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From Partial Differential Equations to generalized Locally Toeplitz sequences

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Abstract

Starting from the Finite Difference discretization of an elliptic second order PDE as $-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial}{\partial x_j}u(x)\right) = b(x)$, we introduce the notion of generalized locally Toeplitz sequence of matrices. The singular value distribution (the eigenvalue distribution in the Hermitian case) is studied and characterized for generalized locally Toeplitz sequences in terms of weighted multidimensional Szegő formulas which extend preceding results due to the second author and concerning the unilevel case. The application of this theoretic analysis to the numerical solution of elliptic PDEs is finally discussed.

1 Prelude

Let f be a d variate complex-valued integrable function, defined over the ipercube Q^d, with $Q = (-\pi, \pi)$ and $d \geq 1$. From the Fourier coefficients of f

$$(1) \quad f_j = \frac{1}{m\{Q^d\}} \int_{Q^d} f(s) e^{-\hat{\mathbf{i}}(j,s)}\, ds, \qquad \hat{\mathbf{i}}^2 = -1, \quad j = (j_1, \ldots, j_d) \in \mathbf{Z}^d$$

with $(j, s) = \sum_{k=1}^{d} j_k s_k$, $n = (n_1, \ldots, n_d)$ and $N(n) = n_1 \cdots n_d$, one can build the sequence of Toeplitz matrices $\{T_n(f)\}$, where $T_n(f) = \{f_{j-i}\}_{i,j=e^T}^{n}$

$\in \mathbf{C}^{N(n)\times N(n)}$, $e^T = (1,\ldots,1) \in \mathbf{N}^d$ is said to be the Toeplitz matrix of order n generated by f (see [24]). Furthermore, throughout the paper when we write $n \to \infty$ with $n = (n_1,\ldots,n_d)$ being a multi-index, we mean that $\min_{1\le j\le d} n_j \to \infty$.

The asymptotic distribution of eigen and singular values of a sequence of Toeplitz matrices has been widely analyzed in the literature, and has been completely characterized in terms of the generating function f (see, for example, [24] and the references there reported). The key-stone of this theory, from which many extensions and other results stemmed, is a celebrated theorem [5, 26] of Szegő, which we state here in the d dimensional version due to Tyrtyshnikov:

Theorem 1.1 (Szegő, [25]) *Suppose f is real-valued and integrable over Q^d, and let $\{T_n(f)\}$ be the sequence of Hermitian Toeplitz matrices generated by f; then for any function F, continuous with bounded support, it holds*

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F\Big(\lambda_j(T_n(f))\Big) = \frac{1}{m\{Q^d\}} \int_{Q^d} F(f(s))\, ds, \tag{2}$$

where $\lambda_j(T_n(f))$, $j = 1,\ldots,N(n)$, are the eigenvalues of $T_n(f)$.

This basic result has been generalized to the case where f is Hermitian matrix-valued (see, for example, [19, 21, 22]) so that the matrices $T_n(f)$ have Hermitian multilevel block Toeplitz structure.

If f is not real but complex-valued, $T_n(f)$ is not necessarily Hermitian: as a consequence, the distribution of eigenvalues is less regular and (2) cannot be extended in the natural way (see [19] for a discussion on possible extensions). In the complex-valued case, however, an analog of Theorem 1.1 exists concerning singular values instead of eigenvalues; following Tyrtyshnikov (who extended previous results due to Avram and Parter [1, 9]), it can be stated as follows

Theorem 1.2 (Tyrtyshnikov, [24, 25]) *If f is complex-valued and integrable over Q^d, then the singular values of $\{T_n(f)\}$ are distributed as $|f(s)|$, that is, for any continuous function F with bounded support it holds*

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F\Big(\sigma_j(T_n(f))\Big) = \frac{1}{m\{Q^d\}} \int_{Q^d} F(|f(s)|)\, ds, \tag{3}$$

where $\sigma_j(T_n(f))$, $j = 1,\ldots,N(n)$ are the singular values of $T_n(f)$.

This result has recently been extended by the second author [19] to the case where f is matrix-valued (also non-square values are allowed); in that case, integrand in (3) is a summation involving F and the singular values of the matrix $f(x)$.

Now we introduce the notion of (unilevel) locally Toeplitz matrix-sequences that lead to a generalization of (unilevel) Toeplitz sequences.

Definition 1.1 *Consider two matrices $A \in \mathbf{C}^{n\times n}$ and $B \in \mathbf{C}^{m\times m}$. The* direct sum $S = A \oplus B \in \mathbf{C}^{(n+m)\times(n+m)}$ *is defined as*

$$\begin{bmatrix} A & O \\ O & B \end{bmatrix}.$$

The tensor product $P = A \otimes B \in \mathbf{C}^{nm\times nm}$ *is defined as the $n \times n$ block matrix with $m \times m$ blocks, whose block $(i,j), i,j = 1,\ldots,m$ is given by $a_{i,j}B$.*

Definition 1.2 [20] *A sequence of matrices $\{A_n\}$, where $A_n \in \mathbf{C}^{n\times n}$, is said to be* Locally Toeplitz *with respect to a pair of functions (a,f), with $a : [0,1] \to \mathbf{C}$ and $f : Q \to \mathbf{C}$, if f is Lebesgue-integrable and, for all sufficient large $m \in \mathbf{N}$, there exists $n_m \in \mathbf{N}$ such that the following splittings hold*

$$A_n = LT_n^m(a,f) + R_{n,m} + N_{n,m} \quad \forall n > n_m \tag{4}$$

with

$$\mathrm{rank}(R_{n,m}) \le c(m), \quad ||N_{n,m}||_F \le \omega(m)\sqrt{n}, \tag{5}$$

where $c(m)$ and $\omega(m)$ are functions of m with $\lim_{m\to\infty} \omega(m) = 0$ and with

$$LT_n^m(a,f) = D_{m,a} \otimes T_{\lfloor n/m \rfloor}(f) \oplus \mathrm{O}_{n \bmod m},$$

where, as usual, $\lfloor n/m \rfloor$ is the integer part of n/m and $n \bmod m = n - m\lfloor n/m \rfloor$ (it is understood that the zero block $\mathrm{O}_{n \bmod m}$ is not present if n is a multiple of m). Moreover $D_{m,a}$ is the $m \times m$ diagonal matrix whose entries are given by $a(j/m))$, $j = 1,\ldots,m$, $T_k(f)$ denotes the Toeplitz matrix of order k generated by f and O_q is the null matrix of order q.
In this case we write in short $\{A_n\} \sim_{\mathrm{LT}} (a,f)$.

For this class of matrix sequences the second author obtained a Szegő-like formula for singular values, namely

Theorem 1.3 (Tilli, [20]) *Assume that $\{A_n\}$ is a sequence of $n \times n$ complex matrices. Let $f \in L^1(Q)$ and a be piecewise continuous over $\Omega_1 = [0,1]$. Then for any continuous function F with bounded support*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F[\sigma_j(A_n)] = \frac{1}{m\{\Omega_1 \times Q\}} \int_{\Omega_1\times Q} F\left(|a\,(x)\,f(s)|\right) dx\, ds \tag{6}$$

holds whenever $\{A_n\}$ is locally Toeplitz with respect to the pair (a,f).

A Szegő-like formula for eigenvalues is obtained if $\{A_n\}$ is Hermitian and locally Toeplitz with respect to (a,f). In that case, as shown in [20], the product $a(x)f(s)$ has to be real almost everywhere.

Theorem 1.4 (Tilli, [20]) *Assume that $\{A_n\}$ is a sequence of $n \times n$ complex matrices. Let $f \in L^1(Q)$ and a be piecewise continuous over $\Omega_1 = [0,1]$. If $\{A_n\}$ is Hermitian and locally Toeplitz with respect to (a, f), then for any continuous function F with bounded support it holds*

$$(7) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} F[\lambda_j(A_n)] = \frac{1}{m\{\Omega_1 \times Q\}} \int_{\Omega_1\times Q} F[a(x)f(s)]\, dx\, ds.$$

2 Introduction

Let $A(x) = (a_{i,j}(x))_{i,j=1}^{d}$ be a $d \times d$ matrix of continuous functions defined over the ipercube $\Omega_d = [0,1]^d$ and let us consider the differential problem

$$(8) \qquad \begin{array}{l} -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial}{\partial x_j}u(x)\right) = b(x) \quad \text{if} \quad x \in \Omega_d^{\circ} = (0,1)^d, \\ \quad + \qquad \text{boundary conditions,} \end{array}$$

where the boundary conditions (Dirichlet, Neumann etc.) are not specified since they do not affect our reasoning here.

If $\circ$ denotes the componentwise Hadamard product [2], then the preceding equation can be conveniently rewritten as follows

$$(9) \qquad \begin{array}{rcl} -e^T[A(x) \circ H_u(x)]e + \text{first order terms} & = & b(x) \quad \text{if} \quad x \in \Omega_d^{\circ} = (0,1)^d, \\ + \qquad \text{boundary conditions} & & \end{array}$$

where $H_u(x)$ is the Hessian matrix of u evaluated at $x \in \Omega_d$ and $e^T = (1, \cdots, 1) \in \mathbf{R}^d$. The discretization of (8) by finite differences (FD) over equispaced d-dimensional grid-sequences $\mathcal{G}$ leads to a sequence of multilevel linear systems [24] whose sequence of coefficient matrices is denoted by

$$(10) \qquad \{A_n(A, \mathcal{F})\}.$$

Here each $A_n(A, \mathcal{F})$ has dimension $N(n) \times N(n)$ with $n = (n_1, \ldots, n_d)$ and $N(n) = n_1 \cdots n_d$; the grid sequence $\mathcal{G}$ is given by $\{\mathcal{G}_{n_1} \times \mathcal{G}_{n_2} \times \cdots \times \mathcal{G}_{n_d}\}_n$, $\mathcal{G}_{n_j} = \left\{\frac{i}{n_j+1}\right\}_{i=1}^{n_j}$ and $\mathcal{F}$ is a "symbolic" $d \times d$ matrix whose entry $\mathcal{F}_{i,j}$ denotes the FD formula used for the discretization of the operator $-\frac{\partial^2}{\partial x_i \partial x_j}$.

Due to the "shift invariance" property that characterizes any differential operator D with constant coefficients, it is customary to represent a FD formula ϕ over a sequence of n-sized equispaced grids as a sequence of Toeplitz matrices $\{T_n(D, \phi)\}$. For $D = \frac{d}{dx}$ and a consistent formula ϕ of minimal precision 2 involving two contiguous discretization points we have $n^{-1}T_n(D, \phi) = T_n(q(s))$ where $q(s) = -1 + e^{\mathbf{i}s}$ is the generating function in the sense of equation (1) with $f_0 = -1$ and $f_1 = 1$. For $D = \frac{d^2}{dx^2}$ and a consistent formula ϕ of precision 2 involving three contiguous discretization points we have $n^{-2}T_n(D,\phi) = T_n(q(s))$ where $q(s) = 2-2\cos(s) = e^{-\hat{\mathbf{i}}s}+2+e^{\hat{\mathbf{i}}s}$ is the generating function in the sense of equation (1) with $f_0 = 2$ and $f_{\pm 1} = -1$. Since the approximate evaluation by ϕ of a given differential operator $D = \frac{d^p}{dx^p}$ applied to a function u at a given grid point involves the evaluation of u at a finite number of continguous grid points, it follows that the Toeplitz sequence $\{n^{-p}T_n(D,\phi)\}$ is associated to a 2π-periodic function with a finite Fourier expansion that is to a trigonometric polynomial. By following the general procedure given in [12] we know that, up to a scaling by a reasonable unimodular function $u(s)$, the generating polynomial $q(s)$ for $D = \frac{d^p}{dx^p}$ and ϕ of precision ν is asymptotic in any neighbourhood of the complex zero to $(\hat{\mathbf{i}}s)^p$ while the error $u(s)q(s) - (\hat{\mathbf{i}}s)^p$ is asymptotic to $(\hat{\mathbf{i}}s)^\nu$ in any neighbourhood of the complex zero. In the multidimensional case the correspondence between formulas for partial derivatives and generating polynomials can be simply handled by using tensor products but we have to impose that the stepsize is up to multiplicative constants the same in each direction i.e. $\exists a_1, a_2, \ldots, a_d \in \mathbf{N}^+$ such that $n_j = va_j$ with v^{-1} being the "finesse" parameter. For $i = (i_1,\ldots,i_d) \in \mathbf{N}^d$ we denote by $\frac{\partial^i}{\partial^i x}$ the operator $\frac{\partial^{i_1}}{\partial^{i_1}x_1}\frac{\partial^{i_2}}{\partial^{i_2}x_2}\cdots\frac{\partial^{i_d}}{\partial^{i_d}x_d} = \frac{\partial^{\|i\|_1}}{\partial^{i_1}x_1\partial^{i_2}x_2\cdots\partial^{i_d}x_d}$. For $D = \sum_{j=1}^t \frac{\partial^{i(j)}}{\partial^{i(j)}x}$ with $j,t \in \mathbf{N}$, $i(j) = (i_1(j),\ldots,i_d(j)) \in \mathbf{N}^d$ and $\|i(1)\|_1 = \|i(2)\|_1 = \cdots = \|i(t)\|_1 = p$, and by considering a formula ϕ obtained by composition of unidimensional formulas, we infer that $v^{-p}T_n(D,\phi) = T_n(q(s))$ with $s = (s_1,\ldots,s_d)$, $n = (n_1,\ldots,n_d)$ with $n_j = va_j$ and

$$q(s_1,\ldots,s_d) = \sum_{j=1}^{t}\prod_{k=1}^{d} a_k^{i_k(j)} q_{i_k(j)}(s_k)$$

where $q_{i_k(j)}(s_k)$ is the generating polynomial of a formula for $\frac{\partial^{i_k(j)}}{\partial^{i_k(j)}x_k}$. Of course from the definition of multilevel Toeplitz matrices is is easy to see that $T_n(f_1(s_1)\cdot f_2(s_2)\cdots f_d(s_d)) = T_{n_1}(f_1(s_1)) \otimes T_{n_1}(f_2(s_2)) \otimes \cdots \otimes T_{n_d}(f_d(s_d))$: for any choice of $i = (i_1,\ldots,i_d) \in \mathbf{N}^d$, the following scheme holds

$$(11) \qquad D = \frac{\partial^i}{\partial^i x} \Longrightarrow \phi \;\overset{\Longrightarrow}{\Longleftarrow}\; q(s_1,\ldots,s_d) = \prod_{k=1}^{d} a_k^{i_k} q_{i_k}(s_k).$$

Owing to (11), it is clear that the matrix sequence $\{A_n(A,\mathcal{F})\}$ considered in (10) can be equivalently represented as $\{A_n(A,P,W_{\mathbf{a}})\}$ where $P_{i,j}$ is the polynomial of the variables s_i and s_j uniquely associated to the formula $\mathcal{F}_{i,j}$ discretizing the operator $-\frac{\partial^2}{\partial x_i \partial x_j}$ over the (sub)sequence $\mathcal{G}$ with $n_1 = n_2 = \cdots = n_d = v$. Furthermore the matrix $W_{\mathbf{a}} = (a_1,\ldots,a_d)(a_1,\ldots,a_d)^T$ is the nonnegative definite dyad constructed by using the weight numbers a_j related to the stepsizes in the different directions.

