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Abstract

There are various ways to prove that, under suitable conditions, the inverse of a Toeplitz matrix can be completely recovered from two of its columns. Toeplitz matrices and their inverses are matrices with displacement structure, i.e., they can be related with matrices of low rank via suitable linear operators. In this note, we show a possible generalization of the result concerning Toeplitz inverses for certain classes of matrices with displacement structure.

1 Introduction

Let T be a square Toeplitz matrix. As well known this means that the entry $T(i,j)$ with i, $j = 1, \ldots, n$ depends only on the difference $i-j$. Clearly the first and last columns of T are sufficient in order to completely recover the other entries of the matrix. If T is nonsingular, in general its inverse $S = T^{-1}$ is not Toeplitz. However, it has been shown that if $S(n,n) \neq 0$, the matrix S can be completely recovered from its first and last column, see [4, 8] and the references therein. As an introductive important example we now review this, making use of the concept of displacement operator [5, 7]. The rest of the paper is devoted to describe a generalization of this property for certain classes of matrices with displacement structure. The results obtained are summarized in theorems 2 and 3.

Before the example, we briefly recall a theorem from [2], which is basic for what follows. Let $A, H \in \mathbb{R}^{n\times n}$. For our purposes we consider linear operators of the form $\mathcal{D}_H(A) = AH - HA$. Given two vectors $a, b \in \mathbb{R}^n$ we define the two Krylov matrices $C_a = (a|Ha|\ldots|H^{n-1}a)$ and $R_b = (b|H^T b|\ldots|(H^T)^{n-1}b)^T$. If for some vector a the matrix C_a is nonsingular (or for some vector b the matrix R_b is nonsingular) then the powers H^i for $i = 0,\ldots,n-1$ are linearly independent and the matrix H is said nonderogatory [6]. We denote with $P(H)$ the subspace of $\mathbb{R}^{n\times n}$ generated by the powers of H. If H is nonderogatory this space has dimension n.

Theorem 1 *Let a matrix H and two vectors a and b be such that C_a and R_b*

are nonsingular. Let A be such that

$$\mathcal{D}_H(A) = \sum_{i=1}^{m} u_i v_i^T.$$

Let F_i be the unique matrix in $P(H)$ such that $F_i a = u_i$ and G_i be the unique matrix in $P(H + ab^T)$ such that $b^T G_i = v_i^T$. Then

$$A = \sum_{i=1}^{m} F_i G_i + F,$$

where F is the matrix in $P(H)$ such that $Fa = Aa$.

In practice, the theorem says that a matrix A can be completely recovered from the vectors u_i, v_i and the vector Aa.

Now we come to the promised example. Let Z be the upper shift matrix, i.e., $Z(i,j) = \delta_{i,j-1}$ and let e_i be the i-th vector of the canonical basis. Setting $H = Z$, $a = e_n$ and $b = e_1$ the matrices C_a and R_b are nonsingular so that theorem 1 can be applied. Specifically we have $R_b = I$, the identity matrix, and $C_a = J$ the reversion matrix such that $J(i,j) = \delta_{i,n-j+1}$. One verifies that

$$\mathcal{D}_Z(T) = Je_1 v^T J - v e_1^T \qquad \text{where} \qquad v = \begin{pmatrix} T(2,1) \\ \vdots \\ T(n,1) \\ 0 \end{pmatrix} \tag{1}$$

Since T is Toeplitz it is persymmetric, i.e., $JT^T J = T$ so that, if it is nonsingular, it has persymmetric inverse. Let $S = T^{-1}$, $w = Sv$ and let $S(:,k)$ be the k-th column of S. By using the persymmetry of S from (1) follows that

$$\mathcal{D}_Z(S) = wS(:,n)^T J - S(:,n) w^T J. \tag{2}$$

In particular

$$-ZS(:,1) = wS(n,n) - w(n)S(:,n).$$

Assuming $S(n,n) \neq 0$ it follows that $w = \frac{1}{S(n,n)}(w(n)S(:,n) - ZS(:,1))$ and by substituting into (2)

$$\mathcal{D}_Z(S) = \frac{1}{S(n,n)}(S(:,n)(ZS(:,1)^T J - ZS(:,1)S(:,n)^T J).$$

Thus, under the assumption $S(n,n) \neq 0$ by virtue of theorem 1 the matrix S can be recovered from its first and last columns. Clearly $S(1,1) = S(n,n)$ due to the persymmetry of S.

