemma 11.17
Let T be a complete k-dense tree and let σ' and σ'' be k-eliminations for T. For each $v \in V(T)$, the following holds:

$$\Phi_T^{\sigma'}(v) = \Phi_T^{\sigma''}(v).$$

Proof. The proof is analogous to the one of Proposition 11.11. For a detailed proof see [Fra01]. □

Therefore, by Lemma 11.17, we know that, for each vertex v of a complete k-dense tree T, the set of vertices $\Phi_T^\sigma(v)$ is independent of the particular k-elimination σ which is used to define it. So, from now on, this set will be denoted simply by $\Phi_T(v)$.

Observation 11.18
For each vertex v of a complete k-dense tree T, the following holds:

$$\mathcal{C}_T(v) \cap \Phi_T(v) = \emptyset.$$

Observation 11.19
Let T be a complete k-dense tree and let v be a vertex of T. For each $t \in \Phi_T(v)$, we have:

$$\Phi_T(t) \subseteq \Phi_T(v).$$

Definition 11.20
Let T be a complete k-dense tree and let σ be a k-elimination for it. For each vertex v of T, the cone *of v, denoted by $\Theta_T(v)$, is the following subset of vertices of T:*

$$\Theta_T(v) := \Phi_T(v) \cup \mathcal{C}_T(v).$$

The set $\Phi_T(v)$ is the body *of the cone, whereas $\mathcal{C}_T(v)$ is the* base.

Lemma 11.21
Let T be a complete k-dense tree and let v be a vertex of T. For each $t \in \Theta_T(v)$ we have:

$$\Theta_T(t) \subseteq \Theta_T(v)$$

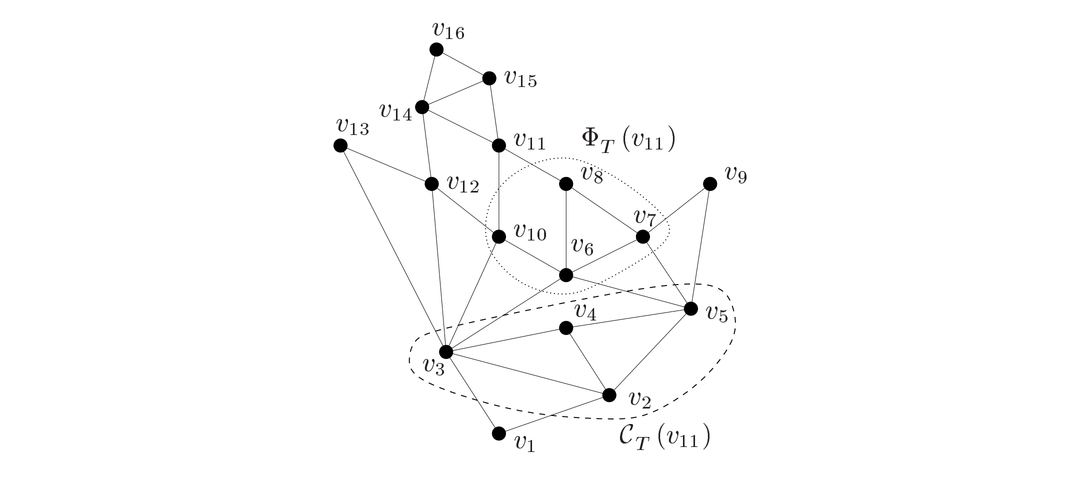

Figure 11.1: The base and the body of the cone of v_{11} are contained in the dotted regions.

PROOF. Let $r \in \Theta_T(t)$. By Definition 11.20, we have two cases.

- $r \in \Phi_T(t)$. By Observation 11.19, we know that $\Phi_T(t) \subseteq \Phi_T(v)$ then $r \in \Phi_T(v)$. Consequently $r \in \Theta_T(v)$.
- $r \in \mathcal{C}_T(t)$. By Observation 11.12, we know that $\mathcal{C}_T(t) \subseteq \mathcal{C}_T(v)$ then $r \in \mathcal{C}_T(v)$. Consequently $r \in \Theta_T(v)$. $\square$

Proposition 11.22
Let T be a complete k-dense tree and v be a vertex not belonging to the core of T. The graph $T_{\{v\}\cup\Theta_T(v)}$ is a non-trivial complete k-dense tree.

PROOF. Let us set $n := |T|$, $n' := |\{v\} \cup \Theta_T(v)|$, $n_c := |\Phi_T(v)|$ and $n_b := |\mathcal{C}_T(v)|$.

By Lemma 11.13, we know that $n' \geq k+3$, then $T_{\{v\}\cup\Theta_T(v)}$ is not a trivial k-dense tree. Moreover $T_{\{v\}\cup\Theta_T(v)}$ is a subgraph of a complete k-dense tree and therefore it cannot contain k-cycles, hence it must be a k-dense forest.

Let σ be a k-elimination for T. For each vertex $d \in V(T)$, let us set:

$$\alpha^\sigma(d) := \{a \in E(T) \mid \exists u.\ u \in \Lambda^\sigma(d) \wedge a = du\}.$$

Since T is a complete k-dense tree, such sets of edges have the following properties:

(i) for each $d_1, d_2 \in V(T)$, if $d_1 \neq d_2$, $\alpha^\sigma(d_1) \cap \alpha^\sigma(d_2) = \emptyset$,

(ii) for each $d \in V(T)$, if $d \notin \Gamma(\sigma)$, $|\alpha^\sigma(d)| = k$.