An interesting and nice correspondence between the differential equation (9) and the sequence $\{A_n(A,P,W_{\mathbf{a}})\}$ is proved in Corollary 6.3: indeed for any continuous function F with bounded support it holds

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\sigma_j(v^{-2}A_n(A,P,W_{\mathbf{a}}))] =$$

$$\frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d\times Q^d} F\left(\left|e^T[A\,(x)\circ P(s)\circ W_{\mathbf{a}}]e\right|\right)\,dx\,ds, \tag{12}$$

$$x\in\Omega_d=[0,1]^d,\ \ s\in Q^d=(-\pi,\pi)^d,$$

where the "discrete operator" $e^T[A\,(x)\circ P(s)\circ W_{\mathbf{a}}]e$ formally adheres to the continuous operator $-e^T[A\,(x)\circ H_u]e$ given in (9) since $P(s)\circ W_{\mathbf{a}}$ is a finite difference (functional) representation of the matrix of operators $-H_u$ over the (sub)sequence $\mathcal{G}$ with $n_j = a_j v$.

If $A(x)$ is symmetric, then we apply the numerical approximation to the operator in divergence form (8) and we arrange the choice of $P_{i,j}$ so that $-P(s)\circ W_{\mathbf{a}}$ is a Hermitian, nonnegative definite dyad of functions (notice that $H_u = [\nabla u]\cdot[\nabla u]^T$ is a sort of dyad with respect to the composition of operators). Consequently the resulting matrices $\{A_n(A,P,W_{\mathbf{a}})\}$ are symmetric and the previous formula can be also stated in the sense of the eigenvalues (see Corollary 6.4) i.e. for any continuous function F with bounded support it holds

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\lambda_j(v^{-2}A_n(A,P,W_{\mathbf{a}}))] =$$

$$\frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d\times Q^d} F\left(e^T[A\,(x)\circ P(s)\circ W_{\mathbf{a}}]e\right)\,dx\,ds, \tag{13}$$

$$x\in\Omega_d=[0,1]^d,\ \ s\in Q^d=(-\pi,\pi)^d.$$

Remark 2.1 *When the differential problem is defined over $\Omega^{'}$ with $\Omega^{'}\subset\Omega_d$ and not necessarily iperrectangular, a linear algebra trick allows us to find ergodic formulas similar to those of* (12) *and* (13)*: the matrix $A_n(A,P,W_{\mathbf{a}})$ is replaced by $A_n(A,P,W_{\mathbf{a}},\Omega^{'})$ of dimension $d_n(\Omega^{'})\times d_n(\Omega^{'})$; the quantity $N(n)$ is substituted with $d_n(\Omega^{'})$ and the set Ω_d is obviously replaced by $\Omega^{'}$ both in the multiplicative constant and in the integration domain. The only limitation over $\Omega^{'}$ concerns the fact that $\Omega^{'}$ is required to be Peano-Jordan measurable* [8, Jordan,pp. 28-29]. *We recall that a subset of $\mathbf{R}^d$ is Peano-Jordan measurable if and only if its characteristic function is integrable in the (multidimensional) Riemann sense.*

Remark 2.2 *When we deal with systems of PDEs, the case of constant coefficients leads to multilevel block Toeplitz matrices generated by Hermitian matrix-valued polynomials so that the ergodic results proved in* [22] *can be used to build up a theory for* multilevel block Locally Toeplitz sequences.

Now we are ready to introduce the notion of multilevel locally Toeplitz sequences.

Definition 2.1 *A sequence of matrices $\{A_n\}$, where $n \in \mathbf{N}^d$, $N(n) = n_1 \cdots n_d$ and $A_n \in \mathbf{C}^{N(n)\times N(n)}$, is called* multilevel Locally Toeplitz *with respect to a pair of functions (a, f), with $a : \Omega_d \to \mathbf{C}$ and $f : Q^d \to \mathbf{C}$, if the separable function $f(s_1, \ldots, s_d) = f_1(s_1) \cdots f_d(s_d)$ is Lebesgue-integrable and, for all sufficient large $m \in \mathbf{N}^d$, there exists $n_m \in \mathbf{N}^d$ such that the following splittings hold*

$$A_n = LT_n^m(a, f) + R_{n,m} + N_{n,m} \quad \forall n > n_m \tag{14}$$

with

$$\begin{aligned} \operatorname{rank}(R_{n,m}) &\le c(m)N(n)(\sum_{j=1}^{d} n_j^{-1}) \\ ||N_{n,m}||_F &\le \omega(m)\sqrt{N(n)}, \end{aligned} \tag{15}$$

where $c(m)$ and $\omega(m)$ are functions of m with $\lim_{m\to\infty} \omega(m) = 0$ and with

$$\begin{aligned} LT_n^m(a, f) \;=\; & (\operatorname*{Diag}_{j_1=1,\ldots,m_1} T_{\lfloor n_1/m_1\rfloor}(f_1) \otimes (\operatorname*{Diag}_{j_2=1,\ldots,m_2} T_{\lfloor n_2/m_2\rfloor}(f_2) \otimes \\ & (\cdots \otimes (\operatorname*{Diag}_{j_{d-1}=1,\ldots,m_{d-1}} T_{\lfloor n_{d-1}/m_{d-1}\rfloor}(f_{d-1}) \otimes \\ & (D_{m_d,a_{j,m,d}} \otimes T_{\lfloor n_d/m_d\rfloor}(f_d) \oplus \mathrm{O}_{n_d \bmod m_d}) \oplus \\ & \mathrm{O}_{(n_{d-1} \bmod m_{d-1})n_d}) \cdots) \oplus \mathrm{O}_{(n_1 \bmod m_1)n_d n_{d-1}\cdots n_2}. \end{aligned}$$

The matrix $D_{m_d,a_{j,m,d}}$ is the $m_d \times m_d$ diagonal matrix constructed as the matrix $D_{m,a}$ in Definition 1.2. Here tjhe function $a_{j,m,d}$ is the projection of the function a over the last component i.e.

$$a_{j,m,d}(y) = a(j_1/m_1, j_2/m_2, \ldots, j_{d-1}/m_{d-1}, y), \quad y \in \Omega_1 = [0, 1].$$

Finally $T_k(f)$ denotes the multilevel Toeplitz matrix of order $N(k)$ generated by f and O_q is the null matrix of order q.
In this case we write in short $\{A_n\} \sim_{\mathrm{LT}} (a, f)$.

Definition 2.2 *Suppose a sequence of matrices $\{A_n\}$ of size d_n is given. We say that $\{\{B_{n,m}\}\}_m$, $m \in \mathbf{N}$ is an approximating class of sequences for $\{A_n\}$ if, for all sufficiently large $m \in \mathbf{N}$, the following splittings hold:*

$$A_n = B_{n,m} + R_{n,m} + N_{n,m}, \quad \forall n > n_m, \tag{16}$$

with

$$\operatorname{rank} R_{n,m} \le d_n\, c(m), \quad ||N_{n,m}||_F \le \sqrt{d_n}\, \omega(m), \tag{17}$$

where n_m, $c(m)$ and $\omega(m)$ depend only on m and, moreover,

$$\lim_{m\to\infty} \omega(m) = 0, \quad \lim_{m\to\infty} c(m) = 0. \tag{18}$$

Definition 2.3 *A sequence of matrices $\{A_n\}$, where $n \in \mathbf{N}^d$, $N(n) = n_1 \cdots n_d$ and $A_n \in \mathbf{C}^{N(n)\times N(n)}$, is* approximable by linear combinations *of multilevel locally Toeplitz sequences with respect to a function κ, with $\kappa \in L^1(\Omega_d \times Q^d)$, if for any $\epsilon > 0$*

- *there exist pairs of functions $\{(a_{i,\epsilon}, f_{i,\epsilon})\}_{i=1}^{N_\epsilon}$ with $f_{i,\epsilon} \in L^1(Q^d)$ separables and $a_{i,\epsilon}$ defined over Ω_d such that $\| \sum_{i=1}^{N_\epsilon} a_{i,\epsilon} f_{i,\epsilon} - \kappa\|_1 < \epsilon$ over $\Omega_d \times Q^d$,*
- *there exist matrix sequences $\{\{A_n^{(i,\epsilon)}\}_n\}_{i=1}^{N_\epsilon}$ such that $\{A_n^{(i,\epsilon)}\} \sim_{\mathrm{LT}} (a_{i,\epsilon}, f_{i,\epsilon})$ and if*
- *$\{\{\sum_{i=1}^{N_\epsilon} A_n^{(i,\epsilon)}\}\}$, $\epsilon = (m+1)^{-1}$, $m \in \mathbf{N}$ is an approximating class of sequences for $\{A_n\}$.*

In this case the sequence $\{A_n\}$ is said to be a generalized Locally Toeplitz sequence *with respect to κ and we write in short $\{A_n\} \sim_{\mathrm{GLT}} \kappa$.*

Remark 2.3 *Given a sequence of matrices $\{A_n\}$, we will write $\{A_n\} \sim_{\mathrm{LT}} (a, f)$ to indicate that $\{A_n\}$ is multilevel locally Toeplitz with respect to a and f. It is understood that each A_n has order $N(n)$, that a is defined over Ω_d, and that f is defined over Q^d with $f(s_1, \ldots, s_d) = f_1(s_1) \cdots f_d(s_d)$ and is integrable; moreover, both a and f are supposed to be complex valued, unless otherwise specified. We call a the* weight function*, and f the* generating function*; moreover, in the splittings* (14)*, the matrices $R_{n,m}$'s are called* rank corrections*, while the $N_{n,m}$'s are called* norm corrections.

If $\{A_n\}$ is a generalized locally Toeplitz sequence i.e. $\{A_n\} \sim_{\mathrm{GLT}} \kappa$ with $\kappa \in L^1(\Omega_d \times Q^d)$, it is evident that the unique function κ has simultaneously the role of weight function *and of* generating function*: we call κ the* kernel function.

Finally it is clear that generalized locally Toeplitz sequences contain the span generated by multilevel locally Toeplitz sequences, since the first space of sequences is a sort of topological closure of the second space.

For the class of generalized locally Toeplitz sequences as in Definition 2.3, we prove a Szegő-like formula for singular values, namely

Theorem 2.1 *Assume that $\{A_n\}$ is a sequence of $n \times n$ complex matrices. Let $\kappa \in L^1(\Omega_d \times Q^d)$. Then for any continuous function F with bounded support*

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\sigma_j(A_n)] =$$

$$\frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F\left(|\kappa(x,s)|\right) dx\, ds \tag{19}$$

$$x \in \Omega_d = [0,1]^d, \ s \in Q^d = (-\pi, \pi)^d$$

holds whenever $\{A_n\}$ is a generalized locally Toeplitz sequence with respect to κ as in Definition 2.3.

A Szegő-like formula for eigenvalues is obtained if $\{A_n\}$ is Hermitian and approximable by multilevel locally Toeplitz sequences with respect to κ. In that case the kernel function $\kappa(x,s)$ has to be real almost everywhere.

Theorem 2.2 *Assume that $\{A_n\}$ is a sequence of $n \times n$ complex matrices. Let $\kappa \in L^1(\Omega_d \times Q^d)$. Then for any continuous function F with bounded support*

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\lambda_j(A_n)] =$$

$$\frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F[\kappa(x,s)]\, dx\, ds \tag{20}$$

$$x \in \Omega_d = [0,1]^d,\ \ s \in Q^d = (-\pi,\pi)^d$$

holds whenever $\{A_n\}$ is a Hermitian generalized locally Toeplitz sequences with respect to κ as in Definition 2.3.