2 Computing the displacement trough the entries

Given a linear space V over $\mathbb{R}$ and an inner product $<\cdot,\cdot>: V\times V\to\mathbb{R}$ the adjoint of a linear operator $\mathcal{L}: V\to V$, is the linear operator, denoted with $\mathcal{L}^T$ such that $<\mathcal{L}(v), w> = <v, \mathcal{L}^T(w)>$ for every v and w in V. The range of a linear operator is orthogonal to the nullspace of its adjoint.

Let tr be the trace operator such that $\mathrm{tr}(A)$ is the sum of the diagonal entries of A. Then

$$<A, B> = \mathrm{tr}(A^T B) \tag{3}$$

is an inner product on $\mathbb{R}^{n\times n}$, as it is easy to verify, moreover $<\mathcal{D}_H(A), B> = <A, \mathcal{D}_{H^T}(B)>$, i.e., the operator $\mathcal{D}_{H^T}$ is the adjoint of the operator $\mathcal{D}_H$ under the inner product (3).

The equation $\mathcal{D}_H(A) = B$ is solvable if and only if B belongs to the space orthogonal to the nullspace of $\mathcal{D}_{H^T}$. Since H is assumed to be nonderogatory the kernel is generated by the powers $(H^i)^T$ for $i = 0, \ldots, n-1$. Hence the equation is solvable if and only if $<(H^i)^T, B> = \mathrm{tr}(H^i B) = 0$ for $i = 0, \ldots, n-1$.

In the particular case where $B = av^T - wb^T$ we have $\mathrm{tr}(H^i B) = \mathrm{tr}(H^i av^T - H^i wb^T) = v^T H^i a - b^T H^i w$. Thus the condition $\mathrm{tr}(H^i B) = 0$ for $i = 0, \ldots, n-1$ becomes, in matrix form, $C_a^T v = R_b w$ implying that only one of the two vectors v or w is free. If the equation $\mathcal{D}_H(A) = av^T - wb^T$ is solvable, then from theorem 1 it follows that $A = G + F$ where $F \in P(H)$ and $Fa = Aa - w$ and $G \in P(H + ab^T)$ and $b^T G = v^T$.

Turning back to our introductive example where $H = Z$, $a = e_n$ and $b = e_1$, the space $P(H)$ is the algebra of the upper triangular Toeplitz matrices, the space $P(H + ab^T)$ is the algebra of the circulant matrices [3]. The equation $\mathcal{D}_H(A) = e_n v^T - we_1^T$ is solvable if and only if $w = Jv$ (as previously noted $C_a = J$ and $R_b = I$). In the case where the equation is solvable A is Toeplitz since it is the sum of a circulant matrix and of a Toeplitz triangular matrix.

However, in general, the equation $\mathcal{D}_H(A) = av^T - wb^T$ does not give us indications on the elements of A needed in order to recover the matrix. We can proceed as follows. First of all we observe that the vectors v and w are only defined up the addition to them of the same scalar multiple of b and a respectively. In addition, it is convenient and not restrictive to assume that H is singular since for our purposes we can add to H an arbitrary scalar matrix.

Let $H^n = p(H)$ where $p(x) = \sum_{i=0}^{n-1} h_i x^i$. Since H is assumed to be singular one has $h_0 = 0$. Observe that $HC_a = C_a F$ where F is the Frobenius matrix having in the last column the coefficients of the polynomial p. Clearly $e_1^T F = 0$. Analogously we obtain $R_b H = F^T R_b$. Let us start from the relation

$$AH - HA = av^T - wb^T \tag{4}$$

then

$$C_a^{-1} AHR_b^{-1} - C_a^{-1} HAR_b^{-1} = C_a^{-1} av^T R_b^{-1} - C_a^{-1} wb^T R_b^{-1}$$

and this implies

$$C_a^{-1}AR_b^{-1}F^T - FC_a^{-1}AR_b^{-1} = e_1v^TR_b^{-1} - C_a^{-1}we_1^T.$$

From the preceding relation it follows that

$$v^TR_b^{-1}e_1 = e_1^TC_a^{-1}w,$$

and

$$-FC_a^{-1}AR_b^{-1}e_1 = e_1v^TR_b^{-1}e_1 - C_a^{-1}w = e_1e_1^TC_a^{-1}w - C_a^{-1}w = (e_1e_1^T - I)C_a^{-1}w.$$

The first component of $C_a^{-1}w$ cannot be recovered from the preceding equation but is arbitrary, since w is determined only up the addition of a scalar multiple of a.

We can summarize the results of this section as follows.