From Definition 11.10 and Definition 11.16 it immediately follows that, for each $d \in \{v\} \cup \Phi_T(v)$:

$$\begin{aligned} \Lambda^\sigma(d) \quad &- \quad V(\mathcal{C}(T)) \subseteq \Phi_T(v) \\ \Lambda^\sigma(d) \quad &\cap \quad V(\mathcal{C}(T)) \subseteq \mathcal{C}_T(v). \end{aligned}$$

Hence, for each $d \in \{v\} \cup \Phi_T(v)$, we have that $\alpha^\sigma(d) \subset E\left(T_{\{v\}\cup\Theta_T(v)}\right)$. Moreover, since $\{v\} \cup \Phi_T(v) \subseteq V(T) - V(\mathcal{C}(T))$, for each $d \in \{v\} \cup \Phi_T(v)$, we know that $d \notin \Gamma(\sigma)$ and then, by (ii), we have $|\alpha^\sigma(d)| = k$.

From Definition 11.10 it immediately follows that the graph $T_{\mathcal{C}_T(v)}$ is a k-tree, hence we have:

$$\left\|T_{\mathcal{C}_T(v)}\right\| = n_b k - \frac{k(k+1)}{2}.$$

Consequently, for the above reasonings and by the property (i), we have:

$$\left\|T_{\{v\}\cup\Theta_T(v)}\right\| \quad = \quad k + \left|\bigcup_{d\in\Phi_T(v)} \alpha^\sigma(d)\right| + \left\|T_{\mathcal{C}_T(v)}\right\| =$$

$$
\begin{aligned}
&= k + n_c k + n_b k - \frac{k(k+1)}{2} = \\
&= n'k - \frac{k(k+1)}{2}
\end{aligned}
$$

The k-dense forest $T_{\{v\}\cup\Theta_T(v)}$ is connected: otherwise, $T_{\{v\}\cup\Theta_T(v)}$ would have $m \geq 2$ connected components $U_1, \ldots, U_m$ and, by Observation 3.5, such components should be k-dense trees; consequently, by Proposition 3.17, the following would hold:

$$
\begin{aligned}
\left\|T_{\{v\}\cup\Theta_T(v)}\right\| &= \left|\bigcup_{i=1\ldots m} E(U_i)\right| = \sum_{i=1\ldots m} \|U_i\| = \\
&\leq \left(\sum_{i=1\ldots m} |U_i|\right) k - m\frac{k(k+1)}{2} = \\
&< n'k - \frac{k(k+1)}{2}
\end{aligned}
$$

against what we have proved previously.

Therefore $T_{\{v\}\cup\Theta_T(v)}$ is a graph of order n', does not contain k-cycles, it is connected and has $n'k - \frac{k(k+1)}{2}$ edges; hence $T_{\{v\}\cup\Theta_T(v)}$ is a non-trivial complete k-dense tree. □

11 (v) The graph $\langle T\rangle$

Definition 11.23
Let T be a complete k-dense tree and let $\sigma := [v \mid \sigma']$ be a k-elimination for T. We will denote by $\langle T\rangle_\sigma$ the directed graph $(V(T), A_T(\sigma))$, where $A_T(\sigma)$ is the set of oriented edges defined in the following way:

$$
A_T(\sigma) := \begin{cases}
\emptyset & |\sigma| = 1 \\
A_T(\sigma') & |\sigma| > 1 \text{ and } v \in V(\mathcal{C}(T)) \\
A_T(\sigma') \cup \{(v,u) \mid u \in (\Lambda^\sigma(v) \cup \Omega_T(v))\} & |\sigma| > 1 \text{ and } v \notin V(\mathcal{C}(T))
\end{cases}
$$

Proposition 11.24
Let T be a complete k-dense tree and let σ' and σ'' be k-eliminations for T. The following holds:

$$\langle T\rangle_{\sigma'} = \langle T\rangle_{\sigma''}.$$

Proof. We will prove the thesis by induction on $|T|$.

If $|T| = 1$, the thesis is trivial.

If $|T| = n > 1$, let us assume that the thesis holds for each complete k-dense tree of order $n-1$. Let f be a k-leaf of T. By Lemma 3.23, we know that the graph $T' := T_{V(T)-\{f\}}$ is a complete k-dense tree of order $n-1$. The two orderings $\sigma^1 := \sigma'_{V(T)-\{f\}}$ and $\sigma^2 := \sigma''_{V(T)-\{f\}}$ are k-eliminations for T' then, by the inductive hypothesis, $\langle T'\rangle_{\sigma^1} = \langle T'\rangle_{\sigma^2}$. By Definition 11.23, it results that:

$$
\begin{aligned}
\langle T\rangle_{\sigma'} &= \left(V\left(\langle T'\rangle_{\sigma^1}\right) \cup \{f\}, E\left(\langle T'\rangle_{\sigma^1}\right) \cup A_f^{\sigma'}\right) \\
\langle T\rangle_{\sigma''} &= \left(V\left(\langle T'\rangle_{\sigma^2}\right) \cup \{f\}, E\left(\langle T'\rangle_{\sigma^2}\right) \cup A_f^{\sigma''}\right)
\end{aligned}
$$