It is crucial to observe that, contrary to multilevel Toeplitz structure, a single matrix A is never generalized locally Toeplitz: the notion of *local Toeplitzness* is an asymptotic one and it is always referred to a *sequence* of matrices $\{A_n\}$.

The paper is organized as follows.

In Section 3 we report some preliminary results which will be widely used throughout.

In Section 4 we study the asymptotic spectrum of multilevel locally Toeplitz sequences and of generalized locally Toeplitz sequences.

To be precise, spectral ergodic results concerning multilevel locally Toeplitz sequences are obtained as particular cases of more general results on distribution, involving finite sums of multilevel locally Toeplitz sequences (see Theorems 4.1 and 4.5) and generalized locally Toeplitz sequences (see Theorems 4.7 and 4.10 or (19) and (20)).

In Section 5 we study some algebraic properties of multilevel locally Toeplitz sequences and we prove that generalized locally Toeplitz sequences of matrices embrace (multilevel) Toeplitz sequences generated by L^1 symbols as a particular case.

Finally in Section 6 we come back to the relations with the discretization of PDEs by showing that these matrix-sequences can be naturally described as sums of multilevel locally Toeplitz matrix-sequences. A concluding section of remarks and open problems ends the paper.

3 Preliminaries

Given a real number $p \geq 1$ and measurable function f defined over a Lebesgue measurable set $\Omega \subset \mathbf{R}^n$, we write $f \in L^p$ if f p-integrable, and we denote by

$$\|f\|_p = \left(\int_\omega |f(x)|^p \, dx \right)^{\frac{1}{p}}$$

the L^p norm of f. We write $f \in L^\infty$ if f is essentially bounded over Ω, and we denote by $\|f\|_\infty$ the essential sup-norm of f (see [10]). We remark that, although the domain of f does not explicitly appear in the symbol $\|f\|_p$, it will always be clear from the context; typically, the domain will be $Q^d = (-\pi, \pi)^d$ if f is a generating function, $\Omega_d = [0,1]^d$ if f is a weight function, and $\Omega_d \times Q^d$ if f is a kernel function. If $Q \subset \mathbf{R}^d$, and $P(x)$ is a proposition depending on $x \in Q$, by statements such as "$P(x)$ for $x \in Q$ a.e." or "$P(x)$ a.e. over Q" we mean that $P(x)$ holds for $x \in Q$ almost everywhere, that is, $P(x)$ is true for all $x \in Q \setminus Q'$, where Q' is a subset of Q and has zero (d-dimensional) Lebesgue measure (see [6], p.86).

We denote by $\mathcal{C}_0$ the space of continuous functions defined on the real line $\mathbf{R}$, having bounded support; this space is relevant to our purpose, since it is the space of *test functions*, that is, the set of functions F appearing in asymptotic formulas like (2) or (3); the idea of using $\mathcal{C}_0$ as the set of test functions is due to Tyrtyshnikov [24], and enables one to study the distribution of eigen and singular values over the whole real line (thus dropping the assumption that $f \in L^\infty$ in the original version of the Szegő theorem, see [5, 24, 25]). Finally, we denote by $\mathcal{C}_0^1$ the subspace of those functions $F \in \mathcal{C}_0$ such that the derivative F' also belongs to $\mathcal{C}_0$.

Given a matrix A of order n, we denote by $\sigma_j(A)$, $j = 1, 2, \ldots, n$ the singular values of A, in non-decreasing order. We also define the Hermitian matrices

$$\operatorname{Re} A = \frac{1}{2}(A + A^*), \quad \operatorname{Im} A = \frac{1}{2\hat{\mathbf{i}}}(A - A^*)$$

where A^* is the Hermitian conjugate of A. Observe that the above notions reduce to those of real and imaginary part when A is a complex number; moreover, it holds $\operatorname{Re} A = A$ and $\operatorname{Im} A = \mathrm{O}_n$ if, and only if, A is Hermitian. In this case, we denote by $\lambda_j(A)$, $j = 1, 2, \ldots, n$ the eigenvalues of A, numbered in non-decreasing order.

For any pair of square matrices A and B, the following elementary properties hold:

$$(21) \qquad \|A \otimes B\|_2 = \|A\|_2 \|B\|_2, \quad \|A \oplus B\|_2 = \max\{\, \|A\|_2, \|B\|_2 \,\},$$

where $\|\cdot\|_2$ denotes the spectral norm. Observe that the symbol $\|\cdot\|_2$ stands for the L^2 norm if referred to a function, while it stands for the spectral norm if referred to a matrix; anyway, the appropriate meaning will always be clear from the context.

The Shatten p-norms [2] will be used in the following.

Definition 3.1 *The space of the $n \times n$ complex matrices is equipped with the Shatten p-norm defined as* [2]

$$\|A\|_{\mathrm{S,p}} = \left[\sum_{j=1}^{n} \sigma_j^p(A)\right]^{1/p}, \quad p \in [1, \infty)$$

and

$$\|A\|_{\mathrm{S},\infty} = \sigma_1(A), \quad p = \infty.$$

For $p = 2$ we find the classical Frobenius norm $\|\cdot\|_F$ and for $p = \infty$ we obtain the so-called "spectral" norm $\|\cdot\|_2$. When choosing $p = 1$, we find the so-called "trace" norm $\|\cdot\|_1$ that for Hermitian nonnegative definite matrices equals the trace of the matrix.

The following basic inequalities involving Toeplitz matrices and their generating functions will be extensively used throughout, without further comments (see [1]):

$$\|T_n(f)\|_2 \le \|f\|_\infty, \quad \|T_n(f)\|_{\mathrm{S,p}} \le n^{1/p}\|f\|_p, \quad \forall n. \tag{22}$$

In this paper, when a sequence of matrices $\{A_n\}$ is involved it will always be understood that each A_n has order $N(n)$ if $n = (n_1, \ldots, n_d)$ is a multi-index (unless otherwise explicitly specified).

The following lemma holds (see Theorem 3.1 from [24] and [14]):

Lemma 3.1 *If two sequences of matrices $\{A_n\}$, $\{\tilde{A}_n\}$ of size n are such that* $\mathrm{rank}(A_n - \tilde{A}_n) \le r(n)$ *with $r(n) = o(n)$, then for every $F \in \mathcal{C}_0$ it holds*

$$\lim_{n\to\infty} \left| \frac{1}{n}\sum_{j=1}^{n} F[\sigma_j(A_n)] - \frac{1}{n}\sum_{j=1}^{n} F[\sigma_j(\tilde{A}_n)] \right| = 0.$$

We will also need the following perturbation result for singular values and eigenvalues, a generalization of the Wielandt-Hoffman Theorem (a proof can be found in [7, 2]):

Theorem 3.2 [2] *Let $p \in [1, \infty)$. For any pair of matrices A, B of order n, we have*

$$\left[\sum_{j=1}^{n} |\sigma_j(A) - \sigma_j(B)|^p\right]^{1/p} \le \|A - B\|_{\mathrm{S,p}}. \tag{23}$$

If A and B are Hermitian, then we also have

$$\left[\sum_{j=1}^{n} |\lambda_j(A) - \lambda_j(B)|^p\right]^{1/p} \leq \|A - B\|_{\mathrm{S,p}}. \tag{24}$$

It is worth observig that labelling eigen and singular values in non-decreasing order is essential, in order to have the above estimates.

Now we investigate some basic spectral properties of the matrix-valued operators LT_n^m, defined according to Definition 2.1.

Remark 3.1 *Given $2p$ functions $a_i : \Omega_d \to \mathbf{C}$, $f_i : Q^d \to \mathbf{C}$ with f_i being separable, $i = 1, 2, \ldots, p$, for any two vectors of natural numbers $n > m$ the singular values of the $N(n) \times N(n)$ matrix $M = \sum_{i=1}^p \mathrm{LT}_n^m(a_i, f_i)$ are easily characterized in terms of the singular values of Toeplitz matrices, as follows. Partitioning M into blocks according to Definition 2.1, the j-th diagonal block, $j = (j_1, \ldots, j_d)$ of M is the matrix of order $N(\lfloor n/m \rfloor) \equiv \prod_{j=1}^d \lfloor n_j/m_j \rfloor$ given by*

$$\sum_{i=1}^{p} a_i(j/m) T_{\lfloor n_1/m_1 \rfloor}((f_i)_1) \otimes \cdots \otimes T_{\lfloor n_d/m_d \rfloor}((f_i)_d) = T_{\lfloor n/m \rfloor}\left(\sum_{i=1}^{p} a_i(j/m)\, f_i\right),$$

$$j/m = (j_1/m_1, \ldots, j_d/m_d), \;\; \lfloor n/m \rfloor = (\lfloor n_1/m_1 \rfloor, \ldots, \lfloor n_d/m_d \rfloor).$$

Therefore, $N(\lfloor n/m \rfloor) N(m) = N(n) \left(\prod_{j=1}^d (1 - (n_j \bmod m_j) n_j^{-1}\right)$ singular values of M are given by

$$\sigma_k\left(T_{\lfloor n/m \rfloor}\left(\sum_{i=1}^{p} a_i(j/m)\, f_i\right)\right),$$

$$j = e^T, \ldots, m, \quad k = 1, 2, \ldots, N(\lfloor n/m \rfloor),$$

while the remaining $N(n)\left(1 - \left(\prod_{j=1}^d (1 - (n_j \bmod m_j) n_j^{-1}\right)\right)$ singular values are null.

Concerning eigenvalues, instead of studying those of M we turn our attention to those of $\mathrm{Re}\, M$ (which reduces to M when M is Hermitian); reasoning as above, and observing that $T_k^*(f) = T_k(\bar{f})$, we obtain that the j-th diagonal block of $\mathrm{Re}\, M$ is given by

$$\mathrm{Re} \sum_{i=1}^{p} a_i(j/m) T_{\lfloor n/m \rfloor}(f_i) = T_{\lfloor n/m \rfloor}\left(\mathrm{Re} \sum_{i=1}^{p} a_i(j/m)\, f_i\right),$$

and hence $N(n) \left(\prod_{j=1}^d (1 - (n_j \bmod m_j) n_j^{-1}\right)$ eigenvalues of $\mathrm{Re}\, M$ are given by

$$\lambda_k\left(T_{\lfloor n/m \rfloor}\left(\mathrm{Re} \sum_{i=1}^{p} a_i(j/m)\, f_i\right)\right),$$

$$j = e^T, \ldots, m, \quad k = 1, 2, \ldots, N(\lfloor n/m \rfloor),$$

while the remaining $N(n)\left(1 - \left(\prod_{j=1}^{d}(1-(n_j \bmod m_j)n_j^{-1}\right)\right)$ *singular values are null.*

For any fixed $m \in \mathbf{N}^d$, sequences of the kind $\{\sum_{i=1}^{p} \mathrm{LT}_n^m(a_i, f_i)\}$ turn out to have a regular asymptotic distribution of singular values, as $n \to \infty$; as we show in the following lemma, a further limit operation is also possible, involving m.

Lemma 3.3 *Given* $2p$ *functions* $a_i : \Omega_d \to \mathbf{C}$, $f_i : Q^d \to \mathbf{C}$ *with* f_i *being separable,* $i = 1, \ldots, p$, *suppose that each* $f_i \in L^1$. *Then, for all* $m \in \mathbf{N}^d$ *and all* $F \in \mathcal{C}_0$ *it holds*

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F\left[\sigma_j\big(\sum_{i=1}^{p} \mathrm{LT}_n^m(a_i, f_i)\big)\right] \tag{25}$$

$$= \frac{1}{N(m)} \sum_{j=e^T}^{m} \frac{1}{m\{Q^d\}} \int_{Q^d} F\left[\left|\sum_{i=1}^{p} a_i(j/m) f_i(s)\right|\right] ds, \quad s \in Q^d = (-\pi, \pi)^d$$

If, moreover, each a_i *is Riemann integrable, then it holds*

$$\lim_{m\to\infty} \frac{1}{N(m)} \sum_{j=e^T}^{m} \frac{1}{m\{Q^d\}} \int_{Q^d} F\left[\left|\sum_{i=1}^{p} a_i(j/m) f_i(s)\right|\right] ds \tag{26}$$

$$= \frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F\left[\left|\sum_{i=1}^{p} a_i(x) f_i(s)\right|\right] dx\, ds,$$
$$x \in \Omega_d = [0,1]^d, \ s \in Q^d = (-\pi, \pi)^d.$$

Proof. For m fixed and $n > m$, setting $N(\lfloor n/m \rfloor) \equiv \prod_{j=1}^{d} \lfloor n_j/m_j \rfloor$, from Remark 3.1 we obtain

$$\frac{1}{N(n)} \sum_{j=1}^{N(n)} F\left[\sigma_j\big(\sum_{i=1}^{p} \mathrm{LT}_n^m(a_i, f_i)\big)\right]$$

$$= \frac{1}{N(n)} \sum_{j=e^T}^{m} \sum_{k=1}^{N(\lfloor n/m \rfloor)} F\left[\sigma_k\left(T_{\lfloor n/m \rfloor}\big(\sum_{i=1}^{p} a_i(j/m) f_i\big)\right)\right]$$
$$+ \left(1 - \left(\prod_{j=1}^{d}(1-(n_j \bmod m_j)n_j^{-1}\right)\right) F(0).$$

The last term tends to zero when $n \to \infty$. Therefore, letting

$$g_j(s) = \sum_{i=1}^{p} a_i(j/m) f_i(s),$$

we obtain

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F\left[\sigma_j(\sum_{i=1}^{p} \mathrm{LT}_n^m(a_i, f_i))\right]$$

$$= \sum_{j=e^T}^{m} \lim_{n\to\infty} \frac{N(\lfloor n/m \rfloor)}{N(n)} \frac{1}{N(\lfloor n/m \rfloor)} \sum_{k=1}^{N(\lfloor n/m \rfloor)} F\left[\sigma_k\left(T_{\lfloor n/m \rfloor}(g_j)\right)\right]$$

$$= \frac{1}{N(m)} \sum_{j=e^T}^{m} \frac{1}{m\{Q^d\}} \int_{Q^d} F\left[|g_j(s)|\right] ds, \tag{27}$$

where the last equality follows from $\lim_{n\to\infty} \lfloor n_r/m_r \rfloor / n_r = 1/m_r$, and from Theorem 1.2 applied $N(m)$ times to the generating functions g_j, $j = (j_1, \ldots, j_d)$, $j_r = 1, 2, \ldots, m_r$.