Theorem 2 *If a matrix A satisfies equation (4) and the matrices C_a and R_b are nonsingular, then A can be completely recovered from the vectors Aa and $AR_b^{-1}e_1$.*

It is clear that an analoguos result holds for the vectors b^TA and $e_1^TC_a^{-1}A$. If a and $R_b^{-1}e_1$ are canonical vectors then A can be recovered from two of its columns. This is the case for Toeplitz matrices where $a = e_n$ and $R_b = I$.

3 Computing the displacement of the inverse from its elements

From (4) we obtain

$$\mathcal{D}_H(A^{-1}) = A^{-1}H - HA^{-1} = A^{-1}wb^TA^{-1} - A^{-1}av^TA^{-1}. \tag{5}$$

First of all we find

$$e_1^TC_a^{-1}A^{-1}av^TA^{-1}R_b^{-1}e_1 = e_1^TC_a^{-1}A^{-1}wb^TA^{-1}R_b^{-1}e_1,$$

and then

$$HA^{-1}R_b^{-1}e_1 = A^{-1}av^TA^{-1}R_b^{-1}e_1 - A^{-1}wb^TA^{-1}R_b^{-1}e_1.$$

Assuming $b^TA^{-1}R_b^{-1}e_1 \neq 0$ and $e_1^TC_a^{-1}A^{-1}a \neq 0$ one finds

$$A^{-1}w = \frac{e_1^TC_a^{-1}A^{-1}w}{e_1^TC_a^{-1}A^{-1}a}A^{-1}a - \frac{1}{b^TA^{-1}R_b^{-1}e_1}HA^{-1}R_b^{-1}e_1. \tag{6}$$

We can exploit equality (6) by introducing some additional hypotheses on H and A. This leads to the following extension of theorem 2.

Theorem 3 *Let the hypotheses of theorem 2 hold. If in addition A and H are persymmetric, A is nonsingular and $b^T A^{-1} R_b^{-1} e_1 \neq 0$ then the matrix A^{-1} can be recovered from the two vectors $A^{-1}a$ and $A^{-1}R_b^{-1}e_1$. The conclusion is the same if A and H are symmetric.*

PROOF. The case where H and A are both persymmetric is the first that we consider since it includes inverses of Toeplitz matrices. If A and H are both persymmetric then $J\mathcal{D}_H(A)^T J = -\mathcal{D}_H(A)$ and this implies $w = Jv$ and $b = Ja$. Thus formula (5) becomes

$$\mathcal{D}_H(A^{-1}) = A^{-1}wa^T A^{-T} J - A^{-1}aw^T A^{-T} J. \tag{7}$$

Since $R_b = C_a^T J$

$$b^T A^{-1} R_b^{-1} e_1 = e_1^T C_a^{-1} A^{-1} a. \tag{8}$$

The substitution of (6) into (7) yields

$$\mathcal{D}_H(A^{-1}) = \frac{1}{e_1^T C_a^{-1} A^{-1} a}(A^{-1}a(A^{-1}R_b^{-1}e_1)^T H^T J - HA^{-1}R_b^{-1}e_1 a^T A^{-T} J),$$

and we get the thesis.

In the case where A and H are both symmetric $\mathcal{D}_H(A)^T = -\mathcal{D}_H(A)$ and this implies $a = b$ and $v = w$. In addition $R_b = C_a^T$ so that again equality (8) holds. The substitution of (6) into (5) again shows that the two vectors $A^{-1}a$ and $A^{-1}R_b^{-1}e_1$ allow us to recover A^{-1}. ∎

An interesting example concerning the second part of theorem 3 is given by the Loewner symmetric matrices [1]. These matrices satisfy a displacement equation of the form $\mathcal{D}_H(A) = av^T - v^T a$ where H is a diagonal matrix with distinct diagonal entries and $a = \sum_{i=1}^n e_i$. The matrix C_a is clearly a nonsingular Vandermonde matrix. Without loss of generality H can be chosen so that $H(1,1) = 0$ and this implies $R_b^{-1}e_1 = e_1$. We deduce that in order to recover both A and A^{-1} we need their first column and the sum of the elements of the other columns. For the recovery of A^{-1} it is required that the sum of the entries of its first column is not zero.

One could ask if it is possible to find a class of symmetric matrices ruled by the displacement equation (4), with H symmetric, and such that the first and the last column are enough in order to recover the matrix. To this end we choose $a = e_1$ and this implies $b = e_1$. However, at this point it is impossible to choose H in such a way that $R^{-1}e_1 = e_n$.

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