where we denote by $A_f^{\sigma'}$ and $A_f^{\sigma''}$ the following sets of oriented edges:

$$
A_f^{\sigma'} := \begin{cases}
\emptyset & f \in V(\mathcal{C}(T)) \\
\left\{(f,u) \;\middle|\; u \in (\Lambda^{\sigma'}(f) \cup \Omega_T(f))\right\} & f \notin V(\mathcal{C}(T))
\end{cases}
$$

$$
A_f^{\sigma''} := \begin{cases}
\emptyset & f \in V(\mathcal{C}(T)) \\
\left\{(f,u) \;\middle|\; u \in (\Lambda^{\sigma''}(f) \cup \Omega_T(f))\right\} & f \notin V(\mathcal{C}(T))
\end{cases}
$$

By inductive hypothesis we know that $V\left(\langle T'\rangle_{\sigma^1}\right) = V\left(\langle T'\rangle_{\sigma^2}\right)$ and that $E\left(\langle T'\rangle_{\sigma^1}\right) = E\left(\langle T'\rangle_{\sigma^2}\right)$ Therefore, in order to prove the thesis, it is sufficient to verify $A_f^{\sigma'} = A_f^{\sigma''}$. We have two cases.

- If $f \in V(\mathcal{C}(T))$, we have obviously that $A_f^{\sigma'} = \emptyset = A_f^{\sigma''}$.
- If $f \notin V(\mathcal{C}(T))$, by Theorem 8.1, we know that $\Lambda^{\sigma'}(f) = \Lambda^{\sigma''}(f)$. Moreover, by Proposition 11.11 and Definition 11.14, we know that $\Omega_T(f)$ is independent of the paricular k-elimination which is used to obtain it. Therefore, in this case as well, we have $A_f^{\sigma'} = A_f^{\sigma''}$. □

The above proposition says that the directed graph $\langle T\rangle_\sigma$ associated to a complete k-dense tree T is independent of the particular k-elimination σ used to define it. Hence, from now on, this graph will be simply denoted by $\langle T\rangle$.

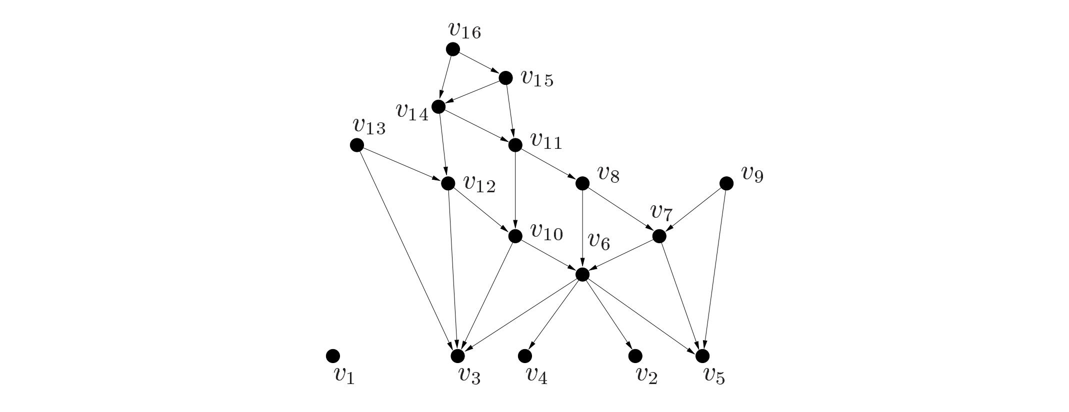

Figure 11.2: The directed graph associated to the complete 2-dense tree in Figure 6.1.

Observation 11.25
Let T be a complete k-dense tree and let f be a k-leaf of T, the graph $\langle T\rangle$ has the following properties:

- *$\langle T\rangle$ does not contain oriented cycles,*
- *the set of vertices $V(\mathcal{C}(T))$ is independent in $\langle T\rangle$,*
- $\left\langle T_{V(T)-\{f\}}\right\rangle = \langle T\rangle_{V(T)-\{f\}}$

11 (vi) Relations between the cone of a vertex and $\langle T\rangle$

Proposition 11.26
Let T be a complete k-dense tree. Let v and t ($v \neq t$) be vertices of T. If there exists an oriented path in $\langle T\rangle$ from v to t then $t \in \Theta_T(v)$.

Proof. If T was a k-tree, we would have that $\langle T\rangle = (V(T), \emptyset)$ then no two distinct vertices would be connected by a path in $\langle T\rangle$.

Since T is not a k-tree, by Lemma 7.3, we know that $|T| \geq k+3$. We will prove the thesis by induction on $|T|$.