Finally, observe that (27) can be thought of as a Riemann sum, on the grid $\{j/m\}_{j=e^T}^{m}$, of the function

$$\phi(x) = \frac{1}{m\{Q^d\}} \int_{Q^d} F\left[\left|\sum_{i=1}^{p} a_i(x) f_i(s)\right|\right] ds, \quad x \in \Omega_d = (0,1)^d. \tag{28}$$

From the dominated convergence theorem it easily follows that ϕ is continuous at every point where all the a_i's are continuous. Since each a_i is Riemann integrable, then ϕ is also Riemann integrable, and (26) immediately follows, as limit of Riemann sums of a bounded and Riemann integrable function. □

Remark 3.2 *In* (27) *we obtained the quantity* $\frac{1}{N(m)} \sum_{j=e^T}^{m} \phi(j/m)$ *where the function* $\phi(\cdot)$ *is the one defined in* (28)*. In order to prove that the former expression represents a Riemann sum we used the fact that each* $a_i(x)$ *is Riemann integrable. However since each* F *is continuous and with bounded support we point out that the statement claimed in Lemma 3.3 still holds if we use the following weaker assumption: for any pair of real numbers* $m \le M$*, the functions* $\max\{\min\{a_i(x), M\}, m\}$ *are Riemann integrable. We stress that this situation occurs in the case where we discretize the Helmotz equation by finite differences over collocation Chebyshev points* [16].

In a similar fashion, we can prove the following statement concerning eigenvalues.

Lemma 3.4 *Given $2p$ functions $a_i : \Omega_d \to \mathbf{C}$, $f_i : Q^d \to \mathbf{C}$ with f_i being separable, $i = 1, \ldots, p$, suppose that each $f_i \in L^1$. Then, for all $m \in \mathbf{N}^d$ and all $F \in \mathcal{C}_0$ it holds*

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{n} F\left[\lambda_j\left(\mathrm{Re} \sum_{i=1}^{p} \mathrm{LT}_n^m(a_i, f_i)\right)\right] \tag{29}$$

$$= \frac{1}{N(m)} \sum_{j=e^T}^{m} \frac{1}{m\{Q^d\}} \int_{Q^d} F\left[\mathrm{Re} \sum_{i=1}^{p} a_i(j/m) f_i(s)\right] ds.$$

If, moreover, each a_i is Riemann integrable, then it holds

$$\lim_{m\to\infty} \frac{1}{N(m)} \sum_{j=e^T}^{m} \frac{1}{m\{Q^d\}} \int_{Q^d} F\left[\mathrm{Re} \sum_{i=1}^{p} a_i(j/m) f_i(s)\right] ds \tag{30}$$

$$= \frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F\left[\mathrm{Re} \sum_{i=1}^{p} a_i(x) f_i(s)\right] dx\, ds,$$
$$x \in \Omega_d = [0,1]^d, \ s \in Q^d = (-\pi, \pi)^d.$$

The proof is omitted since it is similar to that of Lemma 3.3, except that Theorem 1.1 must be used instead of Theorem 1.2.

Lemma 3.5 [14] *Suppose two sequences of matrices $\{A_n\}$ and $\{\tilde{A}_n\}$ of size n are such that*

$$\|A_n - \tilde{A}_n\|_{\mathrm{S,p}} \le n^{1/p} C \tag{31}$$

holds for all sufficiently large n and for a given $p \in [1, \infty)$, where $C > 0$ does not depend on n. Then, for all $F \in \mathcal{C}_0^1$ it holds

$$\left| \frac{1}{n} \sum_{j=1}^{n} F[\sigma_j(A_n)] - \frac{1}{n} \sum_{j=1}^{n} F[\sigma_j(\tilde{A}_n)] \right| \le C\|F'\|_\infty, \quad \forall n. \tag{32}$$

If, moreover, all the matrices involved are Hermitian, then it holds

$$\left| \frac{1}{n} \sum_{j=1}^{n} F[\lambda_j(A_n)] - \frac{1}{n} \sum_{j=1}^{n} F[\lambda_j(\tilde{A}_n)] \right| \le C\|F'\|_\infty, \quad \forall n. \tag{33}$$

Proposition 3.6 *Let d_n an increasing sequence of natural numbers. Suppose a sequence of matrices $\{A_n\}$ of size d_n is given such that $\{\{B_{n,m}\}\}_m$, $m \in \hat{\mathbf{N}} \subset \mathbf{N}$, $\#\hat{\mathbf{N}} = \infty$, is an approximating class of sequences for $\{A_n\}$ in the sense of*

Definition 2.2. Suppose that, for all sufficiently large $m \in \mathbf{N}$ and for all $F \in \mathcal{C}_0$, there exist the limits

$$(34) \quad \lim_{n\to\infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F[\sigma_j(B_{n,m})] = \Phi_m(F), \quad \textit{and} \quad \lim_{m\to\infty} \Phi_m(F) = \Phi(F).$$

Then it necessarily holds

$$(35) \qquad \lim_{n\to\infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F[\sigma_j(A_n)] = \Phi(F), \quad \forall F \in \mathcal{C}_0.$$

Similarly, if all the matrices in (16) *are Hermitian, and for all $F \in \mathcal{C}_0$ there exist the limits*

$$(36) \quad \lim_{n\to\infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F[\lambda_j(B_{n,m})] = \Psi_m(F), \quad \textit{and} \quad \lim_{m\to\infty} \Psi_m(F) = \Psi(F),$$

then it also holds

$$(37) \qquad \lim_{n\to\infty} \frac{1}{d_n} \sum_{j=1}^{d_n} F[\lambda_j(A_n)] = \Psi(F), \quad \forall F \in \mathcal{C}_0.$$

Proof. It is a straightforward generalization of Proposition 2.6 in [20]: the only difference is in the evaluation of the quantity $\alpha_{n,m}$ considered in [20] where we use Lemma 4.2 (part 1.) of [14] instead of Lemma 2.1 of [20]. □

4 Asymptotic spectrum of generalized locally Toeplitz sequences

Now we are in a position to show that the singular values of a multilevel locally Toeplitz sequence have a regular asymptotic distribution on the real line, and this distribution can be explicitly computed in terms of a and f. Indeed, a more general result is proved, concerning finite sums of multilevel locally Toeplitz sequences and generalized locally Toeplitz sequences.

Theorem 4.1 *Suppose that a sequence of matrices $\{A_n\}$ can be decomposed as the sum of p multilevel locally Toeplitz sequences $\{A_n^{(i)}\} \sim_{\mathrm{LT}} (a_i, f_i)$, $i = 1, 2, \ldots, p$, where the a_i's are Riemann integrable over Ω_d and the f_i's belong to $L^1(Q^d)$. Then for all $F \in \mathcal{C}_0$ it holds*

$$(38) \qquad \lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\sigma_j(A_n)] =$$

$$\frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F\left(\left|\sum_{i=1}^{p} a_i(x)\, f_i(s)\right|\right)\, dx\, ds.$$

Proof. By assumption, for each $i = 1, 2, \ldots, p$ we have the splittings

$$A_n^{(i)} = \mathrm{LT}_n^m(a_i, f_i) + R_{n,m}^{(i)} + N_{n,m}^{(i)} \quad \forall n > n_m^{(i)}, \tag{39}$$

where the rank and norm corrections satisfy the conditions

$$\begin{aligned} \operatorname{rank} R_{n,m}^{(i)} &\leq c^{(i)}(m) N(n) \sum_{j=1}^{d} n_j^{-1}, \\ \sum_{j=1}^{d} n_j^{-1} &\leq \frac{1}{\|m\|_\infty \left(\sum_{j=1}^{d} c^{(i)}(m) + 1\right)}, \\ \|N_{n,m}^{(i)}\|_F &\leq \sqrt{N(n)}\, \omega^{(i)}(m), \quad \lim_{m\to\infty} \omega^{(i)}(m) = 0. \end{aligned} \tag{40}$$

Letting $B_{n,m} = \sum_{i=1}^{p} \mathrm{LT}_n^m(a_i, f_i)$, $R_{n,m} = \sum_{i=1}^{p} R_{n,m}^{(i)}$, and $N_{n,m} = \sum_{i=1}^{p} N_{n,m}^{(i)}$, the sequence $\{A_n\}$ trivially satisfies conditions (16), (17) and (18), with $n_m = \max_{i=1}^{p}\{n_m^{(i)}\}$, $c(m) = \frac{1}{\|m\|_\infty}$ and $\omega(m) = \sum_{i=1}^{p} \omega^{(i)}(m)$. If for $F \in \mathcal{C}_0$ we let

$$\Phi_m(F) = \frac{1}{N(m)} \sum_{j=e^T}^{m} \frac{1}{m\{Q^d\}} \int_{Q^d} F\left[\left|\sum_{i=1}^{p} a_i(j/m)\, f_i(s)\right|\right] ds \quad \forall m,$$

and

$$\Phi(F) = \frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F\left[\left|\sum_{i=1}^{p} a_i(x)\, f_i(s)\right|\right] dx\, ds,$$

then by (25) and (26) of Lemma 3.3 we obtain that (34) is also fulfilled. Therefore, we can apply Proposition 3.6 to the sequence of matrices $\{A_n\}$, and observe that (35) reduces to (38). □

The case of a single multilevel locally Toeplitz sequence is obtained as a particular case of the above theorem.

Theorem 4.2 *Suppose the sequence of matrices $\{A_n\}$ is such that $\{A_n\} \sim_{\mathrm{LT}} (a, f)$, where a is Riemann integrable over Ω_d and $f \in L^1(Q^d)$ is separable. Then for all $F \in \mathcal{C}_0$ it holds*

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\sigma_j(A_n)] = \frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F\left(|a(x) f(s)|\right) dx\, ds. \tag{41}$$

Remark 4.1 *Unlike multilevel Toeplitz sequences, the sum of two multilevel locally Toeplitz sequences may fail to be multilevel locally Toeplitz; therefore, Theorem 4.1 does not follow as a consequence of Theorem 4.2.*

Theorem 4.3 *Suppose that a sequence of matrices $\{A_n\}$ can be decomposed, in two different ways, as the sum of p multilevel locally Toeplitz sequences*

$$A_n = \sum_{i=1}^{p} A_n^{(i)} = \sum_{i=1}^{p} \widehat{A}_n^{(i)}, \quad n \in \mathbf{N}^d,$$

$$\{A_n^{(i)}\} \sim_{\mathrm{LT}} (a_i, f_i), \quad \{\widehat{A}_n^{(i)}\} \sim_{\mathrm{LT}} (\hat{a}_i, \hat{f}_i), \quad i = 1, 2, \ldots, p,$$

where all the weight functions are Riemann integrable over Ω_d and the generating functions are separable and belong to $L^1(Q^d)$. Then it holds

$$\sum_{i=1}^{p} a_i(x)\, f_i(s) = \sum_{i=1}^{p} \hat{a}_i(x)\, \hat{f}_i(s), \quad \textit{a.e. over} \quad \Omega_d \times Q^d. \tag{42}$$

Proof. By subtracting the two decompositions from each other, we obtain that the null sequence $\{\mathrm{O}_n\}$ can be decomposed as the sum of $2p$ multilevel locally Toeplitz sequences; therefore, by Theorem 4.1 $\forall F \in \mathcal{C}_0$, we obtain

$$F(0) = \frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F\left[\left|\sum_{i=1}^{p} a_i(x)\, f_i(s) - \sum_{i=1}^{p} \hat{a}_i(x)\, \hat{f}_i(s)\right|\right] dx\, ds.$$

Since F is arbitrary, it is well-known from measure-theory that the above equality implies

$$\left|\sum_{i=1}^{p} a_i(x)\, f_i(s) - \sum_{i=1}^{p} \hat{a}_i(x)\, \hat{f}_i(s)\right| = 0, \quad \text{a.e. over} \quad \Omega_d \times Q^d,$$

and the proof is completed. □

Corollary 4.4 *Suppose that a sequence of Hermitian matrices $\{A_n\}$ can be decomposed as the sum of p multilevel locally Toeplitz sequences $\{A_n^{(i)}\} \sim_{\mathrm{LT}} (a_i, f_i)$, $i = 1, 2, \ldots, p$, where each a_i is Riemann integrable over Ω_d and each $f_i \in L^1(Q^d)$ is separable. Then it holds*

$$\sum_{i=1}^{p} a_i(x)\, f_i(s) \in \mathbf{R}, \quad \textit{a.e. over} \quad \Omega_d \times Q^d.$$

Proof. It is a trivial but general fact that $\{A_n\} \sim_{\mathrm{LT}} (a, f)$ holds if, and only if, $\{A_n^*\} \sim_{\mathrm{LT}} (\bar{a}, \bar{f})$. In our case, since the sequence $\{A_n\}$ is Hermitian, it can also be decomposed as the sum of $\{A_n^{(i)*}\}$, $i = 1, \ldots, p$, where $\{A_n^{(i)*}\} \sim_{\mathrm{LT}} (\bar{a}_i, \bar{f}_i)$; from the previous theorem we then obtain

$$\sum_{i=1}^{p} a_i(x)\, f_i(s) = \sum_{i=1}^{p} \bar{a}_i(x)\, \bar{f}_i(s), \quad \text{a.e. over} \quad \Omega_d \times Q^d,$$

and the proof is completed. $\square$

Concerning the asymptotic distribution of eigenvalues, the analog of Theorem 4.1 can be formulated as follows.

Theorem 4.5 *Suppose that a sequence of matrices $\{A_n\}$ can be decomposed as the sum of p multilevel locally Toeplitz sequences $\{A_n^{(i)}\} \sim_{\rm LT} (a_i, f_i)$, $i = 1, 2, \ldots, p$, where each a_i is Riemann integrable over Ω_d and each $f_i \in L^1(Q^d)$ is separable. Then, for all $F \in \mathcal{C}_0$ it holds*

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\lambda_j(\mathrm{Re}\, A_n)] = \tag{43}$$

$$\frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d\times Q^d} F[\,\mathrm{Re} \sum_{i=1}^{p} a_i(x)\, f_i(s)\,]\, dx\, ds,$$

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\lambda_j(\mathrm{Im}\, A_n)] = \tag{44}$$

$$\frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d\times Q^d} F[\,\mathrm{Im} \sum_{i=1}^{p} a_i(x)\, f_i(s)\,]\, dx\, ds,$$

If, moreover, each $\{A_n\}$ is Hermitian, then it also holds

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\lambda_j(A_n)] = \tag{45}$$

$$\frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d\times Q^d} F[\sum_{i=1}^{p} a_i(x)\, f_i(s)\,]\, dx\, ds.$$

Proof. It is the same reasoning as in the proof of Theorem 4.1 (see also Theorem 3.5 in [20]). $\square$

The case of a single multilevel locally Toeplitz sequence is obtained as a particular case of the above theorem.

Theorem 4.6 *Suppose a sequence of matrices $\{A_n\}$ is given, such that $\{A_n\} \sim_{\rm LT} (a, f)$, where a is Riemann integrable over Ω_d and $f \in L^1(Q^d)$ is separable. Then for all $F \in \mathcal{C}_0$ it holds*

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\lambda_j(\mathrm{Re}\, A_n)] = \tag{46}$$

$$\frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F[\,\mathrm{Re}\, a(x) f(s)\,]\, dx\, ds,$$

(47)
$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\lambda_j(\mathrm{Im}\, A_n)] =$$

$$\frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F[\,\mathrm{Im}\, a(x) f(s)\,]\, dx\, ds.$$

If, moreover, each A_n is Hermitian, then it holds

(48)
$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\lambda_j(A_n)] =$$

$$\frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F[\, a(x) f(s)\,]\, dx\, ds.$$

Now we are in position to prove the most general results concerning the spectral distribution of generalized locally Toeplitz sequences.

Theorem 4.7 *Let $\{A_n\}$ be a generalized locally Toeplitz sequence with respect to $\kappa \in L^1(\Omega_d \times Q^d)$ in the sense of Definition 2.3. Suppose that the functions $a_{i,\epsilon}$ involved in Definition 2.3 are Riemann integrable over Ω_d. Then for all $F \in \mathcal{C}_0$ it holds*

(49)
$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\sigma_j(A_n)] =$$

$$\frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F\big(|\kappa(x,s)|\big)\, dx\, ds.$$

Proof. By the assumptions we have

$$A_n = B_{n,m} + R_{n,m} + N_{n,m}, \quad \forall n > n_m,$$

with

$$\mathrm{rank}\, R_{n,m} \le d_n\, c(m), \quad \|N_{n,m}\|_F \le \sqrt{d_n}\, \omega(m),$$

where n_m, $c(m)$ and $\omega(m)$ depend only on m and, moreover,

$$\lim_{m\to\infty} \omega(m) = 0, \quad \lim_{m\to\infty} c(m) = 0.$$

Here $\epsilon = (m+1)^{-1}$ and $B_{n,m} = \sum_{i=1}^{N_\epsilon} A_n^{(i,\epsilon)}$ with $A_n^{(i,\epsilon)} \sim_{\mathrm{LT}} (a_{i,\epsilon}, f_{i,\epsilon})$ and $\| \sum_{i=1}^{N_\epsilon} a_{i,\epsilon} f_{i,\epsilon} - \kappa\|_1 < \epsilon$ over $\Omega_d \times Q^d$ (as in Definition 2.3). If for $F \in \mathcal{C}_0$ we let

$$\Phi_m(F) = \frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F\left[\left|\sum_{i=1}^{N_\epsilon} a_{i,\epsilon}(x)\, f_{i,\epsilon}(s)\right|\right] dx\, ds, \quad \forall m = m(\epsilon),$$

and

$$\Phi(F) = \frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F\left[|\kappa(x,s)|\right] \, dx \, ds,$$

then by Theorem 4.1 we obtain that (34) is also fulfilled because of the relation

$$\|\sum_{i=1}^{N_\epsilon} a_{i,\epsilon} f_{i,\epsilon} - \kappa\|_1 < \epsilon.$$

Therefore, we can apply Proposition 3.6 to the sequence of matrices $\{A_n\}$, and observe that (35) reduces to (49). □

Remark 4.2 *The right-hand side of* (49) *can be thought of as a generalized version of the right-hand side of* (3)*; in particular, when* $\kappa(x,s) \equiv f(s)$ *the integral in* (49) *reduces to that in* (3)*. This analogy is more than a formal one; indeed, in Theorem 5.2 we will prove that, if* $f \in L^1$*, then* $\{T_n(f)\}$ *is approximable by linear combinations of multilevel locally Toeplitz sequences with respect to* $\kappa(x,s) = f(s)$*, and hence* (49) *reduces to* (3)*. In view of this, Theorem 1.2 can be obtained as a corollary of the above theorem.*

Theorem 4.8 *Suppose that a sequence of matrices* $\{A_n\}$ *is a generalized locally Toeplitz sequence with respect to* $\kappa \in L^1(\Omega_d \times Q^d)$ *and* $\hat{\kappa} \in L^1(\Omega_d \times Q^d)$ *in the sense of Definition 2.3. Then it holds*

$$\kappa(x,s) = \hat{\kappa}(x,s), \quad a.e.\ over \quad \Omega_d \times Q^d. \tag{50}$$

Proof. By subtracting the two decompositions from each other, we obtain that the null sequence $\{\mathrm{O}_n\}$ can be approximated by multilevel locally Toeplitz sequences with respect to the kernel function $\kappa - \hat{\kappa}$; therefore, by Theorem 4.7 $\forall F \in \mathcal{C}_0$, we obtain

$$F(0) = \frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F\left[\,|\kappa(x,s) - \hat{\kappa}(x,s)|\,\right] \, dx \, ds.$$

Since F is arbitrary, it is well-known from measure-theory that the above equality implies

$$|\kappa(x,s) - \hat{\kappa}(x,s)| = 0, \quad \text{a.e. over} \quad \Omega_d \times Q^d,$$

and the proof is completed. □

Corollary 4.9 *Suppose that a sequence of Hermitian matrices* $\{A_n\}$ *is a generalized locally Toeplitz sequence with respect to* $\kappa \in L^1(\Omega_d \times Q^d)$*. Then it holds*

$$\kappa(x,s) \in \mathbf{R}, \quad a.e.\ over \quad \Omega_d \times Q^d.$$

Proof. It is a trivial but general fact that $\{A_n\} \sim_{\rm GLT} \kappa$ holds if, and only if, $\{A_n^*\} \sim_{\rm GLT} \bar{\kappa}$. In our case, since the sequence $\{A_n\}$ is Hermitian, it is simultaneously generalized locally Toeplitz with respect to κ and to $\bar{\kappa}$; from the previous theorem we then obtain

$$\kappa(x,s) = \bar{\kappa}(x,s), \quad \text{a.e. over} \quad \Omega_d \times Q^d,$$

and the proof is completed. □

Theorem 4.10 *Let $\{A_n\}$ be a generalized locally Toeplitz sequence with respect to $\kappa \in L^1(\Omega_d \times Q^d)$ in the sense of Definition 2.3. Suppose that the functions $a_{i,\epsilon}$ involved in Definition 2.3 are Riemann integrable over Ω_d. Then, for all $F \in \mathcal{C}_0$ it holds*

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\lambda_j(\operatorname{Re} A_n)] = \frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F[\operatorname{Re}\kappa(x,s)]\, dx\, ds, \tag{51}$$

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\lambda_j(\operatorname{Im} A_n)] = \frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F[\operatorname{Im}\kappa(x,s)]\, dx\, ds, \tag{52}$$

If, moreover, each $\{A_n\}$ is Hermitian, then it also holds

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\lambda_j(A_n)] = \frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F[\kappa(x,s)]\, dx\, ds. \tag{53}$$

Proof. It is a simple adaptation of the former proof. □

Remark 4.3 *When $f \in L^1(Q^d)$ is real-valued, the above theorem is a generalization of Theorem 1.1; indeed, as we will prove in Theorem 5.2, $\{T_n(f)\}$ is approximable by multilevel locally Toeplitz sequences with respect to f, i.e., $\{T_n(f)\}$ is a generalized locally Toeplitz sequence with respect to $\kappa(x,s) = f(s)$ and we can apply the above theorem. Then the right-hand side of* (53) *reduces to that of* (2).

5 Further properties of generalized locally Toeplitz sequences

In the following theorem we show that any multilevel Toeplitz sequence $\{T_n(f)\}$, generated by $f \in L^1(Q^d)$, is also approximable by multilevel locally Toeplitz sequences with respect to the set f. Therefore, Toeplitz sequences are a particular case of generalized locally Toeplitz.

Theorem 5.1 *If $f : Q^d \to \mathbf{C}$ is a polynomial with $f(s) = f_1(s_1) \cdots f_d(s_d)$ being separable, then it holds $\{T_n(f)\} \sim_{\mathrm{LT}} (1, f)$.*

Proof. First we observe that

$$T_n(f) = T_{n_1}(f_1) \otimes T_{n_2}(f_2) \otimes \cdots \otimes T_{n_d}(f_d). \tag{54}$$

Second we recall that each $\{T_{n_j}(f_j)\} \sim_{\mathrm{LT}} (1, f_j)$ since each f_j is $L^2(Q)$ (see Theorem 4.1 of [20]). Now combining the splitting given in (4) with conditions (5) for each $T_{n_j}(f_j)$, and equation (54) we obtain a splitting of $T_n(f)$ as (14) satisfying for n large enough the conditions given in (15). □

Theorem 5.2 *If $f : Q^d \to \mathbf{C}$ belongs to $L^1(Q^d)$, then it holds that $\{T_n(f)\}$ is approximable by multilevel locally Toeplitz sequences with respect to $\kappa(x, s) = f(s)$.*

Proof. For any positive ϵ, there exists a multivariate trigonometric polynomial p_ϵ such that $\|f - p_\epsilon\|_1 \le \epsilon$. Of course each polynomial can be written as a finite sum of separable polynomials $\{q_{j,\epsilon}\}_j$. Moreover it is known that $\{\{T_n(p_\epsilon)\}\}$, $\epsilon = (m+1)^{-1}$, $m \in \mathbf{N}$, is an approximating class of sequences for $\{T_n(f)\}$ in the sense of Definition 2.2 (see [25, 15, 22]). Finally by Theorem 5.1 we have that $\{T_n(q_{j,\epsilon})\} \sim_{\mathrm{LT}} (1, q_{j,\epsilon})$ so that $\{T_n(f)\}$ is approximable by multilevel locally Toeplitz sequences with respect to the kernel $\kappa(x, s) = f(s)$ i.e. $\{T_n(f)\} \sim_{\mathrm{GLT}} f$. □

Now we investigate the behavior of linear combinations of multilevel locally Toeplitz sequences under sums and products. Of course there is stability under sums and, in general, under linear combinations. On the other hand, it is not reasonable to expect that the product of any two multilevel locally Toeplitz sequences is also multilevel locally Toeplitz or can be expressed as a linear combination of multilevel locally Toeplitz; indeed, when expanding the product of two multilevel locally Toeplits sequences according to their respective splittings (14), the product of the two norm-corrections might grow too fast, that is, its Frobenius norm might fail to be $O(\sqrt{N(n)})$. A sufficient condition, however, is the following.