If $|T| = k+3$, by Lemma 6.3, we know that there exists a k-leaf $d \notin V(\mathcal{C}(T))$. Moreover, by Lemma 7.3, we know that $|\mathcal{C}(T)| \geq k+2$. Since $|T| = k+3$, d is the unique k-leaf of T and the unique vertex not in the core of T. Let σ be a k-elimination for T. Since d is the unique k-leaf of T, we have that $\sigma(1) = d$. From Definition 11.23 it follows directly that, for each edge $a \in E(\langle T\rangle)$, $t(a) = d$. This implies that $v = d$ and that v is adjacent to t in $\langle T\rangle$. Again by Definition 11.23, we have that $t \in \Lambda^\sigma(d) \cup \Omega_T(d)$ hence, by Definitions 11.10, 11.14 and 11.16, we can conclude that $t \in \Theta_T(d)$.

If $|T| = n > k+3$, let us assume that the theis holds for each complete k-dense tree with $n-1$ vertici.

Since T is not a k-tree, by Lemma 6.3, we know that there is a k-leaf f not in the core of T. The graph $T' := T_{V(T)-\{f\}}$ is a complete k-dense tree with $n-1$ vertices. Since f is a k-leaf of T, there is a k-elimination σ^f for T with $\sigma^f(1) = f$.

Let P be the path in $\langle T \rangle$ from v to t. We have two cases.

- $f \notin \{v, t\}$. Therefore $v, t \in V(T')$. Since $\langle T' \rangle = \langle T_{V(T)-\{f\}} \rangle = \langle T \rangle_{V(T)-\{f\}}$ and since all the edges of $\langle T \rangle_{\sigma^f} = \langle T \rangle$ incident on f have f as tail, the oriented path from v to t in $\langle T \rangle$ is contained in $\langle T' \rangle$ as well. Then, by the inductive hypothesis, we have: $t \in \Theta_{T'}(v)$. Since $f \notin V(\mathcal{C}(T))$, it results that $f \notin \mathcal{C}_T(v)$ and $\mathcal{C}_T(v) = \mathcal{C}_{T'}(v)$. By Lemma 11.17 and by definition of σ^f, we have that $f \notin \Phi_T^{\sigma^f}(v) = \Phi_T(v)$ and this implies $\Phi_T(v) = \Phi_{T'}(v)$. Consequently $\Theta_T(v) = \Theta_{T'}(v)$, hence $t \in \Theta_T(v)$.
- $f \in \{v, t\}$. By definition of σ^f, Definition 11.23, and Proposition 11.24, we know that all the edges of $\langle T \rangle_{\sigma^f} = \langle T \rangle$, incident on f, have f as tail and that all the vertices, adjacent to f in $\langle T \rangle$, belong to the set $V := \Lambda^{\sigma^f}(f) \cup \Omega_T(f)$.

  All this implies that $f = v$ and that the vertex successor of v in P, say v', belongs to the set V and, finally, that the subpath $P' := v'P$ is contained in $\langle T' \rangle$. By inductive hypothesis, we know that $t \in \Theta_{T'}(v')$. Moreover, for the same reasons of the previous case, we have that $\Theta_{T'}(v') = \Theta_T(v')$. Since $v' \in V$ and $V \subseteq \Theta_T(v)$, it is obvious that $v' \in \Theta_T(v)$. Therefore, from Lemma 11.21 it follows that $\Theta_T(v') \subseteq \Theta_T(v)$ and $t \in \Theta_T(v)$. □

Proposition 11.27
Let T be a complete k-dense tree. For each $v \in V(T)$ and for each $t \in \Theta_T(v)$, we have:

(i) there is an oriented path in $\langle T \rangle$ from v to t which is contained in $\langle T \rangle_{\{v\} \cup \Theta_T(v)}$,

(ii) no oriented path in $\langle T \rangle$ from v to t exists which is not contained in $\langle T \rangle_{\{v\} \cup \Theta_T(v)}$.

Proof. Let v be a vertex of T. If T is a k-tree we have that $\langle T \rangle = (V(T), \emptyset)$ and $\Theta_T(v) = \{v\}$. Hence the thesis holds.

If T is not a k-tree then, by Lemma 7.3, we know that $|T| \geq k+3$. We will prove the thesis by induction on $|T|$.

If $|T| = k+3$ then, by Lemma 6.3, there is a k-leaf $d \notin V(\mathcal{C}(T))$. Moreover, by Lemma 7.3, we know that $|\mathcal{C}(T)| \geq k+2$. Since $|T| = k+3$, d is certainly the unique k-leaf of T and it is the unique vertex not included in $V(\mathcal{C}(T))$. We have two cases.

- $v \neq d$. Then $v \in V(\mathcal{C}(T))$ and $\Theta_T(v) = \{v\}$, hence the thesis is obviously true.
- $v = d$. Then $adj_T(v) \subset V(\mathcal{C}(T))$, since d is the unique k-leaf of T, all the k-leaves of $\mathcal{C}(T)$ are contained in $adj_T(v)$ and then, by Proposition 11.4, the following holds:

$$\begin{aligned} \mathcal{C}_T(v) &= V\left(min_{\mathcal{C}(T)}^{k\text{-tree}}\left(\bigcup_{u \in \Lambda^\sigma(v)} \mathcal{C}_T^\sigma(u)\right)\right) = V\left(min_{\mathcal{C}(T)}^{k\text{-tree}}(adj_T(v))\right) = \\ &= V(\mathcal{C}(T)) \end{aligned}$$

  hence $\Theta_T(v) = \Phi_T(v) \cup \mathcal{C}_T(v) = \mathcal{C}_T(v) = V(\mathcal{C}(T))$. By Definition 11.23, for each vertex s in $V := adj_T(v) \cup \Omega_T(v)$ there is an edge $a \in E(\langle T \rangle)$ such that:

$$t(a) = v \quad \text{and} \quad h(a) = s.$$

  However, since $\Omega_T(v) = \mathcal{C}_T(v) - adj_T(v)$, we have that $V = \mathcal{C}_T(v) = \Theta_T(v)$ hence the condition (i) is true. The condition (ii) is obviously respected since $\Theta_T(v) = V(\langle T \rangle) - \{d\}$.