Theorem 5.3 *Suppose $\{A_n\} \sim_{\rm LT} (a,f)$ and $\{\tilde A_n\} \sim_{\rm LT} (\tilde a,\tilde f)$, where $a,\tilde a,f,\tilde f$ are essentially bounded and f and $\tilde f$ are separable. If there exists a constant $C>0$ such that*

$$\|N_{n,m}\|_2 \le C, \tag{55}$$

holds in the splittings (14)*, then it holds*

$$\{A_n\tilde A_n\} \sim_{\rm LT} (a\tilde a, f\tilde f), \qquad \{\tilde A_n A_n\} \sim_{\rm LT} (a\tilde a, f\tilde f), \tag{56}$$

where the new spittings in (14) *concerning $\{A_n\tilde A_n\}$ and $\{\tilde A_n A_n\}$ satisfy* (55)*.*

Proof. Due to the tensorial decomposition of $T_k(f) = T_{k_1}\otimes T_{k_2}\otimes\cdots\otimes T_{k_d}$ and of $T_k(\tilde f) = T_k(\tilde f_1)\otimes T_k(\tilde f_2)\otimes\cdots\otimes T_k(\tilde f_d)$ the claimed thesis is simply reduced to the unilevel case where $d=1$. Since this result with $d=1$ has been proved in Theorem 4.4 of [20] the proof is over. $\square$

Now we investigate the stability of generalized locally Toeplitz sequences under linear combinations and products. The stability under linear combinations is trivial: in order to analyze the stability under products we introduce the following lemma (whose statements are contained in Lemma 3.1 of [16])

Lemma 5.4 *Consider a sequence $\{A_n\}$, where $A_n \in \mathbf{C}^{n\times n}$, $n\in\hat{\mathbf{N}}\subset\mathbf{N}$. Then the following are equivalent.*

- *$\forall p \in [1,+\infty)$, there exists a function $x_p(s)$ with $\lim_{s\to 0} x_p(s) = 0$ so that $\forall\epsilon>0$ $\exists n_\epsilon(p)\in\mathbf{N}$ such that $\forall n\ge n_\epsilon(p)$ it holds $A_n = N_{n,\epsilon}(p) + R_{n,\epsilon}(p)$, with $\|N_{n,\epsilon}(p)\|_{\rm S,p}\le\epsilon$ and $\operatorname{rank} R_{n,\epsilon}(p)\le x_p(\epsilon)n$.*
- *There exists a function $x(s)$ with $\lim_{s\to 0} x(s) = 0$ so that $\forall\epsilon>0$ $\exists n_\epsilon\in\mathbf{N}$ such that $\forall n\ge n_\epsilon$ it holds $A_n = N_{n,\epsilon} + R_{n,\epsilon}$, with $\|N_{n,\epsilon}\|_2\le\epsilon$ and $\operatorname{rank} R_{n,\epsilon}\le x(\epsilon)n$.*

Theorem 5.5 *Suppose $\{A_n\}\sim_{\rm GLT}\kappa$ and $\{\tilde A_n\}\sim_{\rm GLT}\tilde\kappa$, where the functions $\kappa,\tilde\kappa$ and $a_{i,\epsilon},\tilde a_{i,\epsilon},f_{i,\epsilon},\tilde f_{i,\epsilon}$ appearing in Definition 2.3 are essentially bounded and*

$$\Big\|\sum_{i=1}^{N_\epsilon} a_{i,\epsilon} f_{i,\epsilon}\Big\|_\infty, \quad \Big\|\sum_{i=1}^{\tilde N_\epsilon} \tilde a_{i,\epsilon}\tilde f_{i,\epsilon}\Big\|_\infty \le C. \tag{57}$$

If there exists a constant $C^{(t,\epsilon)}>0$ such that

$$\|N_{n,m}^{(t,\epsilon)}\|_2 \le C^{(t,\epsilon)}, \quad t\in\{i,j\} \tag{58}$$

holds in the splittings in the splittings (14) *for $\{A_n^{(i,\epsilon)}\}$, $i=1,\ldots,N_\epsilon$ and for $\{\tilde A_n^{(j,\epsilon)}\}$, $j=1,\ldots,\tilde N_\epsilon$, then we obtain*

$$\{A_n\tilde A_n\}\sim_{\rm GLT}\kappa\tilde\kappa,\ \{\tilde A_n A_n\}\sim_{\rm GLT}\kappa\tilde\kappa \tag{59}$$

in the generalized sense where the new spittings in (14) *concerning the decompositions of $\{A_n\tilde A_n\}$ and $\{\tilde A_n A_n\}$ satisfy* (58)*.*

Proof. From the assumptions $\{A_n\} \sim_{\mathrm{GLT}} \kappa$ and $\{\tilde{A}_n\} \sim_{\mathrm{GLT}} \tilde{\kappa}$ and from (58), for any positive ϵ, we know that

$$A_n = \sum_{i=1}^{N_\epsilon} A_n^{(i,\epsilon)} + R_{n,\epsilon} + N_{n,\epsilon}, \quad \forall n > n_\epsilon,$$

$$\tilde{A}_n = \sum_{i=1}^{\tilde{N}_\epsilon} \tilde{A}_n^{(i,\epsilon)} + \tilde{R}_{n,\epsilon} + \tilde{N}_{n,\epsilon}, \quad \forall n > n_\epsilon,$$

with

$$\operatorname{rank} R_{n,\epsilon} \le N(n)\, c(\epsilon), \quad \|N_{n,\epsilon}\|_F \le \sqrt{N(n)}\, \omega(\epsilon),$$
$$\operatorname{rank} \tilde{R}_{n,\epsilon} \le N(n)\, c(\epsilon), \quad \|\tilde{N}_{n,\epsilon}\|_F \le \sqrt{N(n)}\, \omega(\epsilon),$$

where n_ϵ, $c(\epsilon)$ and $\omega(\epsilon)$ depend only on ϵ and, moreover,

$$\lim_{\epsilon \to 0} \omega(\epsilon) = 0, \quad \lim_{\epsilon \to 0} c(\epsilon) = 0.$$

since $\|\cdot\|_F = \|\cdot\|_{\mathrm{S},2}$, by Lemma 5.4 we can arrange the choice of the splittings so that the inequalities

$$\|N_{n,\epsilon}\|_2 \le \epsilon, \qquad \|\tilde{N}_{n,\epsilon}\|_2 \le \epsilon$$

hold too.

Here $\{A_n^{(i,\epsilon)}\} \sim_{\mathrm{LT}} (a_{i,\epsilon}, f_{i,\epsilon})$ and $\{\tilde{A}_n^{(i,\epsilon)}\} \sim_{\mathrm{LT}} (\tilde{a}_{i,\epsilon}, \tilde{f}_{i,\epsilon})$ with

$$A_n^{(i,\epsilon)} = LT_n^m(a_{i,\epsilon}, f_{i,\epsilon}) + R_{n,m}^{(i,\epsilon)} + N_{n,m}^{(i,\epsilon)} \quad \forall n > n_{m,\epsilon,i}$$

$$\tilde{A}_n^{(i,\epsilon)} = LT_n^m(\tilde{a}_{i,\epsilon}, \tilde{f}_{i,\epsilon}) + \tilde{R}_{n,m}^{(i,\epsilon)} + \tilde{N}_{n,m}^{(i,\epsilon)} \quad \forall n > n_{m,\epsilon,i}$$

where

$$\operatorname{rank} R_{n,m}^{(i,\epsilon)} \le c^{(i,\epsilon)}(m) N(n) (\sum_{j=1}^d n_j^{-1}), \tag{60}$$
$$\|N_{n,m}^{(i,\epsilon)}\|_F \le \omega^{(i,\epsilon)}(m) \sqrt{N(n)}, \tag{61}$$
$$\|N_{n,m}^{(i,\epsilon)}\|_2 \le C^{(i,\epsilon)}, \tag{62}$$

$$\operatorname{rank} \tilde{R}_{n,m}^{(i,\epsilon)} \le c^{(i,\epsilon)}(m) N(n) (\sum_{j=1}^d n_j^{-1}), \tag{63}$$
$$\|\tilde{N}_{n,m}^{(i,\epsilon)}\|_F \le \omega^{(i,\epsilon)}(m) \sqrt{N(n)}, \tag{64}$$
$$\|\tilde{N}_{n,m}^{(i,\epsilon)}\|_2 \le C^{(i,\epsilon)} \tag{65}$$

where $c^{(i,\epsilon)}(m)$ and $\omega^{(i,\epsilon)}(m)$ are functions of m with $\lim_{m\to\infty} \omega^{(i,\epsilon)}(m) = 0$ and $C^{(i,\epsilon)}$ being constant independent of m. Moreover we have

$$\|\sum_{i=1}^{N_\epsilon} a_{i,\epsilon} f_{i,\epsilon} - \kappa\|_1 < \epsilon, \qquad \|\sum_{i=1}^{\tilde{N}_\epsilon} \tilde{a}_{i,\epsilon} \tilde{f}_{i,\epsilon} - \tilde{\kappa}\|_1 < \epsilon$$

over $\Omega_d \times Q^d$ with $f_{i,\epsilon},\ \tilde{f}_{i,\epsilon}$ separables and $f_{i,\epsilon},\ a_{i,\epsilon},\ \tilde{f}_{i,\epsilon},\ \tilde{a}_{i,\epsilon} \in L^\infty$.

Now we prove that

$$\{A_n \tilde{A}_n\} \sim_{\mathrm{GLT}} \kappa\tilde{\kappa}$$

in the generalized sense where the new spittings in (14) concerning the decompositions of $\{A_n \tilde{A}_n\}$ satisfy (58). The proof regarding $\{\tilde{A}_n A_n\}$ is similar.

By the preceding relations we have

$$A_n \tilde{A}_n = \sum_{i=1}^{N_\epsilon} \sum_{j=1}^{\tilde{N}_\epsilon} A_n^{(i,\epsilon)} \tilde{A}_n^{(j,\epsilon)} + R_{n,\epsilon}\tilde{A}_n + N_{n,\epsilon}\tilde{A}_n, \quad \forall n > n_\epsilon.$$

First of all we analyze the rank and the norm of the terms $R_{n,\epsilon}\tilde{A}_n + N_{n,\epsilon}\tilde{A}_n$. We have

$$\text{rank}\, R_{n,\epsilon}\tilde{A}_n \le \text{rank}\, R_{n,\epsilon} \le N(n)\, c(\epsilon) \tag{66}$$

and

$$\begin{aligned} N_{n,\epsilon}\tilde{A}_n &= N_{n,\epsilon}\left(\sum_{j=1}^{\tilde{N}_\epsilon} \tilde{A}_n^{(j,\epsilon)} + \tilde{R}_{n,\epsilon} + \tilde{N}_{n,\epsilon}\right) \\ &= N_{n,\epsilon}\left(\sum_{j=1}^{\tilde{N}_\epsilon} LT_n^m(\tilde{a}_{j,\epsilon}, \tilde{f}_{j,\epsilon}) + \tilde{R}_{n,m}^{(j,\epsilon)} + \tilde{N}_{n,m}^{(j,\epsilon)}\right) \\ &\quad + N_{n,\epsilon}\tilde{R}_{n,\epsilon} + N_{n,\epsilon}\tilde{N}_{n,\epsilon} \end{aligned}$$

so that

$$\text{rank}\, N_{n,\epsilon}\tilde{R}_{n,\epsilon} \le \text{rank}\, \tilde{R}_{n,\epsilon} \le N(n)\, c(\epsilon), \tag{67}$$

$$\|N_{n,\epsilon}\tilde{N}_{n,\epsilon}\|_2 \le \epsilon^2, \tag{68}$$

$$\|N_{n,\epsilon}\tilde{N}_{n,\epsilon}\|_2 \le \epsilon\ \omega(\epsilon)\sqrt{N(n)}. \tag{69}$$

Now by using (57) we obtain that

$$\|N_{n,\epsilon} \sum_{j=1}^{\tilde{N}_\epsilon} LT_n^m(\tilde{a}_{j,\epsilon}, \tilde{f}_{j,\epsilon})\|_2 \quad \le \quad \epsilon\ \|\sum_{j=1}^{\tilde{N}_\epsilon} \tilde{a}_{j,\epsilon}\tilde{f}_{j,\epsilon}\|_\infty \le \epsilon\ C$$

and

$$\|N_{n,\epsilon}\sum_{j=1}^{\tilde N_\epsilon} LT_n^m(\tilde a_{j,\epsilon},\tilde f_{j,\epsilon})\|_F \le \sqrt{N(n)}C\ \epsilon.$$

On the the other hand we have

$$\text{rank}\, N_{n,\epsilon}\tilde R_{n,m}^{(j,\epsilon)} \le \text{rank}\, \tilde R_{n,m}^{(j,\epsilon)}, \tag{70}$$

$$\|N_{n,\epsilon}\tilde N_{n,m}^{(j,\epsilon)}\|_2 \le \epsilon\|\tilde N_{n,m}^{(j,\epsilon)}\|_2, \quad \|N_{n,\epsilon}\tilde N_{n,m}^{(j,\epsilon)}\|_F \le \epsilon\|\tilde N_{n,m}^{(j,\epsilon)}\|_F. \tag{71}$$