If $|T| = n > k+3$, let us suppose that the thesis holds for each complete k-dense tree with $n-1$ vertices.

Since T is not a k-tree, by Lemma 6.3, we know that there is a k-leaf f of T not included in the core. The graph $T' := T_{V(T)-\{f\}}$ is a complete k-dense tree with $n-1$ vertices. We have two cases.

- $v \neq f$. From the inductive hypothesis it follows that for each $t \in \Theta_{T'}(v)$:
  - ∗ there is an oriented path P in $\langle T' \rangle$ from v to t which is contained in $\langle T' \rangle_{\{v\} \cup \Theta_{T'}(v)}$,
  - ∗ no oriented path in $\langle T' \rangle$ from v to t exists, which is not contained in $\langle T' \rangle_{\{v\} \cup \Theta_{T'}(v)}$.

  Since $f \notin V(\mathcal{C}(T))$, it is obvious that $f \notin \mathcal{C}_T(v)$ and $\mathcal{C}_T(v) = \mathcal{C}_{T'}(v)$. Moreover, since f is a k-leaf of T, there is a k-elimination σ^f for T with $\sigma^f(1) = f$ hence, by Lemma 11.17:
  $$f \notin \Phi_T^{\sigma^f}(v) = \Phi_T(v)$$
  from which it follows that $\Phi_T(v) = \Phi_{T'}(v)$. Then $\Theta_T(v) = \Theta_{T'}(v)$ and, by Observation 11.25, we know that: $\langle T' \rangle = \langle T_{V(T)-\{f\}} \rangle = \langle T \rangle_{V(T)-\{f\}}$, then the path P is also contained in $\langle T \rangle_{\{v\} \cup \Theta_{T'}(v)} = \langle T \rangle_{\{v\} \cup \Theta_T(v)}$. Hence, v satisfies the condition (i) in $\langle T \rangle$. From Definition 11.23 it follows that all the edges of $\langle T \rangle$ incident in f, must have f as tail, then no oriented path in $\langle T \rangle$ can have f as inner vertex, hence v satisfies the condition (ii) in $\langle T \rangle$.

- $v = f$. Let σ be a k-elimination for T such that $\sigma(1) = v$. Let us consider a vertex $t \in \Theta_T(v)$. We have two cases.
  - $t \in \Omega_T(v)$. Therefore, by Definition 11.23, there exists an edge $a \in E(\langle T \rangle)$ such that:
    $$t(a) = v \quad \text{and} \quad h(a) = t.$$
    Hence the condition (i) is respected. Let us suppose that there exists in $\langle T \rangle$ an oriented path P, from v to t, not contained in $\langle T \rangle_{\{v\} \cup \Theta_T(v)}$. By Definition 11.23 and Proposition 11.24, we know that any vertex adjacent to v in $\langle T \rangle_\sigma = \langle T \rangle$ belongs to the set $(\Lambda^\sigma(v) \cup \Omega_T(v)) \subseteq \Theta_T(v)$. Therefore the successor of v in the path P, denoted by v', belongs to $\Theta_T(v)$. From this observation, it follows that the oriented subpath $P' := v'P$, from v' to t, contained in $\langle T' \rangle$ and not contained in $\langle T \rangle_{\{v\} \cup \Theta_T(v)}$, is not even contained in $\langle T' \rangle_{\{v'\} \cup \Theta_{T'}(v')}$, as $\Theta_{T'}(v') \subseteq \Theta_T(v)$. Since P' is an oriented path from v' to t, from Proposition 11.26 it follows that $t \in \Theta_{T'}(v')$. By the inductive hypothesis, we know that no oriented path from v' to t exists in $\langle T' \rangle$ not contained in $\langle T' \rangle_{\{v'\} \cup \Theta_{T'}(v')}$. So we have a contradiction.
  - $t \in \Phi_T(v)$ or $t \in \mathcal{C}_T(v) - \Omega_T(v)$. If $t \in adj_T(v)$, the proof is analogous to the one of the previous case. Let us suppose that $t \notin adj_T(v)$. In this case, by the Definitions 11.10, 11.14 and 11.16, there must be a vertex $u \in adj_T(v)$ such that $t \in \Theta_T(u)$. By the inductive hypothesis, for each $v' \in V(T')$ such that $t \in \Theta_{T'}(v')$, we know that:
    - ∗ there is an oriented path P' in $\langle T' \rangle$ from v' to t which is contained in $\langle T' \rangle_{\{v'\} \cup \Theta_{T'}(v')}$,
    - ∗ no oriented path in $\langle T' \rangle$ from v' to t exists, which is not contained in $\langle T' \rangle_{\{v'\} \cup \Theta_{T'}(v')}$.

    Since $v \notin V(\mathcal{C}(T))$, it is obvious that $v \notin \mathcal{C}_T(u)$ hence we have that $\mathcal{C}_T(u) = \mathcal{C}_{T'}(u)$. By the assumptions, we know that v is a k-leaf of T hence there is a k-elimination σ^v for T with $\sigma^v(1) = v$. By Lemma 11.17, we have:
    $$v \notin \Phi_T^{\sigma^v}(u) = \Phi_T(u)$$
    and this implies that $\Phi_T(u) = \Phi_{T'}(u)$. This is enough to conclude that $\Theta_T(u) = \Theta_{T'}(u)$. By Observation 11.25, we know that $\langle T' \rangle = \langle T_{V(T)-\{v\}} \rangle = \langle T \rangle_{V(T)-\{v\}}$

hence the path P' is contained in $\langle T\rangle_{\{u\}\cup\Theta_{T'}(u)} = \langle T\rangle_{\{u\}\cup\Theta_T(u)}$. Therefore u satisfies the condition (i) in $\langle T\rangle$, and the path: $P := vP'$ is oriented from v to t and is contained in $\langle T\rangle_{\{v\}\cup\Theta_T(v)}$.

Let us suppose that there exists in $\langle T\rangle$ an oriented path P'' from v to t not contained in $\langle T\rangle_{\{v\}\cup\Theta_T(v)}$. By Definition 11.23 and Proposition 11.24, any vertex, adjacent to v in $\langle T\rangle_\sigma = \langle T\rangle$, belongs to the set $(\Lambda^\sigma(v)\cup\Omega_T(v)) \subseteq \Theta_T(v)$. Then, the successor of v in the path P'', denoted by v'', belongs to $\Theta_T(v)$. Hence the subpath $P''' := v''P''$, oriented from v'' to t, is contained in $\langle T'\rangle$ but not contained in $\langle T\rangle_{\{v\}\cup\Theta_T(v)}$ and therefore it is not even contained in $\langle T'\rangle_{\{v''\}\cup\Theta_{T'}(v'')}$, since $\Theta_{T'}(v'') \subseteq \Theta_T(v)$.

If $v'' = u$, we have immediately a contradiction with the inductive hypothesis.

Conversely, if $v'' \neq u$ then, since P''' is an oriented path from v'' to t, by Proposition 11.26, we have that $t \in \Theta_{T'}(v'')$. So, by the inductive hypothesis, we know that there not exists in $\langle T'\rangle$ an oriented path from v'' to t, not contained in $\langle T'\rangle_{\{v''\}\cup\Theta_{T'}(v'')}$, against what we have assumed for P'''. □

11 (vii) Complete description of the partial order $\preccurlyeq_T$

Theorem 11.28
Let T be a complete k-dense tree and let x and y be vertices of T. The following holds:

$$x \preccurlyeq_T y \text{ if and only if there is in } \langle T\rangle \text{ an oriented path from } y \text{ to } x.$$

Proof. ($\Leftarrow$) Let us suppose that there exists in $\langle T\rangle$ an oriented path P going from y to x. If $|P| = 0$, the thesis is obviously true. Therefore, let us assume that $|P| \geq 1$.

By Observation 11.25, we know that $\langle T\rangle$ does not contain oriented cycles then the assumption made implies that $x \neq y$. Moreover T is not a k-tree. Otherwise, we would have that $T = \mathcal{C}(T)$ and that $\langle T\rangle = (V(T), \emptyset)$, against the assumption that there is a path between the two distinct vertices x and y.

Since T is not a k-tree, by Lemma 7.3 we know that $|T| \geq k+3$. We will prove the ($\Leftarrow$) part of the thesis by induction on $|T|$.

If $|T| = k+3$, by Lemma 7.3 and Lemma 6.3, there is in T a unique k-leaf d and such k-leaf does not belong to the core of T. Moreover, by Lemma 7.3, we know that d is the unique vertex of T not included in $\mathcal{C}(T)$. Finally, by Observation 11.25, we know that there not exists two vertices of the core of T adjacent in $\langle T\rangle$. Hence, from above, it follows that $d \in \{x, y\}$. However, d is a k-leaf of T and, by Definition 11.23, we know that, for each edge $a \in E(\langle T\rangle)$, $t(a) = d$ then it results that $d = y$. Since $d = y$ is the unique k-leaf of T, by Definition 3.24 and Proposition 3.26, we have that, for each k-elimination σ for T, $\sigma(1) = d$, hence $x \preccurlyeq_T y$.

If $|T| = n > k+3$, let us suppose that the ($\Leftarrow$) part of the thesis holds for each complete k-dense tree with -1 vertices. Since T is not a k-tree, by Lemma 6.3, we know that there is a k-leaf f of T not included in the core. The graph $T' := T_{V(T)-\{f\}}$ is a complete k-dense tree with $n-1$ vertices. We have two cases.

- $f \notin \{x, y\}$. By inductive hypothesis, we know that $x \preccurlyeq_{T'} y$, that is, for each k-elimination σ' for T', we have that $\langle\sigma'\rangle^{-1}(y) < \langle\sigma'\rangle^{-1}(x)$.