Therefore we have deduced that

$$\begin{aligned} A_n\tilde A_n &= \sum_{i=1}^{N_\epsilon}\sum_{j=1}^{\tilde N_\epsilon}\left(A_n^{(i,\epsilon)}\tilde A_n^{(j,\epsilon)} + \frac{N_{n,\epsilon}}{N_\epsilon}\tilde R_{n,m}^{(j,\epsilon)} + \frac{N_{n,\epsilon}}{N_\epsilon}\tilde N_{n,m}^{(j,\epsilon)}\right) \\ &\quad + \hat R_{n,\epsilon} + \hat N_{n,\epsilon}, \quad \forall n > n_\epsilon \end{aligned} \tag{72}$$

where by (66), (67), (68) and (69) we have

$$\begin{aligned} \text{rank}\,\hat R_{n,\epsilon} &\le 2c(\epsilon)\ N(n), \\ \|\hat N_{n,\epsilon}\|_2 &\le \epsilon^2 + \epsilon\ C, \\ \|\hat N_{n,\epsilon}\|_F &\le \epsilon(\omega(\epsilon)+C)\sqrt{N(n)}. \end{aligned}$$

At this point it remains to be analyzed the terms

$$A_n^{(i,\epsilon)}\tilde A_n^{(j,\epsilon)} + \frac{N_{n,\epsilon}}{N_\epsilon}\tilde R_{n,m}^{(j,\epsilon)} + \frac{N_{n,\epsilon}}{N_\epsilon}\tilde N_{n,m}^{(j,\epsilon)}.$$

We recall that $\{A_n^{(i,\epsilon)}\}$ and $\{\tilde A_n^{(j,\epsilon)}\}$ satisfy the assumption of Theorem 5.3 so that $A_n^{(i,\epsilon)}\tilde A_n^{(j,\epsilon)} \sim_{\mathrm{LT}} (a_i\tilde a_j, f_i\tilde f_j)$ with

$$\begin{aligned} A_n^{(i,\epsilon)}\tilde A_n^{(j,\epsilon)} &= LT_n^m(a_{i,\epsilon}\tilde a_{j,\epsilon}, f_{i,\epsilon}\tilde f_{j,\epsilon}) + \\ &\quad \hat R_{n,m}^{(i,j,\epsilon)} + \hat N_{n,m}^{(i,j,\epsilon)} \quad \forall n > n_{m,i,j,\epsilon} \end{aligned}$$

where

$$\text{rank}\,\hat R_{n,m}^{(i,j,\epsilon)} \le \hat c^{(i,\epsilon)}(m)N(n)\Big(\sum_{j=1}^d n_j^{-1}\Big), \tag{73}$$

$$\|\hat N_{n,m}^{(i,j,\epsilon)}\|_F \le \hat\omega^{(i,j,\epsilon)}(m)\sqrt{N(n)}, \tag{74}$$

$$\|\hat N_{n,m}^{(i,j,\epsilon)}\|_2 \le \hat C^{(i,j,\epsilon)}, \tag{75}$$

with $\hat c^{(i,j,\epsilon)}(m)$ and $\hat\omega^{(i,j,\epsilon)}(m)$ being functions of m with $\lim_{m\to\infty}\hat\omega^{(i,j,\epsilon)}(m) = 0$ and $\hat C^{(i,j,\epsilon)}$ being constant independent of m. Furthermore, owing to (70) and (71), it follows that

$$\left\{A_n^{(i,\epsilon)}\tilde A_n^{(j,\epsilon)} + \frac{N_{n,\epsilon}}{N_\epsilon}\tilde R_{n,m}^{(j,\epsilon)} + \frac{N_{n,\epsilon}}{N_\epsilon}\tilde N_{n,m}^{(j,\epsilon)}\right\} \sim_{\mathrm{LT}} (a_i\tilde a_j, f_i\tilde f_j).$$

Finally we have to check that

$$\sum_{i=1}^{N_\epsilon}\sum_{j=1}^{\tilde{N}_\epsilon} a_{i,\epsilon} f_{i,\epsilon} \tilde{a}_{j,\epsilon} \tilde{f}_{j,\epsilon}$$

approximates in $L^1(\Omega_d \times Q^d)$ norm the function $\kappa\tilde{\kappa}$.

Call $p_\epsilon = \sum_{i=1}^{N_\epsilon} a_{i,\epsilon} f_{i,\epsilon}$ and $\tilde{p}_\epsilon = \sum_{i=1}^{\tilde{N}_\epsilon} \tilde{a}_{i,\epsilon} \tilde{f}_{i,\epsilon}$.

$$\begin{aligned} \|p_\epsilon \tilde{p}_\epsilon - \kappa\tilde{\kappa}\|_1 &\le \|p_\epsilon(\tilde{p}_\epsilon - \tilde{\kappa}) + \tilde{\kappa}(p_\epsilon - \kappa)\|_1 \\ &\le \epsilon\|p_\epsilon\|_1 + \epsilon\|\tilde{\kappa}\|_1 \\ &= \epsilon\|\kappa + (p_\epsilon - \kappa)\|_1 + \epsilon\|\tilde{\kappa}\|_1 \\ &\le \epsilon(\|\kappa\|_1 + \epsilon) + \epsilon\|\tilde{\kappa}\|_1 \end{aligned}$$

By joining all the previous relationships we have demonstrated that (72) reduces to

$$A_n \tilde{A}_n \quad = \quad \sum_{i=1}^{N_\epsilon}\sum_{j=1}^{\tilde{N}_\epsilon} B_n^{(i,j,\epsilon)} + \hat{R}_{n,\epsilon} + \hat{N}_{n,\epsilon}, \quad \forall n > n_\epsilon$$

with $\{B_n^{(i,j,\epsilon)}\} \sim_{\mathrm{LT}} (a_i \tilde{a}_j, f_i \tilde{f}_j)$ and therefore the claimed thesis is proved. $\square$

Remark 5.1 *It is worth observing that the additional assumption* (55) *is always satisfied by a sequence of multilevel Toeplitz matrices* $\{T_n(f)\}$*, generated by* $f \in L^\infty$ *(this is a straightforward consequence of the Szegő theory). Therefore, when the generating functions are bounded, the product of any generalized locally Toeplitz sequence with multilevel Toeplitz sequences is also a generalized locally Toeplitz sequence; in particular, the product of a finite number of multilevel Toeplitz sequences is a generalized locally Toeplitz sequence (we recall that, in general, the product of two Toeplitz matrices is not a Toeplitz matrix).*

Example 5.6 (Algebra generated by multilevel Toeplitz sequences) In [23] Tyrtyshnikov proved that the singular values of a finite product of multilevel Toeplitz sequences are distributed as the magnitude of the product of their generating functions, that is

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F\Big(\sigma_j\left(T_n(f_1)\cdots T_n(f_p)\right)\Big) = \frac{1}{m\{Q^d\}} \int_{Q^d} F\left(|f_1(s)\cdots f_p(s)|\right) ds$$

whenever all the f_i's belong to L^∞. We can extend it to handle all sequences in the algebra generated by Toeplitz sequences, as follows. Let $\mathcal{T}_\infty$ be the vector space of all multilevel Toeplitz sequences $\{T_n(f)\}$ with $f \in L^\infty$, and let $\mathcal{A}_\infty$ be the algebra of multilevel sequences generated by $\mathcal{T}_\infty$; it is clear that any sequence $\{A_n\} \in \mathcal{A}_\infty$ can be written as

$$A_n = \sum_\alpha \prod_\beta T_n(f_{\alpha\beta}), \quad f_{\alpha\beta} \in L^\infty$$

(if necessary, one can let $f_{\alpha\beta} \equiv 1$ for some indices α, β, observing that $T_n(1) = \mathrm{I}_{N(n)}$ does not affect the product). From Theorem 5.2, Theorem 5.5, and Remark 5.1, we know that, for each α, the sequence $\{\prod_\beta T_n(f_{\alpha\beta})\}$ is a generalized locally Toeplitz with respect to the kernel $\kappa(x,s) = \prod_\beta f_{\alpha\beta}(s)$, and hence from Theorem 4.7 we can conclude that

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F\Big(\sigma_j\left(\sum_\alpha \prod_\beta T_n(f_{\alpha\beta})\right)\Big) =$$

$$\frac{1}{m\{Q^d\}} \int_{Q^d} F\Big(\Big|\sum_\alpha \prod_\beta f_{\alpha\beta}(s)\Big|\Big)\, ds;$$

concerning eigenvalues, from Theorem 4.10 we obtain

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F\Big(\lambda_j\left(\mathrm{Re} \sum_\alpha \prod_\beta T_n(f_{\alpha\beta})\right)\Big) =$$

$$\frac{1}{m\{Q^d\}} \int_{Q^d} F\Big(\mathrm{Re}\ \sum_\alpha \prod_\beta f_{\alpha\beta}(s)\Big)\, ds,$$

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F\Big(\lambda_j\left(\mathrm{Im} \sum_\alpha \prod_\beta T_n(f_{\alpha\beta})\right)\Big) =$$

$$\frac{1}{m\{Q^d\}} \int_{Q^d} F\Big(\mathrm{Im}\ \sum_\alpha \prod_\beta f_{\alpha\beta}(s)\Big)\, ds.$$

6 Applications to discretized second-order elliptic boundary-value problems

In this section we come back to the PDEs given in (8) with the aim of proving that a finite difference discretization of such equations leads to generalized locally Toeplitz matrix-sequences. For the sake of simplicity we start the analysis by referring to the equations in the form (9).

Since the operators of finite differences are linear and the considered PDEs are linear it follows that the matrix $A_n(A, P, W_{\mathbf{a}})$ considered in the introduction can be decomposed as follows

$$(76) \qquad A_n(A, P, W_{\mathbf{a}}) = \sum_{i,j=1}^{d} \tilde{A}_n(a_{i,j}E_{i,j}, P_{i,j}E_{i,j}, a_i a_j E_{i,j}) + \mathcal{A}_n(A)$$

where $E_{i,j} = e_i e_j^T$, $(e_k)_t = \delta_{k-t}$ with δ being the Kronecker delta. The matrix $\mathcal{A}_n(A)$ denotes the discretization of the first-order differential terms appearing in (9) and $\tilde{A}_n(B, P, W_{\mathbf{a}})$ denotes the discretization of the operator $-e^T[B(x)\circ H_u]e$, for $B(x) = (b_{i,j}(x))_{i,j=1}^d$, $b_{i,j} : \Omega_d \to \mathbf{C}$.

Lemma 6.1 *Let $a_{i,j}$ be continuous. The sequence*

$$\{v^{-2}\tilde{A}_n(a_{i,j}E_{i,j}, P_{i,j}E_{i,j}, a_i a_j E_{i,j})\}$$

with $n = (n_1, \ldots, n_d)$, $n_j = va_j$ and $a_j \in \mathbf{N}^+$ for $j = 1, \ldots, d$, is a multilevel locally Toeplitz sequence with respect to $(a_{i,j}(x), a_i a_j P_{i,j}(s))$.

Proof. The matrix $v^{-2}\tilde{A}_n(a_{i,j}E_{i,j}, P_{i,j}E_{i,j}, a_i a_j E_{i,j})$ can be written as

$$D_{n,a_{i,j}} T_n(a_i a_j P_{i,j}) = a_i a_j D_{n,a_{i,j}} T_n(P_{i,j})$$

where $P_{i,j} = p_i(s_i)p_j(s_j)$ is a separable polynomial and

$$D_{n,a_{i,j}} = \operatorname*{Diag}_{l_1=1,\ldots,m_1} \operatorname*{Diag}_{l_2=1,\ldots,m_2} \cdots \operatorname*{Diag}_{l_d=1,\ldots,m_d} a_{i,j}(l/m).$$

By direct inspection we have

$$\{D_{n,a_{i,j}}\} \sim_{\mathrm{LT}} (a_{i,j}, 1)$$

and by Theorem 5.1 we have $\{T_n(P_{i,j})\} \sim_{\mathrm{LT}} (1, P_{i,j})$. Therefore by Theorem 5.3 we conclude $\{a_i a_j D_{n,a_{i,j}} T_n(P_{i,j})\} \sim_{\mathrm{LT}} (a_{i,j}, a_i a_j P_{i,j})$. □

Theorem 6.2 *Let $a_{i,j}$ be continuous for any i and j. Suppose that the multi-index $n = (n_1, \ldots, n_d)$ is such that $n_j = va_j$ and $a_j \in \mathbf{N}^+$ for any j. Then the matrix-sequences $\{v^{-2}A_n(A, P, , W_{\mathbf{a}})\}$ and $\{v^{-2}\tilde{A}_n(A, P, W_{\mathbf{a}})\}$ can be expressed as the sum of d^2 multilevel locally Toeplitz sequences.*

Proof. Observe that $v^{-2}\mathcal{A}_n(A)$ can be written as a term of spectral norm infinitesimal as v^{-1}. Now, it is a straightforward consequence of the decomposition (76) and of Lemma 6.1. □

Corollary 6.3 *Let $a_{i,j}$ be continuous for any i and j. Suppose that the multi-index $n = (n_1, \ldots, n_d)$ is such that $n_j = va_j$ and $a_j \in \mathbf{N}^+$ for any j. For any continuous function F with bounded support it holds*

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\sigma_j(v^{-2}A_n(A, P, W_{\mathbf{a}}))] =$$

$$\frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d \times Q^d} F\left(\left|e^T[A(x) \circ P(s) \circ W_{\mathbf{a}}]e\right|\right) dx\, ds, \tag{77}$$

$$x \in \Omega_d = [0,1]^d,\ s \in Q^d = (-\pi, \pi)^d.$$

Proof. By Theorem 6.2, we know that $\{v^{-2}A_n(A,P,W_{\mathbf{a}})\}$ is a generalized locally Toeplitz sequence with respect to $|e^T[A(x)\circ P(s)\circ W_{\mathbf{a}}]e|$. The application of Theorem 4.7 concludes the proof. $\square$