  This must hold also for the k-eliminations of T. In fact, if there was a k-elimination σ for T such that $\langle\sigma\rangle^{-1}(x) < \langle\sigma\rangle^{-1}(y)$, then for the k-elimination for T' $\sigma'' := \sigma_{V(T)-\{f\}}$ we would have that $\langle\sigma''\rangle^{-1}(x) < \langle\sigma''\rangle^{-1}(y)$, although $x \preccurlyeq_{T'} y$.

- $f \in \{x, y\}$. By Definition 11.23 and Corollary 8.2, we have that any edge of $\langle T\rangle$, incident on f has f as tail hence, by the hypothesis on the orientation of P, it results that $f = y$.

  We can distinguish three cases:

  - $\|P\| \geq 2$. Let y' be the vertex of P is adjacent to y. From Definition 11.23 it follows that $y' \notin V(\mathcal{C}(T))$. In fact, any edge in $E(\langle T\rangle)$ incident in a vertex h of the core of T, must have h as head; however if $\|P\| \geq 2$, for y' there exist one vertex y'' of P, and one edge $a \in E(P)$ such that:

$$t(a) = y' \quad \text{and} \quad h(a) = y''.$$

Since $\langle T' \rangle = \langle T_{V(T)-\{f\}} \rangle = \langle T \rangle_{V(T)-\{f\}}$ the subpath $\mathring{y}\ P$ of P is an oriented path of $\langle T' \rangle$ connecting y' and x. Then, by the inductive hypothesis, we have that $x \preccurlyeq_{T'} y'$. From this, it follows that $x \preccurlyeq_T y'$. In fact, if there was a k-elimination σ for T such that $\langle \sigma \rangle^{-1}(x) < \langle \sigma \rangle^{-1}(y')$ then, for the k-elimination $\sigma' := \sigma_{V(T)-\{f\}}$ for T' we would have $\langle \sigma' \rangle^{-1}(x) < \langle \sigma' \rangle^{-1}(y)$, against what we have inferred from the inductive hypothesis.

Finally, by Corollary 8.2, we know that, for each k-elimination σ'' for T, $\langle \sigma'' \rangle^{-1}(y) < \langle \sigma'' \rangle^{-1}(y')$ and hence we can conclude that $x \preccurlyeq_T y$.

– $\|P\| = 1$ and $x \in adj_T(f)$. From Corollary 8.2 it immediately follows that $x \preccurlyeq_T y$.
– $\|P\| = 1$ and $x \notin adj_T(f)$. By Definition 11.23, we have that $x \in \Omega_T(f)$. Let us suppose that there exists a k-elimination σ for T such that $\langle \sigma \rangle^{-1}(x) < \langle \sigma \rangle^{-1}(f)$. Let z be the vertex such that:

$$\langle \sigma \rangle^{-1}(z) = min \left\{ \langle \sigma \rangle^{-1}(v) \mid v \in \mathcal{C}_T(f) \right\}.$$

By Proposition 11.27 and Definition 11.20, we know that, for each $t \in \mathcal{C}_T(f)$, there is in $\langle T \rangle$ an oriented path from f to t. Therefore, for the results of the two previous cases, for each vertex $r \in \mathcal{C}_T(f) - \Omega_T(f)$, we have that $r \preccurlyeq_T f$ and so then $\langle \sigma \rangle^{-1}(f) < \langle \sigma \rangle^{-1}(r)$. Then, the vertex z previously defined must belong to $\Omega_T(f)$. On the other hand, by the hypothesis on x and σ, and for the choice of z, we have that:

$$deg_{T_{\mathcal{C}_T(f)}}(z) = k,$$

against the thesis of Lemma 11.15.

($\Rightarrow$) Let us suppose that $x \preccurlyeq_T y$ and $x \neq y$ (if $x = y$, the proof is trivial) that is, for each k-elimination σ, $\langle \sigma \rangle^{-1}(x) > \langle \sigma \rangle^{-1}(y)$. Let us suppose that $x \notin \Theta_T(y)$. By Proposition 11.22, we know that the graph $T^y := T_{\{y\} \cup \Theta_T(y)}$ is a non-trivial complete k-dense tree. Let σ^y be a k-elimination for T^y. By Proposition 11.7, we know that there is a k-elimination σ' for T such that:

$$\sigma' = app\left(\sigma'_{V(T)-V(T^y)}, \sigma^y\right).$$

Since $x \notin \Theta_T(y)$, we have that $\langle \sigma' \rangle^{-1}(x) < \langle \sigma' \rangle^{-1}(y)$, against the first assumption.

Therefore $x \in \Theta_T(y)$ and hence, by Proposition 11.27, we know that there is in $\langle T \rangle$ an oriented path from y to x. □

Theorem 11.29
Let T be a complete k-dense tree and let x and y be vertices of T. The following holds:

$$x \preccurlyeq_T y \textit{ if and only if } x \in \Theta_T(y).$$

Proof. By Theorem 11.28, Proposition 11.26 and Proposition 11.27. □

11 (viii) Construction of $\langle T \rangle$

Algorithm for the construction of $min_R^{k\text{-tree}}(V')$

Theorem 11.30
Let R be a k-tree and let $V' \subseteq V(R)$. The Algorithm 11.1, applied to R and V', finds $min_R^{k\text{-tree}}(V')$.