If $A(x)$ is symmetric, then we arrange the choice of $P_{i,j}$ so that $P(s)$ is a Hermitian, nonnegative definite dyad and consequently the previous formula can be also stated in the sense of the eigenvalues. We proceed as follows. We leave the differential operator in divergence form as in (8) and for any derivative appearing in the differential operator we use the same finite difference formula [12]. To clarify our point we consider a simple example of a centered finite difference formula discretizing the first derivative of a two times continuously differentiable function u (for more details see [18]):

$$\frac{\partial}{\partial x_t}u(x)|_{x=\bar{x}} = \frac{u(\bar{x}+h_te_t)-u(\bar{x}-h_te_t)}{2h_t}+O(h_t^2)$$

so that

$$\begin{aligned}\frac{\partial^2}{\partial x_i\partial x_j}u(x)|_{x=\bar{x}} \quad &= \quad \frac{1}{h_ih_j}[u(\bar{x}+h_ie_i+h_je_j)-u(\bar{x}+h_ie_i-h_je_j)\\ &\quad -u(\bar{x}-h_ie_i+h_je_j)+u(\bar{x}-h_ie_i-h_je_j)]\\ &\quad +O(h_i^2)+O(h_j^2).\end{aligned}$$

Now setting $n_j = va_j$, $h_j=(n_j+1)^{-1}$ and $q(t)=e^{\hat{\mathbf{i}}t}-e^{-\hat{\mathbf{i}}t}$ the trigonometric polynomial generating the Toeplitz sequence associated to the (univariate) formula $v'(\bar{x}) = \frac{v(\bar{x}+h)-v(\bar{x}-h)}{2h}+O(h^2)$, $x\in\mathbf{R}$ we deduce that

$$P_{i,j}(s) = -a_ia_jq(s_i)q(s_j).$$

But $\overline{q(t)} = -q(t)$ and consequently $P_{i,j}(s) = a_ia_j\overline{q(s_i)}q(s_j) = P_{j,i}(s)$ and $P_{i,i}(s)=a_i^2|q(s_i)|^2 = a_i^2(2-2\cos(s_i))$. It is evident that

$$P(s)=\mathbf{q}(s)\mathbf{q}^H(s),\quad \mathbf{q}_i(s)=q(s_i)$$

and therefore $P(s)$ is a nonnegative definite dyad of polynomials. Finally $P(s)\circ W_{\mathbf{a}} = [\mathbf{a}\circ\mathbf{q}(s)][\mathbf{a}\circ\mathbf{q}(s)]^H$ is the nonnegative definite dyad of polynomials representing the discretization of the operator $-H_u=[\nabla u]\cdot[\nabla u]^T$ over $\mathcal{G}$ where $\cdot$ is the usual multiplication between matrices but where the product of the entries is the "composition" of the corresponding differential operators.

As a consequence the discretized matrix $A_n(A,P,W_{\mathbf{a}})$ can be written as

$$A_n(A,P,W_{\mathbf{a}}) = \sum_{i,j=1}^{d}\hat{A}_n(a_{i,j}E_{i,j},P_{i,j}E_{i,j},a_ia_jE_{i,j}) \tag{78}$$

where $\hat{A}_n(B, P, W_{\mathbf{a}})$ denotes the discretization of the operator

$$-\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}\left(b_{i,j}(x)\frac{\partial}{\partial x_j}u(x)\right)$$

for $B(x) = (b_{i,j}(x))_{i,j=1}^d$, $b_{i,j} : \Omega_d \to \mathbf{C}$.

The following facts are true:

F1. $\hat{A}_n(a_{i,j}E_{i,j}, P_{i,j}E_{i,j}, a_i a_j E_{i,j}) = \left[\hat{A}_n(a_{j,i}E_{j,i}, P_{j,i}E_{j,i}, a_i a_j E_{j,i})\right]^T$ if $a_{i,j} = a_{j,i}$.

F2. The matrix $\hat{A}_n(a_{i,i}E_{i,i}, P_{i,i}E_{i,i}, a_i^2 E_{i,i})$ is symmetric semidefinite if $a_{i,i} \geq 0$ and the matrix $\hat{A}_n(a_{i,i}E_{i,i}, P_{i,i}E_{i,i}, a_i^2 E_{i,i})$ is a positive definite multilevel Toeplitz matrix generated by a nonnegative polynomial $a_i^2|q(s_i)|^2$ if $a_{i,i}$ is the constant function 1 (see Theorems 3.4 and 3.6 of [12]).

F3. $\|v^{-2}(\tilde{A}_n(A, P, W_{\mathbf{a}}) - \hat{A}_n(A, P, W_{\mathbf{a}}))\|_2 \to 0$ if v tends to infinity (in particular the spectral norm of the difference goes to zero as $\omega_A(v^{-1}) = \max_{i,j}\omega_{a_{i,j}}(v^{-1})$ where $\omega_g(\cdot)$ denotes the modulus of continuity of g.

Corollary 6.4 *Let $A(x) = (a_{i,j}(x))_{i,j=1}^d$ be symmetric and let $a_{i,j}$ be continuous for any i and j. Suppose that the multi-index $n = (n_1, \ldots, n_d)$ is such that $n_j = va_j$ and $a_j \in \mathbf{N}^+$ for any j and suppose that any derivative appearing in the differential operator* (8) *is discretized by using the same finite difference formula. Then for any continuous function F with bounded support it holds*

$$\lim_{n\to\infty}\frac{1}{N(n)}\sum_{j=1}^{N(n)} F[\lambda_j(v^{-2}A_n(A, P, W_{\mathbf{a}}))] =$$

$$\frac{1}{m\{\Omega_d \times Q^d\}}\int_{\Omega_d\times Q^d} F\left(e^T[A(x)\circ P(s)\circ W_{\mathbf{a}}]e\right)dx\,ds, \tag{79}$$

$$x \in \Omega_d = [0,1]^d,\ s \in Q^d = (-\pi, \pi)^d.$$

Proof. Owing to **F1.** and **F2.** it follows that each $A_n(A, P, W_{\mathbf{a}})$ is Hermitian. By Theorem 6.2, we know that $\{v^{-2}\tilde{A}_n(A, P, W_{\mathbf{a}})\}$ is a generalized locally Toeplitz sequence with respect to $e^T[A(x)\circ P(s)\circ W_{\mathbf{a}}]e$. From the relation **F3.** and by the definition of generalized locally Toeplitz sequence we infer that $\{v^{-2}\hat{A}_n(A, P, W_{\mathbf{a}})\}$ is a generalized locally Toeplitz sequence as well with respect to the same kernel. Finally by invoking Theorem 4.10 the proof is concluded. □

Remark 6.1 *Suppose that problems as* (8) *are given with $A(x)$ being piecewise continuous, i.e. there exists a constant number c of disjoint Peano-Jordan measurable sets Ω_j with $\Omega_d = \cup_{j=1}^c \Omega_j$ such that the restriction of $A(x)$ over each Ω_j*

is continuous. In this case it is possible to prove by approximation techniques that the resulting discretizing matrix-sequence $\{v^{-2}\hat{A}_n(A,P,W_{\mathbf{a}})\}$ is a generalized locally Toeplitz sequence. We notice that under these assumptions the functions $a_{i,j}$ are Riemann integrable as required by the definition of multilevel locally Toeplitz sequences.

Suppose now that the differential problem is defined over a bounded subset $\Omega^{'}$ of $\mathbf{R}^d$ with $\Omega^{'}$ not necessarily iperrectangular. Without loss of generality we can always suppose that $\Omega^{'} \subset \Omega_d$. If it is not the case a linear change of coordinates leads to the desired inclusion.

Given $A(x)$ defined over $\Omega^{'}$ we consider its extension $\hat{A}(x)$ over Ω_d in the following way: $\hat{A}(x) = A(x)$ if $x \in \Omega^{'}$ and zero otherwise.

With these premise it is easy to see that $A_n(A,P,W_{\mathbf{a}},\Omega^{'})$ is a submatrix of dimension $d_n(\Omega^{'}) \times d_n(\Omega^{'})$ of $A_n(\hat{A},P,W_{\mathbf{a}})$ while the other rows and columns are zero since the function $\hat{A}(x)$ vanishes outside $\Omega^{'}$. Therefore, in light of Remark 6.1, the application of Corollary 6.3 yields the following formula:

$$\lim_{n\to\infty} \frac{1}{N(n)} \sum_{j=1}^{N(n)} F[\sigma_j(v^{-2}A_n(\hat{A},P,W_{\mathbf{a}}))] =$$

$$\frac{1}{m\{\Omega_d \times Q^d\}} \int_{\Omega_d\times Q^d} F\left(\left|e^T[\hat{A}(x)\circ P(s)\circ W_{\mathbf{a}}]e\right|\right) dx\, ds,$$

$$x \in \Omega_d = [0,1]^d, \ s \in Q^d = (-\pi,\pi)^d.$$

i.e.

$$\lim_{n\to\infty} \frac{d_n(\Omega^{'})}{N(n)} \left(\frac{1}{d_n(\Omega^{'})} \sum_{j=1}^{d_n(\Omega^{'})} F[\sigma_j(v^{-2}A_n(A,P,W_{\mathbf{a}},\Omega^{'}))] \right) + \left(1 - \frac{d_n(\Omega^{'})}{N(n)}\right) F(0)$$

$$= \frac{m\{\Omega^{'}\times Q^d\}}{m\{\Omega_d \times Q^d\}} \left(\frac{1}{m\{\Omega^{'}\times Q^d\}} \int_{\Omega^{'}\times Q^d} F\left(\left|e^T[A(x)\circ P(s)\circ W_{\mathbf{a}}]e\right|\right) dx\, ds \right) +$$

$$\frac{m\{\left(\Omega_d\backslash\Omega^{'}\right)\times Q^d\}}{m\{\Omega_d\times Q^d\}} F(0), \quad x\in\Omega^{'}, \ s \in Q^d = (-\pi,\pi)^d.$$

The above formulas are not completely satisfactory because we would like to obtain a formula involving only $\Omega^{'}$, $d_n(\Omega^{'})$ and $A(x)$. To do this we use the following lemma.

Lemma 6.5 [17] *If $\Omega^{'}$ is Peano-Jordan measurable (i.e. its characteristic function is integrable in the (multidimensional) Riemann sense), then*

$$\lim_{n\to\infty} \frac{d_n(\Omega^{'})}{N(n)} = \frac{m\{\Omega^{'}\}}{m\{\Omega_d\}} = \frac{m\{\Omega^{'}\times Q^d\}}{m\{\Omega_d\times Q^d\}}.$$

By using the latter lemma, we have

$$\lim_{n\to\infty} \frac{1}{d_n(\Omega')} \sum_{j=1}^{d_n(\Omega')} F[\sigma_j(v^{-2}A_n(A,P,W_{\mathbf{a}},\Omega'))] =$$
$$\frac{1}{m\{\Omega' \times Q^d\}} \int_{\Omega' \times Q^d} F\left(\left|e^T[A\,(x) \circ P(s) \circ W_{\mathbf{a}}]e\right|\right) dx\, ds$$
$$x \in \Omega',\; s \in Q^d = (-\pi,\pi)^d.$$

Remark 6.2 *The results presented in this section have a theoretical interest in its own in order to understand the spectral behaviour of these matrix-sequences and its relationships with the properties of the continuous problems* (8)*. However another important aspect is of practical nature. The considered analysis provides a theoretical tool in order to devise and to analyze optimal and superlinear preconditioners for the numerical solution of linear systems arising from the discretization of PDEs as* (8) *by preconditioned conjugate gradient methods (for more details see* [11, 12, 13]*).*

7 Concluding remarks and further applications

The generalized locally Toeplitz sequences are spectrally described by a kernel $\kappa(x,s)$. Other problems as linear integral equations are spectrally described by a kernel. The question is: is this connection deeper? Can we describe the discretization of linear integral equations in terms of generalized locally Toeplitz sequences?

Let us consider the following problem: find u such that

$$\alpha u(x) + \int_0^\tau \kappa(x,s)u(s)ds = f(x) \tag{80}$$

where $0 < \tau \leq \infty$, $\alpha \in \mathbf{R}$, $x, s \in [0,\tau]$, $u, g \in C([0,\tau])$ and $\kappa \in C([0,\tau]^2)$.

If $\kappa(x,s)$ is shift invariant namely $\kappa(x,s) = \eta(x-s)$, then the discretization of (80) which is now a Wiener-Hopf equation leads to a sequence of Toeplitz linear systems. If in addition $\alpha = 0$, then we find a kind of equation that models the blurring in image restoration problems with "spatially invariant" kernel [3].

If $\kappa(x,s)$ is locally shift invariant i.e. $\kappa(x,s) = d(s)\eta(x-s)$, then the discretization of (80) is described by linear systems whose n-th coefficient matrix is expressible as the product of a Toeplitz matrix times a diagonal "sampling" matrix: the similarity with locally Toeplitz sequences is remarkable. If in addition $\alpha = 0$, then we find a kind of equation that models the blurring in image restoration problems with "spatially variant" kernel [3].

There is the open problem of understanding how this can help in the spectral analysis of these matrix-sequences and in the numerical solution of (80) by preconditioned conjugate gradient methods (see [4, 3]).

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