Algorithm 11.1 Construction of $min_R^{k\text{-tree}}(V')$

Input: $\Rightarrow$ R, k-tree,

$\Rightarrow$ V', set of vertices.

Output: $\Leftarrow$ set of the vertices of $min_R^{k\text{-tree}}(V')$.

$R' \leftarrow R$
$L \leftarrow \{l \in V(R) \mid deg_R(l) \leq k\}$
while $L \not\subseteq V'$ **do**
  $f \leftarrow \langle$*scegli* $l \in (L - V')\rangle$
  $A \leftarrow adj_{R'}(f)$
  $R' \leftarrow R'_{V(R')-\{f\}}$
  $L \leftarrow (L - \{f\}) \cup \{v \in A \mid deg_{R'}(v) \leq k\}$
end while
return $V(R')$

Algorithm 11.2 Construction of $\langle T\rangle$ for a complete k-dense tree T

Input: $\Rightarrow$ T, complete k-dense tree,

$\Rightarrow$ σ, k-elimination for T.

Output: $\Leftarrow$ $\langle T\rangle$.

Data: - Dip, array of $|T|$ sets of vertices. For each i, we have that $Dip[i] = \mathcal{C}_T(\sigma(i))$.

$n \leftarrow |T|$
$A \leftarrow \emptyset$
$N \leftarrow \langle$*Trova il core di* $T\rangle$
for $i = n$ downto 1 **do**
  $v \leftarrow \sigma(i)$
  if $v \in V(N)$ **then**
    $Dip[i] = \{v\}$
  else
    $Dip[i] = V\left(min_N^{k\text{-tree}}\left(\bigcup_{u \in \Lambda^\sigma(v)} Dip[\langle\sigma\rangle^{-1}(u)]\right)\right)$
    $C \leftarrow Dip[i] - \bigcup_{u \in \Lambda^\sigma(v)} Dip[\langle\sigma\rangle^{-1}(u)]$
    $A \leftarrow A \cup \{(v,u) \mid u \in (\Lambda^\sigma(v) \cup C)\}$
  end if
end for
return $(V(T), A)$

Proof. It follows directly from the proof of Proposition 11.8. For a detailed proof, see [Fra01]. $\square$

Theorem 11.31
Let R be a k-tree, the Algorithm 11.1 can be implemented to run in $O(nk)$ time, where $n := |R|$.

Proof. The step 2 requires an $O(n)$ time. The body of the cycle of step 3 is iterated n times at most. In step 6, the degree of at most k vertices is updated, hence this step requires $O(k)$ time. Step 7 also requires $O(k)$ time, because the informations on the degrees are updated. In total, the execution of Algorithm 11.1 requires $O(kn)$ time. $\square$

Algorithm for the construction of $\langle T \rangle$

Theorem 11.32
Let T be a complete k-dense tree. For each k-elimination σ for T, Algorithm 11.2, applied to T and σ, constructs the graph $\langle T \rangle$.

Proof. It follows directly from Definition 11.10, Definition 11.14 and Definition 11.23. For a detailed proof see [Fra01]. $\square$

Theorem 11.33
Let T be a complete k-dense tree. The Algorithm 11.2 can be implemented in order to run in $O(kcn)$ time, where $c := |\mathcal{C}(T)|$ and $n := |T|$.

Proof. The step 3 requires an $O\left(k^2 n\right)$ time (by Algorithm 10.3). The body of the cycle of step 4 is iterated n times. The condition of step 6 requires at most an $O(c)$ time. The step 9 requires an $O(kc)$ time (by Algorithm 11.1). The operation on sets can be implemented so that the step 10 is executed in $O(kc)$ time. The step 11 is a simple union of disjoint sets. By Lemma 7.3, we know that $c \geq k+2$, then the execution of Algorithm 11.2 requires $O(kcn)$ time. $\square$

References

[Die97] Reinhard Diestel. *Graph Theory.* Springer, 1997.

[Fra01] Gianni Franceschini. Dense tree: una generalizzazione del concetto di albero. Master's thesis, Università degli Studi di Pisa, Facoltà di Scienze Matematiche Fisiche e Naturali, Corso di Laurea in Informatica, 2001.

[Fre82] Eugene C. Freuder. A sufficient condition for backtrack-free search. *Journal of ACM*, pages 24–32, 1982.

[Klo94] T. Kloks. Treewidth. Computations and Approximations. *Lecture Notes in Computer Science 842*, 1994.

[LMP00] Fabrizio Luccio, Antonio Mesa, e Linda Pagli. A distributed tree data structure. *Proceedings ISADS*, 2000.

[LP99] Fabrizio Luccio e Linda Pagli. Dense trees: a new structure for interconnection. *Distributed Data and Structures*, 1999.

[Mat68] D. W. Matula. A min-max theorem for graphs with applications to graph coloring. *SIAM Rev.*, 10:481–482, 1968.

[PFLP01] Lino Flores Pacheco, Gianni Franceschini, Fabrizio Luccio, e Linda Pagli. Decomposition of k-Dense Trees. *Distributed Data and Structures 3*, 2001.

[SW68] G. Szekeres e H. S. Wilf. An inequality for the chromatic number of a graph. *J. Combinatorial Theory*, 4:1–3, 1968.